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LIPSCHITZ B-PREINVEX FUNCTIONS AND NONSMOOTH MULTIOBJECTIVE PROGRAMMING*

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Abstract: In this paper, a class of B-preinvex functions introduced by Bector et al. [2] are considered. Necessary and sufficient conditions, under which a locally Lipschitz function is B-preinvex, are established in terms of the Clarke subdifferentiable. Moreover, a sufficient optimality condition of efficient solutions for a nonsmooth multiobjective programming problem involving B-preinvex functions is obtained. Finally, weak and strong duality theorems are proved for Mond-Weir type dual under B-preinvexity assumption.

Key words: *B-preinvex function, regularity, nonsmooth multiobjective programming, efficient solution, optimality condition, duality*

Mathematics Subject Classification: *90C26, 90C29, 90C46*

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1 Introduction

In optimization theory, the convexity plays an important role in deriving optimality conditions and duality for the nonlinear programming problem. Various classes of generalized of convex functions have appeared in the literature for the purpose of weaking the limitation of convexity. Hanson [5] introduced a class of functions which were called invex by Craven [4] as a generalization of convexity. Later, Weir and Mond [13] introduced preinvex functions, and they studied how and where preinvex functions can replace convex functions in optimization problem.

On the other hand, Bector and Singh [1] considered a class of functions, called B-vex functions, which are also generalization of convex function. They also studied the properties of differentiable B-vex functions. Later, the concept of B-vexity of functions was extend to B-invex and B-preinvex functions by Bector et al. [2], to explicitly B-preinvex functions by Yang et al. [14], to semilocally B-preinvex functions by Stancu-Minasian [11], to semi-B-preinvex functions by Long and Peng [8]. For a differentiable programming problem involving B-vex and B-invex functions, sufficient optimality conditions and duality results for Mond-Weir duality were otained by Bector et al. [2]. Recently, Li et al. [7] obtained necessary and sufficient optimality conditions for a nonsmooth single-objective programming involving B-vex functions. In [12], Suneja et al. obtained some properties of B-preinvex

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functions. Very recently, Stancu-Minasian [11] derived some sufficient optimality conditions and duality results for nonsmooth single-objective programming problems under semilocally B-preinvexity.

In this paper, we consider the B-preinvex functions introduced by Bector et al. [2]. In terms of the Clarke subdifferentiable, some characterizations of B-preinvex are derived under suitable conditions. Moreover, a sufficient optimality condition is obtained for a nonsmooth multiobjective programming problem involving B-preinvex functions. Finally, weak and strong duality theorems are proved for Mond-Weir type dual under B-preinvexity assumption.

2 Preliminaries

Throughout this paper, let R^n be the n-dimensional Euclidean space with Euclidean norm $\|\cdot\|$ and the usual inner product $\langle\cdot,\cdot\rangle$, respectively. Let X be a nonempty subset of R^n and denote by R_+ the set of nonnegative real numbers. Suppose that $f : X \to R$, $\eta : X \times X \to X$, and $b : X \times X \times [0,1] \to R_+$ such that $\lambda b(x,y,\lambda) \in [0,1]$ for all $x, y \in X$ and $\lambda \in [0,1]$.

We now recall some definitions as follows.

Definition 2.1 ([13])**.** A set X is said to be invex at $y \in X$ with respect to η if, for all $x \in X$ and $\lambda \in [0,1]$, we have

$$y + \lambda\eta(x,y) \in X.$$

The set X is said to be invex with respect to η if X is invex at each $y \in X$ with respect to same η.

Definition 2.2 ([2])**.** Let X be a nonempty invex set in R^n with respect to η. The function f is said to be B-preinvex at $y \in X$ with respect to η and b if for all $x \in X$ and $\lambda \in [0,1]$, one has

$$f(y + \lambda\eta(x,y)) \le \lambda b(x,y,\lambda)f(x) + (1 - \lambda b(x,y,\lambda))f(y).$$

f is said to be B-preinvex on X with respect to η and b if it is B-preinvex at each $y \in X$ with respect to same η and b.

Remark 2.3. Every preinvex function with respect to η is B-preinvex with respect to η, b, where

$$b(x,y,\lambda) = 1.$$

However, the converse is not true, as can be seen from the following example.

Example 2.4. Let $f : R \to R$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \le 0; \\ -x & \text{if } x > 0. \end{cases}$$

Then, f is B-preinvex with respect to η, b, where

$$\eta(x,y) = \begin{cases} 0 & \text{if } x \le 0, y > 0; \\ x - y & \text{if } x \le 0, y \le 0; \\ x - y & \text{if } x > 0, y > 0; \\ y - x & \text{if } x > 0, y \le 0 \end{cases}$$

and

$$b(x, y, \lambda) = \begin{cases} \lambda & \text{if } x \leq 0, y > 0; \\ 1 & \text{if } x \leq 0, y \leq 0; \\ 1 & \text{if } x > 0, y > 0; \\ 0 & \text{if } x > 0, y \leq 0. \end{cases}$$

But f is not preinvex with respect to η, because

$$f(y + \lambda\eta(x, y)) > \lambda f(x) + (1 - \lambda)f(y), \text{ for } x = 1,\ y = 0,\ \lambda = \frac{1}{2}.$$

Remark 2.5. Every B-vex function with respect to b is B-preinvex with respect to η, b, where

$$\eta(x, y) = x - y.$$

But the converse is not true.

Example 2.6. The function f considered in Example 2.4 is B-preinvex with respect to η, b, but f is not a B-vex function with respect to b, because

$$f(\lambda x + (1 - \lambda)y) > \lambda b(x, y, \lambda)f(x) + (1 - \lambda b(x, y, \lambda))f(y), \text{ for } x = 0,\ y = 1,\ \lambda = \frac{1}{2}.$$

Let f be a real-valued function defined on an open subset U of R^n. f is said to be directionally differentiable at a point $u \in U$ if the one-sided directional derivative

$$f'(x; v) = \lim_{t \downarrow 0} \frac{[f(x + tv) - f(x)]}{t}$$

exists for every $v \in R^n$.

A real-valued function $f : R^n \to R$ is said to be locally Lipscitz at a point $u \in R^n$ if there exists a number $K > 0$ such that

$$|f(x) - f(y)| \leq K\|x - y\|,$$

for all x, y in a neighbourhood of u. A function f is said to be locally Lipschitiz on R^n if it is locally Lipschitz at each point of R^n.

Let f be a locally Lipschitz function at x, the Clarke [3] generalized directional derivative of f at x in the direction $v \in R^n$ is defined by

$$f^\circ(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}$$

and the Clarke [3] generalized gradient of f at x is denoted by

$$\partial f(x) = \{\xi \in R^n \mid f^\circ(x; v) \geq \langle \xi, v \rangle, \forall\, v \in R^n\}.$$

It follows that

$$f^\circ(x; v) = \max\{\langle \xi, v \rangle \mid \xi \in \partial f(x)\},\ \forall\, v \in R^n.$$

When f is locally Lipschitz at x, f is said to be regular [3] at x if it is directionally differentiable at x and if

$$f^\circ(x; v) = f'(x; v),\ \forall\, v \in R^n.$$

Mohan and Neogy [9] introduced Condition C defined as follows.

Condition C ([9]). The vector-valued function $\eta : X \times X \to X$ is said to satisfy Condition C if for any $x, y \in X$ and $\lambda \in [0, 1]$,

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y),$$
$$\eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y).$$

And they proved that a differentiable function which is invex with respect to η is also preinvex under Condition C. Mohan and Neogy [9] also gave an example which shows that Condition C may hold for a general class of function η, rather than just for the trivial case of $\eta(x, y) = x - y$.

3 Characterizations of B-Preinvex Functions

In this section, we obtain necessary and sufficient conditions for a locally Lispschitz function to be B-preinvex functions.

Theorem 3.1. *Let f be a locally Lipschitz function on X. Suppose that*

(i) *f is B-preinvex with respect to η, b at $y \in X$, and $\lim_{\lambda\downarrow 0} b(x, z, \lambda) = b(x, z, 0)$ for any $x, z \in X$;*

(ii) *f is regular at y.*

Then, for all $x \in X$,

$$b(x, y, 0)[f(x) - f(y)] \geq \langle \eta(x, y), \xi \rangle, \ \forall\, \xi \in \partial f(y). \tag{3.1}$$

Proof. Since f is B-preinvex at $y \in X$, for all $x \in X$ and $\lambda \in (0, 1)$, one has

$$f(y + \lambda\eta(x, y)) \leq \lambda b(x, y, \lambda)f(x) + (1 - \lambda b(x, y, \lambda))f(y),$$

which implies

$$\frac{f(y + \lambda\eta(x, y)) - f(y)}{\lambda} \leq b(x, y, \lambda)[f(x) - f(y)].$$

Taking limit as $\lambda \downarrow 0$, and noting that f is regular at y, we get

$$\begin{aligned} b(x, y, 0)[f(x) - f(y)] &= \lim_{\lambda\downarrow 0} b(x, y, \lambda)[f(x) - f(y)] \\ &\geq \lim_{\lambda\downarrow 0} \frac{f(y + \lambda\eta(x, y)) - f(y)}{\lambda} \\ &= f'(y; \eta(x, y)) \\ &= f^\circ(y; \eta(x, y)) \\ &= \max\{\langle \xi, \eta(x, y)\rangle \mid \xi \in \partial f(y)\}, \end{aligned}$$

that is

$$b(x, y, 0)[f(x) - f(y)] \geq \langle \eta(x, y), \xi \rangle, \ \forall\, \xi \in \partial f(y).$$

This completes the proof. □

Remark 3.2. It is worth nothing that the assumption that the function f is regular at $y \in X$ in Theorem 3.1 is essential. To illustrate this point, we give an example.

Example 3.3. The function f considered in Example 2.4 is B-preinvex with respect to η, b defined there. Moreover, f is globally Lipschitz of rank 1 on R and is directionally differentiable at each point in R and

$$\partial f(0) = [-1, 0], \quad f'(0; -1) = 0,$$

$$f^\circ(0; -1) = \max\{\langle \xi, -1 \rangle \mid \xi \in [-1, 0]\} = 1.$$

Therefore, f is not regular at 0. Now, let

$$x = 1, \quad y = 0, \quad \xi = -1 \in \partial f(0).$$

It follows that

$$b(x, y, 0)[f(x) - f(y)] = 0,$$

$$\langle \eta(x, y), \xi \rangle = \langle y - x, \xi \rangle = 1$$

and so

$$b(x, y, 0)[f(x) - f(y)] < \langle \eta(x, y), \xi \rangle.$$

Hence, relation (3.1) is not hold.

Corollary 3.4. *Let f be a locally Lipschitz function on X. Suppose that*

(i) *f is B-preinvex with respect to η, b on X, and $\lim_{\lambda \downarrow 0} b(x, y, \lambda) = b(x, y, 0)$ for any $x, y \in X$;*

(ii) *f is regular on X;*

(iii) *η is a skew function, i.e., $\eta(x, y) + \eta(y, x) = 0$, for all $x, y \in X$.*

Then, for all $x, y \in X$,

$$b(x, y, 0)\langle \eta(x, y), \zeta \rangle - b(y, x, 0)\langle \eta(x, y), \xi \rangle \geq 0, \ \forall \, \xi \in \partial f(y), \ \forall \, \zeta \in \partial f(x).$$

Proof. By Theorem 3.1, for any $x, y \in X$, we have

$$b(x, y, 0)[f(x) - f(y)] \geq \langle \eta(x, y), \xi \rangle, \ \forall \, \xi \in \partial f(y), \tag{3.2}$$

$$b(y, x, 0)[f(y) - f(x)] \geq \langle \eta(y, x), \zeta \rangle, \ \forall \, \zeta \in \partial f(x). \tag{3.3}$$

Multiplies both side of (3.2) by $b(y, x, 0)$ and of (3.3) by $b(x, y, 0)$, adding them and using $\eta(x, y) + \eta(y, x) = 0$, we obtain

$$b(x, y, 0)\langle \eta(x, y), \zeta \rangle - b(y, x, 0)\langle \eta(x, y), \xi \rangle \geq 0, \ \forall \, \xi \in \partial f(y), \ \forall \, \zeta \in \partial f(x).$$

This completes the proof. □

We now give a sufficient condition for a locally Lipschitz function to be B-preinvex.

Theorem 3.5. *Let f be a locally Lipschitz function on X. Suppose that*

(i) *there exists a function* $k : X \times X \to R_{++}$ *(the set of positive real numbers) such that for all* $x, y \in X$*,*

$$k(x,y)[f(x) - f(y)] \geq \langle \eta(x,y), \xi \rangle, \ \forall\, \xi \in \partial f(y); \tag{3.4}$$

(ii) *the function* $\eta : X \times X \to X$ *satisfies Condition C.*

Then, f *is a B-preinvex function on* X *with respect to* η *and* b*, where*

$$b(x,y,\lambda) = \frac{k(x, y + \lambda\eta(x,y))}{\lambda k(x, y + \lambda\eta(x,y)) + (1-\lambda)k(y, y + \lambda\eta(x,y))}.$$

Proof. We first note that b in the theorem satisfies the condition that

$$\lambda b(x,y,\lambda) \in [0,1], \quad \text{for all } x, y \in X \text{ and } \lambda \in [0,1].$$

Now, for any $x, y \in X$ and $\lambda \in (0,1)$, we take $\xi \in \partial f(y + \lambda\eta(x,y))$. By (3.4) and Condition C, we have

$$\begin{aligned} & k(x, y + \lambda\eta(x,y))[f(x) - f(y + \lambda\eta(x,y))] \\ \geq & \langle \eta(x, y + \lambda\eta(x,y)), \xi \rangle \\ = & (1-\lambda)\langle \eta(x,y), \xi \rangle, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} & k(y, y + \lambda\eta(x,y))[f(y) - f(y + \lambda\eta(x,y))] \\ \geq & \langle \eta(y, y + \lambda\eta(x,y)), \xi \rangle \\ = & (-\lambda)\langle \eta(x,y), \xi \rangle. \end{aligned} \tag{3.6}$$

Multiplying (3.5) by λ and (3.6) by $1-\lambda$, and adding them, we get

$$\lambda k(x, y+\lambda\eta(x,y))[f(x) - f(y+\lambda\eta(x,y))] + (1-\lambda)k(y, y+\lambda\eta(x,y))[f(y) - f(y+\lambda\eta(x,y))] \geq 0.$$

It follows that

$$\begin{aligned} & \lambda k(x, y + \lambda\eta(x,y))f(x) + (1-\lambda)k(y, y + \lambda\eta(x,y))f(y) \\ \geq & [\lambda k(x, y + \lambda\eta(x,y)) + (1-\lambda)k(y, y + \lambda\eta(x,y))]f(y + \lambda\eta(x,y)). \end{aligned} \tag{3.7}$$

Dividing both sides of (3.7) by

$$\lambda k(x, y + \lambda\eta(x,y)) + (1-\lambda)k(y, y + \lambda\eta(x,y))$$

and taking

$$b(x,y,\lambda) = \frac{k(x, y + \lambda\eta(x,y))}{\lambda k(x, y + \lambda\eta(x,y)) + (1-\lambda)k(y, y + \lambda\eta(x,y))},$$

we have the conclusion. This completes the proof. □

4 Optimality Conditions

In the sequel, for any $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T \in R^n$, we will use the following conventions:

$$
\begin{aligned}
&x > y \Leftrightarrow x_i > y_i, \quad i = 1, 2, \cdots, n; \\
&x \geqq y \Leftrightarrow x_i \geqq y_i, \quad i = 1, 2, \cdots, n; \\
&x \geq y \Leftrightarrow x_i \geqq y_i, \quad i = 1, 2, \cdots, n, \text{ but } x \neq y, \\
&x \not\geq y \text{ is the negation of } x \geq y.
\end{aligned}
$$

Consider the following multiobjective programming problem:

$$
\begin{aligned}
\text{(MP)} \quad \text{Minimize} \quad & f(x) = (f_1(x), f_2(x), \cdots, f_p(x))^T \\
\text{s.t.} \quad & x \in S = \{x \in X : g(x) = (g_1(x), g_2(x), \cdots, g_m(x))^T \leqq 0\},
\end{aligned}
$$

where $X \subseteq R^n$ is an open set, the functions $f : X \to R^p$ and $g : X \to R^m$ are locally Lipschitz functions.

In the sequel, unless specified otherwise, let $\eta(x, y) \neq 0$ for all $x \neq y$.

Definition 4.1. A point $x_0 \in S$ is said to be an efficient solution for (MP) if there does not exist $x \in S$ such that $f(x) \leq f(x_0)$.

Now we establish the following sufficient optimality condition for (MP).

Theorem 4.2. *Let $x_0 \in S$. Suppose that*

(i) *f_i is B-preinvex with respect to η, b_{f_i} at x_0 for $i = 1, 2, \cdots, p$, and g_j is B-preinvex with respect to η, b_{g_j} at x_0 for $j = 1, 2, \cdots, m$;*

(ii) *$\lim_{\lambda \downarrow 0} b_{f_i}(x, y, \lambda) = b_{f_i}(x, y, 0) > 0$ and $\lim_{\lambda \downarrow 0} b_{g_j}(x, y, \lambda) = b_{g_j}(x, y, 0) \geq 0$ for any $x, y \in X$;*

(iii) *f_i, $i = 1, 2, \cdots, p$, and g_j, $j = 1, 2, \cdots, m$, are regular at x_0.*

If there exist $\mu^* \in R^p$, $\mu^* > 0$, and $\beta^* \in R^m$ with $\beta^* \geqq 0$ such that

$$0 \in {\mu^*}^T \partial f(x_0) + {\beta^*}^T \partial g(x_0), \tag{4.1}$$

$${\beta^*}^T g(x_0) = 0, \tag{4.2}$$

then, x_0 is an efficient solution for (MP).

Proof. Suppose by contradiction that x_0 is not an efficient solution for (MP). Then, there exists $x \in S$ such that $f(x) \leq f(x_0)$. Since $\mu^* > 0$ and $b_{f_i} > 0$, $i = 1, 2, \cdots, p$,

$${\mu^*}^T b_f(x, x_0, 0)[f(x) - f(x_0)] < 0, \tag{4.3}$$

where

$$
\begin{aligned}
b_f(x, x_0, 0)[f(x) - f(x_0)] = (&b_{f_1}(x, x_0, 0)[f_1(x) - f_1(x_0)], b_{f_2}(x, x_0, 0)[f_2(x) - f_2(x_0)], \cdots, \\
&b_{f_p}(x, x_0, 0)[f_p(x) - f_p(x_0)])^T.
\end{aligned}
$$

It follows from (4.1) that there exist $\xi \in \partial f(x_0)$ and $\zeta \in \partial g(x_0)$ such that

$$\mu^{*T}\xi + \beta^{*T}\zeta = 0.$$

By hypotheses (i), (ii), (iii), (4.2) and Theorem 3.1, we get

$$\begin{aligned}\mu^{*T} b_f(x, x_0, 0)[f(x) - f(x_0)] &\geqq \langle \eta(x, x_0), \mu^{*T}\xi \rangle \\ &= \langle \eta(x, x_0), -\beta^{*T}\zeta \rangle \\ &\geqq -\beta^{*T} b_g(x, x_0, 0)[g(x) - g(x_0)] \\ &= -\beta^{*T} b_g(x, x_0, 0) g(x) \\ &\geqq 0,\end{aligned}$$

which contradicts (4.3), where

$$\begin{aligned} b_g(x, x_0, 0)[g(x) - g(x_0)] = (& b_{g_1}(x, x_0, 0)[g_1(x) - g_1(x_0)], b_{g_2}(x, x_0, 0)[g_2(x) - g_2(x_0)], \cdots, \\ & b_{g_m}(x, x_0, 0)[g_m(x) - g_m(x_0)])^T. \end{aligned}$$

This completes the proof. □

We now give an example to illustrate Theorem 4.2.

Example 4.3. Consider the problem

$$\begin{aligned} \text{(MP)} \quad \text{Minimize} \quad & f(x) = (f_1(x), f_2(x)) \\ \text{s.t.} \quad & x \in S = \{x \in R : g(x) \leq 0\}, \end{aligned}$$

where $f_i : R \to R$, $i = 1, 2$, and $g : R \to R$ are given by

$$f_1(x) = \begin{cases} -x & \text{if } x < 0; \\ 0 & \text{if } x \geq 0; \end{cases}$$

$$f_2(x) = \begin{cases} 0 & \text{if } x < 0; \\ x & \text{if } x \geq 0; \end{cases}$$

and $g(x) = x$. Let $\eta(x, x_0) = x - x_0$ and

$$b_{f_1}(x, x_0, \lambda) = \begin{cases} \lambda & \text{if } x \geq 0, x_0 \geq 0, \lambda > 0; \\ 1 & \text{otherwise}; \end{cases}$$

$$b_{f_2}(x, x_0, \lambda) = 1,$$

$$b_g(x, x_0, \lambda) = \begin{cases} 1 & \text{if } x \geq 0, x_0 \geq 0; \\ 0 & \text{otherwise}. \end{cases}$$

Then f_i, $i = 1, 2$, is B-preinvex with respect to η, $b_{f\,i}$ at $x_0 = 0$ and g is B-preinvex with respect to η, b_g at $x_0 = 0$. The set of feasible solutions of (MP) is given by $S = (-\infty, 0]$. It is easy to see that the hypotheses (iii) of Theorem 4.2 is satisfied at $x_0 = 0$ and

$$\partial f_1(0) = [-1, 0], \ \partial f_2(0) = [0, 1], \ \partial g(0) = 1.$$

Let $\mu^* = (1, 1)$ and $\beta^* = \frac{1}{2}$. Then

$$0 \in [-\frac{1}{2}, \frac{3}{2}] = \mu^{*T}\partial f(0) + \beta^{*T}\partial g(0).$$

Also $\beta^{*T} g(0) = 0$. Therefore (4.1) and (4.2) hold.

Thus by Theorem 4.2, $x_0 = 0$ is an efficient solution of (MP).

5 Duality

In this section, we shall prove weak and strong duality results for (MP) and (DMP). We consider the following Mond-Weir type dual [10] for problem (MP):

$$\text{(DMP) Maximize} \quad f(y)$$

$$\text{s.t.} \quad 0 \in \mu^T \partial f(y) + \beta^T \partial g(y), \tag{5.1}$$

$$\beta^T g(y) \geqq 0, \tag{5.2}$$

$$\mu \geq 0,\ \beta \geqq 0,\ y \in X. \tag{5.3}$$

Theorem 5.1 (Weak Duality). *Let x be feasible for (MP) and (x_0, μ, β) be feasible for (DMP). Suppose that*

(i) *$\mu > 0$, f_i is B-preinvex with respect to η, b_{f_i} at x_0 for $i = 1, 2, \cdots, p$, and g_j is B-preinvex with respect to η, b_{g_j} at x_0 for $j = 1, 2, \cdots, m$;*

(ii) *$\lim_{\lambda \downarrow 0} b_{f_i}(x, y, \lambda) = b_{f_i}(x, y, 0) > 0$ and $\lim_{\lambda \downarrow 0} b_{g_j}(x, y, \lambda) = b_{g_j}(x, y, 0) \geq 0$ for any $x, y \in X$;*

(iii) *f_i, $i = 1, 2, \cdots, p$, and g_j, $j = 1, 2, \cdots, m$ are regular at x_0.*

Then, the following cannot hold:

$$f(x) \leq f(x_0).$$

Proof. Suppose by contradiction that

$$f(x) \leq f(x_0).$$

Since $\mu > 0$ and $b_{f_i} > 0$, $i = 1, 2, \cdots, p$,

$$\mu^T b_f(x, x_0, 0)[f(x) - f(x_0)] < 0,$$

where

$$b_f(x, x_0, 0)[f(x) - f(x_0)] = (b_{f_1}(x, x_0, 0)[f_1(x) - f_1(x_0)], b_{f_2}(x, x_0, 0)[f_2(x) - f_2(x_0)], \cdots, b_{f_p}(x, x_0, 0)[f_p(x) - f_p(x_0)])^T.$$

By hypotheses (i), (iii) and Theorem 3.1, we have

$$\langle \eta(x, x_0), \mu^T \xi \rangle < 0,\ \forall\, \xi \in \partial f(x_0). \tag{5.4}$$

Since x is feasible for (MP) and (x_0, μ, β) is feasible for (DMP), it follows that

$$g(x) - g(x_0) \leqq 0.$$

By hypotheses (i), (ii), (iii) and Theorem 3.1, we get

$$\langle \eta(x, x_0), \beta^T \zeta \rangle \leqq 0,\ \forall\, \zeta \in \partial g(x_0). \tag{5.5}$$

Combining (5.4) and (5.5) yields

$$\langle \eta(x, x_0), \mu^T \xi + \beta^T \zeta \rangle < 0,\ \forall\, \xi \in \partial f(x_0),\ \zeta \in \partial g(x_0),$$

which contradicts (5.1). This completes the proof. □

Theorem 5.2 (Strong Duality). *Let x_0 be an efficient solution for (MP) at which a suitable constraint qualification [6] holds. Then there exist $\mu_0 \in R^p$, $\beta_0 \in R^m$ such that (x_0, μ_0, β_0) is feasible for (DMP). If the conditions of Theorem 4.2 hold, then (x_0, μ_0, β_0) is an efficient solution of (DMP).*

Proof. Since x_0 is an efficient solution for (MP) at which a suitable constraint qualification [6] is satisfied, there exist $\mu_0 \in R^p$, $\beta_0 \in R^m$ such that

$$\begin{aligned}&0 \in {\mu_0}^T \partial f(x_0) + {\beta_0}^T \partial g(x_0),\\ &{\beta_0}^T g(x_0) = 0,\\ &\mu \geq 0,\ \beta \geqq 0.\end{aligned}$$

Thus, (x_0, μ_0, β_0) is a feasible solution to (DMP). Suppose that (x_0, μ_0, β_0) is not an efficient solution for (DMP). Then, there exists (x, μ, β) feasible for (DMP) such that

$$f(x_0) \leq f(x),$$

which contradicts Theorem 4.2. Therefore, (x_0, μ_0, β_0) is an efficient solution of (DMP). This completes the proof. □

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References

[1] C.R. Bector and C. Singh, *B*-vex function, *J. Optim. Theory Appl.* 71 (1991) 237–253.

[2] C.R. Bector, S.K. Suneja and C.S. Lalitha, Generalized *B*-vex functions and generalized *B*-vex programming, *J. Optim. Theory Appl.* 76 (1993) 561–576.

[3] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983.

[4] B.D. Craven, Invex functions and constrained local minima, *Bull. Austr. Math. Soc.* 24 (1981) 357–366.

[5] M.A. Hanson, On sufficiency of the Kuhn-Tucker conditions, *J. Math. Anal. Appl.* 80 (1981) 545–550.

[6] A. Jourani, Constraint qualification and Lagrange multipliers in nondifferentiable programming problems, *J. Optim. Theory Appl.* 81 (1994) 533–548.

[7] X.F. Li, J.L. Dong and Q.H. Liu, Lipschitz *B*-vex functions and nonsmooth programming, *J. Optim. Theory Appl.* 93(1997) 557-574.

[8] X.J. Long and J.W. Peng, Semi-*B*-preinvex functions, *J. Optim. Theory Appl.* 131 (2006) 301–305.

[9] S.R. Mohan and S.K. Neogy, On invex sets and preinvex functions, *J. Math. Anal. Appl.* 189 (1995) 901–908.

[10] B. Mond and T. Weir, Generalized concavity and duality, in: *Generalized Concavity in Optimization and Economics*, S. Schaible and W.T. Ziemba (eds.), Academic Press, New York, NY, 1981, pp. 263-279

[11] I.M. Stancu-Minasian, Optimality and duality in nonlinear programming involving semilocally B-preinvex and related functions, *European J. Oper. Res.* 173 (2006) 47–58.

[12] S.K. Suneja, C. Singh and C.R. Bector, Generalization of preinvex and B-vex functions, *J. Optim. Theory Appl.* 76 (1993) 577–587.

[13] T. Weir and B. Mond, Preinvex functions in multiple-objective optimization, *J. Math. Anal. Appl.* 136 (1988) 29–38.

[14] X.M. Yang, X.Q. Yang and K.L. Teo, Explicitly B-preinvex functions, *J. Comput. Appl. Math.* 146 (2002) 25–36.

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