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OPTIMAL PORTFOLIOS WITH STRESS ANALYSIS AND THE EFFECT OF A CVAR CONSTRAINT

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Abstract: Risk-constrained allocation of risky assets in financial portfolios is particularly important in situations when asset returns appear to have large fluctuations. This problem is addressed here. The asset price is assumed to be driven by a Brownian motion perturbed by a compound Poisson process. This resembles a price process perturbed by an exogenous factor which may cause large movements in price. The jump size of the Poisson process and the rate of jump define, respectively, a scenario and its occurrence probability. The stress testing is conducted to evaluate the performance and assess the resilience of the portfolio subject to exceptional but major events. We examine how a conditional-value-at-risk constraint exerts an influence on the portfolio composition.

Key words: *optimal portfolio, stress testing, conditional-value-at-risk, jump-diffusion*

Mathematics Subject Classification: *49L20, 49M25*

1 Introduction

In finance, optimal portfolio allocation and risk management are two important problems. While maximizing wealth or consumption is often practised, potential risk in the downturn should not be ignored. Otherwise, the investment could suffer from a severe consequence when a shock occurs. Especially, when large movements of stock prices are encountered due to some unexpected factors which could cause a heavy tailed loss. We will resort to stress testing to get useful information on a firm's risk exposure. Stress testing ([1, 7, 14, 19]) can be considered as a procedure that aims to identify possible losses which may accrue under extreme movements of asset prices by constructing scenarios. We implement it by the methods in Berkovwitz [5], which suggest assigning probabilities to scenarios that are identified by stress testing.

To construct scenarios which are reflecting real situations in a portfolio model, a compound Poisson process is incorporated into the stock price evolution. If the investor only wishes to maximize the utilities of her wealth or consumption, it becomes a stochastic optimal control problem involving a jump diffusion model ([12, 13, 15]). However, risk constraint is not considered in these models. When value-at-risk (VaR) is used as the a risk constraint, the optimal portfolio policy was obtained in a static setting ([4, 9]) as well as in a dynamic one ([21, 22, 23]). One objective here is to show how the stressed risk constraint exerts an

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effect on the optimal portfolios. Here, conditional value at risk (CVaR), instead of VaR, is employed as a risk measure and as the risk constraint. Compared with VaR, CVaR, which is characterized by conditional expectation of losses above VaR, is not only able to quantify the extreme loss beyond VaR effectively but also is a coherent risk measure with desirable properties such as subadditivity and convexity ([16, 17]).

In this paper, by utilizing the dynamic programming principle, the optimal portfolio problem is reduced to the problem of solving a Hamilton-Jacobi-Bellman (HJB) equation with a CVaR constraint. The method of Lagrange multiplier is adopted to deal with the CVaR constraint. An efficient numerical method is proposed to solve the HJB-equation, leading to the solution of the optimal constrained portfolio. The CVaR constraint will influence the investment decision dynamically. The risky investments will be reduced whenever the CVaR constraint becomes active. In this way, the risk in a portfolio will be reduced even for a fairly wide range of scenarios. It illustrates the practicality of the risk-constrained portfolios.

The rest of the paper is organized as follows. In Section 2, the model is presented. In Section 3, we construct a set of scenarios to test the CVaR dynamically. In Section 4, we formulate optimal portfolio problem incorporating the CVaR constraint. In Section 5, numerical examples are presented to show how the CVaR constraint affects the optimal portfolio and utility function. The final section summarizes this paper.

2 Continuous-Time Optimal Portfolios

Assume that an agent have two investment opportunities: a risky free asset and a risky asset. The investor wishes to maximize the total expected utility of consumption or wealth over a given time horizon $[0, T]$ with an initial wealth x_0. We proceed to formulate our model as follows:

i) Let $B(t)$ be the deterministic price process of the risk free asset, say a bond. It is written as

$$dB(t) = rB(t)dt \tag{2.1}$$

with a fixed interest rate r.

ii) Let $S(t)$ denote the price process of the risky asset. It is assumed that $S(t)$ evolves according to

$$dS(t) = S(t)(\mu dt + \sigma dW(t) + dJ(t)) \tag{2.2}$$

where μ is the mean appreciation rate, σ is the dispersion coefficient, $W(t)$ is a standard Wiener process, $J(t)$ is a compound Poisson process. We assume that $J(t)$ takes the form $\sum_{k=1}^{N(t)} Y_k$, where, $N(t)$ is a Poisson process with rate λ, which denotes the number of extreme events (sudden jump of the dynamics of price process)that have occurred up to time t. And $Y_i, i \geq 1$, are independent and identically distributed random variables which reflects that how severe the extreme event can be when it occurs. We also assume that for each $k = 1, \ldots, N(t)$, the value of Y_k is greater than or equal to -1, so that one jump does not make the underlying worthless (see [12]).

Let $\omega(t)$ be the amount of money invested in the risky asset at time t. Then, the dynamics of a portfolio, which consists of $B(t)$, $S(t)$ and the consumption $c(t)$, is given by

$$\begin{aligned} dX(t) &= \frac{X(t) - \omega(t)}{B(t)} dB(t) + \omega(t)\frac{dS(t)}{S(t)} - c(t)dt \\ &= (\omega(t)(\mu - r) + rX(t) - c(t))dt + \omega(t)\sigma dW(t) + \omega(t)dJ(t). \end{aligned} \tag{2.3}$$

$\boxed{3}$ Stress Testing of the Loss and CVaR

Before conducting the stress testing and applying risk constraint, we first derive the return (loss) over $[t, t+\Delta t)$ when a compound Poisson process is added into the stock evolution. Assume that Δt is small and the portfolio is not being adjusted in this small interval. Let

$$\alpha = -r, \quad \theta(t) = \frac{\omega(t)(\mu - r) - c(t)}{-r}. \tag{3.1}$$

Then, the dynamics (2.3) becomes

$$dX(t) = \alpha(\theta(t) - X(t))dt + \omega(t)\sigma dW(t) + \omega(t)dJ(t). \tag{3.2}$$

Define

$$Y(t) = e^{\alpha t}X(t). \tag{3.3}$$

Then (3.2) becomes

$$dY(t) = \alpha\theta(t)e^{\alpha t}dt + e^{\alpha t}\omega(t)\sigma dW(t) + e^{\alpha t}\omega(t)dJ(t). \tag{3.4}$$

By integrating (3.4) over $[t, t+\Delta t)$, we obtain

$$\begin{aligned} & Y(t+\Delta t) - Y(t) \\ = \ & \alpha \int_t^{t+\Delta t} \theta(s)e^{\alpha s}ds + \int_t^{t+\Delta t} e^{\alpha s}\omega(s)\sigma dW(s) + \int_t^{t+\Delta t} e^{\alpha s}\omega(s)dJ(s). \end{aligned} \tag{3.5}$$

Assume that Δt is so small that $\omega(s)$, $c(s)$, $\theta(s)$ and $e^{\alpha s}$ can be approximated by $\omega(t)$, $c(t)$, $\theta(t)$ and $e^{\alpha t}$ (with errors less than $0(\Delta t^{\frac{3}{2}})$), respectively. Then, (3.5) becomes

$$\begin{aligned} & Y(t+\Delta t) - Y(t) \\ = \ & \theta(t)(e^{\alpha(t+\Delta t)} - e^{\alpha t}) + \int_t^{t+\Delta t} e^{\alpha t}\omega(t)\sigma dW(s) + \int_t^{t+\Delta t} e^{\alpha t}\omega(s)dJ(s). \end{aligned} \tag{3.6}$$

From (3.3), we obtain

$$\begin{aligned} & X(t+\Delta t) \\ = \ & e^{-\alpha\Delta t}(X(t) - \theta(t)) + \theta(t) + e^{-\alpha\Delta t}\omega(t)\int_t^{t+\Delta t}(\sigma dW(s) + dJ(s)). \end{aligned} \tag{3.7}$$

Let the return, adjusted for the future value of the current portfolio value consistent with (3.7),

$$\Delta X(t) = X(t+\Delta t) - e^{r\Delta t}X(t). \tag{3.8}$$

Then, it follows from (3.1) and (3.7) that the loss is given by

$$\begin{aligned} -\Delta X(t) &= e^{-\alpha\Delta t}\theta(t) - \theta(t) - e^{-\alpha\Delta t}\omega(t)(\Delta P(t) + \Delta Q(t)) \\ &= a_1\omega(t)(\Delta P(t)) + \Delta Q(t)) + a_2\omega(t) + bc(t) \end{aligned} \tag{3.9}$$

where

$$a_1 = -e^{r\Delta t}, \quad a_2 = \frac{\mu - r}{r}(e^{r\Delta t} - 1), \quad b = -\frac{1}{r}(e^{r\Delta t} - 1), \tag{3.10}$$

$$\Delta P(t) = \int_t^{t+\Delta t} \sigma dW(s) \tag{3.11}$$

and

$$\Delta Q(t) = \int_t^{t+\Delta t} dJ(s). \tag{3.12}$$

It is clear from the Markov property of jump-diffusion process that, at time t, $\Delta P(t) \sim N(0, \Delta t\sigma^2)$, and $\Delta Q(t)$ has a compound Poisson distribution given by

$$\Delta Q(t) = (\sum_{k=1}^{N(t+\Delta t)} Y_k - \sum_{k=1}^{N(t)} Y_k) \sim \sum_{k=1}^{N(\Delta t)} Y_k. \tag{3.13}$$

Let $f_t(x)$ denote the density function of the loss, and define

$$Z = a_1\omega(t)\Delta P(t) + a_2\omega(t) + bc(t).$$

Then, from the answer to Problem 14 of Section 1.8 given in [11], it follows that

$$f_t(x) = e^{-\lambda\Delta t}\sum_{k=0}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\int \Phi(z)\cdot\Phi^{k*}(x-z)dz. \tag{3.14}$$

Here $\Phi(x)$ is the density function of $a_2\omega(t) + bc(t)$, $\Phi^{k*}(x)(k \geq 1)$ is the density function of $\sum_{i=1}^{k} a_i\omega(t)Y_i$ and $\Phi^{0*}(x)$ is the dirac function.

$\boxed{3.1}$ Stress Testing

In (3.9), for a given portfolio, b is a negative constant and a_2 is positive assuming that $\mu > r$. Therefore, the loss mainly comes from the movements of stock prices. In the dynamics of (3.9), $\Delta P(t)$ has a normal distribution and $\Delta Q(t)$ captures the extreme losses.

For a sufficiently small time interval Δt, the jumps with i $(i \geq 2)$ jumps can be neglected. Then, the scenarios can be constructed as follows. Let the event with a jump in this interval be the scenario and its probability denoted by α. Thus, the loss can be tested as a normal model with a scenario being incorporated.

It follows from (3.14) that the combined loss density function is approximated by

$$f_c \sim (1-\alpha)f + \alpha f_s = (1-\lambda\Delta t)f_s + \lambda\Delta t f_s,$$

where f and f_s is the first two term of the right hand of (3.14).

$\boxed{3.2}$ Conditional Value at Risk

Although VaR is a popular risk measure, it has some significant shortcomings. First, it doesn't shed light on the size of extreme losses exceeding the VaR. As a result, the risk manager doesn't know the potential loss she may suffer when VaR is violated. Furthermore, the distribution of the return isn't elliptical and may lack of convexity. To overcome these drawbacks, conditional value at risk (CVaR), was introduced to the optimization modeling in [20].

Let $g(x)$ be the loss associated with a portfolio. Assume that $g(x)$ is the loss induced by x. Its occurrence probability is denoted by $p(x)$.

The probability of $g(x)$ not exceeding a threshold a is given by

$$\Psi(a) = \int_{g(x)\leq a} p(x)dx. \tag{3.15}$$

Then, for a specified probability level k in (0, 1), the values of the VaR and the CVaR for the loss random variable are, respectively, given by

$$VaR = \min(a \mid \Psi(a) \geq k) \tag{3.16}$$

and

$$CVaR = \frac{1}{1-k} \int_{g(x) \geq VaR} g(x)p(x)dx, \tag{3.17}$$

By definition, $CVaR$ comes out as the conditional expectation of the loss associated with x relative to that loss being VaR or greater. Thus, it catch the extreme loss, which will be shown in Section 5. Thus, it is important to evaluate the CVaR and impose it as a constraint. Moreover, the VaR, if employed as a risk measure in our model, is nonadditive [3, 17].

The general VaR is

$$VaR = a_1'|\omega(t)| + a_2\omega(t) + bc(t), \tag{3.18}$$

where a_1'=$aF_c^{-1}(k)$ with

$$F_c^{-1}(k) = \min(a \mid \int_{-\infty}^{a} f_c(x)dx \geq k). \tag{3.19}$$

The general CVaR is

$$CVaR = a_1''|\omega(t)| + a_2\omega(t) + bc(t) \tag{3.20}$$

where a_1''=$aH(k)$ and

$$H(k) = \frac{1}{1-k} \int_{x \geq F_c^{-1}(k)} x f_c(x)dx. \tag{3.21}$$

4 Optimal Problem with Risk Constraint

Assume that an agent needs continuous consumption over a given period of time and the CVaR can not exceed a level R. Then, the final optimal portfolio problem with the CVaR constraint can be formally stated as :

$$\max_{\omega(t),c(t)} E\left[\int_0^T U_1(t, c(t))dt\right] + U_2(T, X(T)) \tag{4.1}$$

subject to

$$dX(t) = (\omega(t)'(\mu - re) + rX(t) - c(t))dt + \omega(t)'(\sigma dW(t) + dJ(t)), \tag{4.2}$$

$$a_1''|\omega(t)| + a_2\omega(t) + bc(t) \leq R. \tag{4.3}$$

To solve the optimal portfolio problem, we make use of the dynamic programming method developed in [18, 10] to find a solution to the HJB-equation. Define the value function as

$$V(x,t) = \sup_{\omega(x,t),c(x,t)} E\left[\int_t^T U_1(s, c(x,s))ds + U_2(T, x(T))\right], \tag{4.4}$$

where x is a possible state of X_t. Denote

$$G(x, \omega(x,t), c(x,t)) \equiv \omega(x,t)(\mu - r) + rx - c(x,t)$$

and

$$H(\omega(x,t)) \equiv \omega^2(x,t)\sigma^2.$$

Then, the corresponding HJB equation is given by

$$\begin{aligned}&\frac{\partial V}{\partial t} + \sup_{\omega(x,t),c(x,t)} (U_1(t, c(x,t)) + G(x, \omega(x,t), c(x,t))\frac{\partial V}{\partial x} + \frac{1}{2}H(\omega(x,t))\frac{\partial^2 V}{\partial x^2} \\ &+\lambda E(V(x+\omega(x,t)Y, t) - V(x,t))) = 0 \end{aligned} \tag{4.5}$$

with the boundary conditions

$$V(x,T) = U_2(T, X(T)), \quad V(0,t) = 0, \tag{4.6}$$

subjects to the CVaR constraint

$$a_1''|\omega(t)| + a_2\omega(t) + bc(t) \le R. \tag{4.7}$$

For the optimal portfolio problem with the CVaR constraint, it requires first to solve the following the static optimization problem

$$\begin{aligned}&\max_{\omega(x,t),c(x,t)} (U_1(t, c(x,t)) + G(x, \boldsymbol{\omega}(x,t), c(x,t))\frac{\partial V}{\partial x} + \frac{1}{2}H(\omega(x,t))\frac{\partial^2 V}{\partial x^2} \\ &+\lambda E(V(x+\omega(x,t)Y, t) - V(x,t))) \end{aligned} \tag{4.8}$$

subject to (3.4).

Introducing the Lagrange function, we obtain

$$\begin{aligned}L(\omega(x,t), c(x,t), \lambda(x,t)) &= U_1(t, c(x,t)) + G(x, \omega(x,t), c(x,t))\frac{\partial V}{\partial x} + \frac{1}{2}H(\omega(x,t))\frac{\partial^2 V}{\partial x^2} \\ &+\lambda E(V(x+\omega(x,t)Y, t) - V(x,t)) - \lambda_1(x,t)(R - a_1''\omega(x,t) - a_2\omega(x,t) - bc(x,t)) \\ &-\lambda_2(x,t)(R - (-a_1'')\omega(x,t) - a_2'\omega(x,t) - bc(x,t)). \end{aligned} \tag{4.9}$$

Then, the first order necessary conditions of optimality for the static optimization problem are given by

$$\begin{aligned}&(\mu - r)\frac{\partial V}{\partial x} + \Sigma\omega(x,t)\frac{\partial^2 V}{\partial x^2} + \lambda EV_\omega(x,t) \\ &+\lambda_1(x,t)\left(a_1'' + a_2\right) + \lambda_2(x,t)\left(-a_1' + a_2\right) = 0,\end{aligned} \tag{4.10}$$

$$\frac{\partial U_1}{\partial c} = \frac{\partial V}{\partial x} - \lambda_1(x,t)b - \lambda_2(x,t)b, \tag{4.11}$$

$$\lambda_1(x,t)(R - a_1''\omega(x,t) - a_2\omega(x,t) - bc(x,t)) = 0, \tag{4.12}$$

$$\lambda_2(x,t)(R - (-a_1'')\omega(x,t) - a_2\omega(x,t) - bc(x,t)) = 0, \tag{4.13}$$

$$\lambda_1(x,t) \le 0 \tag{4.14}$$

$$\lambda_2(x,t) \le 0 \tag{4.15}$$

(4.10) is used for finding $\omega_{opt}(x,t)$ and (4.11) is used to solve for $c_{opt}(x,t)$ and (4.12) and (4.13) are applied to solve for $\lambda_1(x,t)$ and $\lambda_2(x,t)$ whenever $\lambda_1(x,t) \neq 0$ and $\lambda_2(x,t) \neq 0$. Substituting $\omega_{opt}(x,t)$ and $c_{opt}(x,t)$ into (4.5) gives

$$\begin{aligned}&\frac{\partial V}{\partial t} + U_1(t, c_{opt}(x,t)) + G(x, \omega_{opt}(x,t), c_{opt}(x,t))\frac{\partial V}{\partial x} + \frac{1}{2}H(\omega_{opt}(x,t))\frac{\partial^2 V}{\partial x^2}\\ &+\lambda E(V(x+\omega_{opt}(x,t)Y,t) - V(x,t)) = 0 \end{aligned} \tag{4.16}$$

which can, in principle, be solved for the value function $V_{opt}(x,t)$. However, in view of the nonlinearity in $c_{opt}(x,t)$ and $\omega_{opt}(x,t)$, the first order conditions, let alone the HJB equation, are highly nonlinear. Thus, numerical methods are required to be used for finding $\omega_{opt}(x,t)$, $c_{opt}(x,t)$, $\lambda_1(x,t)$, $\lambda_2(x,t)$ and $V_{opt}(x,t)$ iteratively.

5 Numerical Results

In this section, we consider a classical utility function

$$U_1(t,x) = U_2(t,x) = e^{-\delta t}x^\gamma \qquad \delta > 0\ ,\ 0 < \gamma < 1, \tag{5.1}$$

where δ is the discount factor. By using a trial function

$$V(x,t) = e^{-\delta t}h(t)x^\gamma \qquad \delta > 0\ ,\ 0 < \gamma < 1, \tag{5.2}$$

which separates the x and t variables, the HJB equation reduces to a Bernoulli equation for $h(t)$ which is an ordinary differential equation.

5.1 Algorithms

When the CVaR constraint is imposed, although the variation of $V(x,t)$ in x is still well modeled by the term x^γ, the function h will depend on x as well because of ω and c. However, we shall show later numerically that for some reasonable values of the parameters, h is a slow varying function of x and its derivatives with respect to x is therefore very small.

In the following sections, to illustrate the effect of the jumps Y and simplify the calculations, Y is assumed to a deterministic constant y. Let the utility function be defined by (5.1). Then the HJB equation for the value function is given by

$$\begin{aligned}&\frac{\partial V}{\partial t} + e^{-\delta t}c_{opt}^\gamma(x,t) + (\omega_{opt}(x,t)(\mu - r) + rx - c_{opt}(x,t))\frac{\partial V}{\partial x} + \frac{1}{2}\omega_{opt}^2(x,t)\sigma^2\frac{\partial^2 V}{\partial x^2}\\ &+\lambda e^{-\delta t}h(x,t)((x+\omega(x,t)y)^\gamma - x^\gamma) = 0. \end{aligned} \tag{5.3}$$

Neglecting the derivatives of h with respect to x, we obtain

$$\frac{\partial V}{\partial x} = \gamma e^{-\delta t}h(t)x^{\gamma-1}, \quad \frac{\partial^2 V}{\partial x^2} = \gamma(\gamma-1)e^{-\delta t}h(t)x^{\gamma-2}, \tag{5.4}$$

$$\frac{\partial V}{\partial t} = e^{-\delta t}h'(t)x^\gamma - \delta e^{-\delta t}h(t)x^\gamma. \tag{5.5}$$

Then, substituting (5.4) and (5.5) into (5.3), dividing by $e^{-\delta t}x^\gamma$ and rearranging gives

$$h'(x,t) + A(\omega_{opt}(x,t), x)h(t) + B(c_{opt}(x,t), h(t)) = 0 \tag{5.6}$$

with the boundary condition

$$h(x,T) = 1, \tag{5.7}$$

where

$$\begin{aligned} A(\omega_{opt}(x,t),x) &= \gamma\left(\frac{\omega_{opt}(x,t)}{x}(\mu-r)+r\right)+\frac{1}{2}\frac{\omega_{opt}^2(x,t)}{x^2}\sigma^2\gamma(\gamma-1)-\delta \\ &\quad +\lambda((1+\frac{\omega_{opt}(x,t)}{x}y)^\gamma-1), \qquad (5.8)\\ B(c_{opt}(x,t),h(t)) &= \frac{c_{opt}^\gamma(x,t)}{x^\gamma}-\frac{\gamma h(x,t)c_{opt}(x,t)}{x}. \qquad (5.9)\end{aligned}$$

We transform (5.3) into a convenient form

$$g'(t)+(1-\beta)A(\omega_{opt}(x,t),x)g(t)+(1-\beta)B(c_{opt}(x,t),g(t))g(t)^\gamma = 0$$

with the boundary condition

$$g(T) = 1,$$

where

$$g(t) = h(t)^{1-\beta}, \quad \beta = -\frac{\gamma}{1-\gamma}. \tag{5.10}$$

Dividing the computational domain into a grid of $N_x \times N_t$ mesh points, the final algorithm can be summarized as follows:
1) For the unconstrained case, $\lambda_1^{(0)}(x,t) = \lambda_2^{(0)}(x,t) = 0$. Thus (5.6) is reduced to

$$g'(t)+(1-\beta)Ag(t)+(1-\beta)B = 0, \tag{5.11}$$

where

$$\begin{aligned} A = \gamma\left(\frac{\omega_{opt}^{(0)}(x,t)}{x}(\mu-r)+r\right)+\frac{1}{2}\frac{\omega_{opt}^{(0)}(x,t)^2}{x^2}\sigma^2\gamma(\gamma-1)-\delta \\ +\lambda((1+\frac{\omega_{opt}^{(0)}}{x}y)^\gamma-1) \qquad (5.12)\end{aligned}$$

and $B = 1-\gamma$.

Form the optimality condition (4.10), $\omega_{opt}^{(0)}(x,t)$ is calculated from

$$(\mu-r)\gamma x^{\gamma-1}h(t)+\omega_{opt}^{(0)}(x,t)\sigma^2\gamma(\gamma-1)x^{\gamma-2}h(t)+\lambda h(t)\gamma(x+\omega_{opt}^{(0)}(x,t)y)^{\gamma-1}y = 0.$$

Dividing by $\gamma h(t)x^{\gamma-1}$ yields

$$(\mu-r)+\frac{\omega_{opt}^{(0)}(x,t)}{x}\sigma^2(\gamma-1)+\lambda(1+\frac{\omega_{opt}^{(0)}(x,t)}{x}y)^{\gamma-1}y = 0. \tag{5.13}$$

By Newton's method, $\frac{\omega_{opt}^{(0)}(x,t)}{x}$ is the function of time t.

The explicit solution of Bernoulli equation (5.11) is given by

$$g(t) = (\frac{B}{A}+1)e^{\frac{1}{B}A(T-t)}-\frac{B}{A}. \tag{5.14}$$

Thus, $c_{opt}^{(0)}(x,t)$ has a simple form of $xg^{-1}(t)$. The solution of the unconstrained problem will be used as an initial guess in the iterative algorithm. Set iterative index k=0.

2) For $x = [0, \Delta x, \cdots, N_x \Delta x]$ and $t = [(N_t - 1)\Delta t, \cdots, \Delta t, 0]$, for notational simplicity, omit (x, t) in all the functions involved. Then, we calculate $\omega_{opt}^{(k+1)}$, $\lambda_1^{(k+1)}$, $\lambda_2^{(k+1)}$ and $c_{opt}^{(k+1)}$ from

$$\begin{aligned}&(\mu - r\mathbf{e})V_x^{(k)}(x) + \sigma^2 \omega_{opt}^{(k+1)} V_{xx}^{(k)}(x) + \lambda E V_\omega^{(k)}(x + \omega y, t)\\&+\lambda_1^{(k)}(a_1'' + a_2) + \lambda_2^{(k)}(-a_1'' + a_2) = 0,\end{aligned} \tag{5.15}$$

$$\lambda_1^{(k+1)}(R - (a_1'' + a_2)\omega_{opt}^{(k+1)} - bc_{opt}^{(k)}) = 0, \tag{5.16}$$

$$\lambda_1^{(k+1)}(R - (-a_1'' + a_2)\omega_{opt}^{(k+1)} - bc_{opt}^{(k)}) = 0, \tag{5.17}$$

$$\gamma(c_{opt}^{(k+1)})^{\gamma-1} = (\gamma x^{\gamma-1} h^{(k)} - e^{\delta t})(\lambda_1^{(k+1)} + \lambda_2^{(k+1)})b \tag{5.18}$$

respectively.

3) For $x = [0, \Delta x, \cdots, N_x \Delta x]$ and $n = [N_t - 1, \cdots, 0]$, solve

$$g_n^{(k+1)} = g_{n+1}^{(k+1)} + \Delta t(1-\beta)A(\omega_{opt}^{(k+1)}, x)g_{n+1}^{(k+1)} + (1-\beta)B(c_{opt}^{(k+1)}, g_n^{(k)})(g_n^{(k)})^\gamma. \tag{5.19}$$

4) Return to 2) with $k =: k + 1$ until convergence.

In the following subsections, we will first carry out a stress test to check how CVaR catch the extreme loss with the constructed scenario. Then we explore the effect of the CVaR constraint. Numerical experiments are carried out in the environment of Matlab 7 and Fortran 90.

In our calculations, unless otherwise specified, the following parameters are used: $\delta = 0.2$, $\gamma = 0.5$, the appreciation rate $\mu = 0.2$, the diffusion volatility $\sigma = 0.5$, the interest rate $r = 0.1$, the discount factor $\delta = 0.2$, time T is fixed as 20 years, $N_t = 1000$ and Δt is equal to 1/52 year (i.e. about one week), $\Delta x = 2$ and $N_x = 500$.

5.2 Stress Test

In this subsection, stress test is conducted to evaluate the risk exposure in a scenario. We consider a scenario, where $\Delta t = 0.1$ year, $\lambda = 1$ and the jump magnitude is a constant of -0.9 (i.e. the stock price may drop by 90 percent with 0.1 probability in 0.1 year). Figure 5.1.1 depicts density function of $a_1\omega(t)(\Delta P(t) + \Delta Q(\Delta t))$ in two situations: a normal economy (f) and a scenario characterized by a negative jump (f_c). It is easily seen from the heavy tail of f_c, bigger losses are more likely to occur in this situation. As a result, the investor will suffer a bigger mean loss above the same VaR, which will be shown by Figure 5.1.2. Let $CVaR(f)$ and $CVaR(f_c)$ denote, respectively, the CVaRs based on f and f_c. Figure 5.1.2 shows that $CVaR(f_c)$ is much larger than $CVaR(f)$ for high confidence levels. Thus, to capture the loss exceeding the VaR, it is reasonable to use CVaR as the risk measure.

5.3 The Result with CVaR Constraint

In this subsection, the CVaRs, optimal risky investments and the value functions are compared between the cases with and without constraint, respectively. Let $t = 10$, $x = 400$ and the risk level R is constrained to 120. The figures are plotted when the jump rate and jump magnitude vary between [0,0.4] and (-1,1], respectively, which denotes different economic conditions.

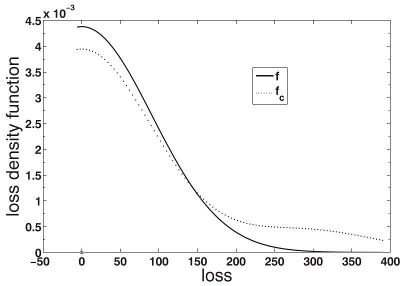

Figure 5.1.1: A typical loss distribution density with jumps

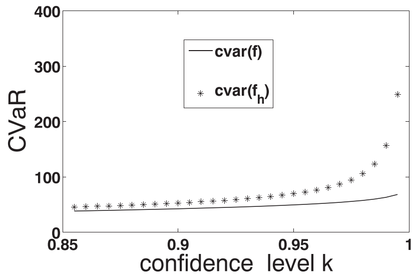

Figure 5.1.2: CVaR for different confidence levels

Figure 5.2.1 describes the value of CVaRs. And Figure 5.2.1′ shows that we have stabilized the CVaRs once they are active.

Figure 5.2.2 and Figure 5.2.2′ plots the risky investment in the cases without and with constraint, respectively. It is easily seen from these two figures that when the constraint is not active, the allocation in risky assets are same as the unconstrained case. However, when once the CVaR constraint is active, the risky investment should be cut back to meet the risk level constraint.

From Figure 5.2.3 and Figure 5.2.3′, it is clearly observed that $v(x,t)$ is decreased once the constraint is active. It is trivial since the investment is constrained.

Compared with the case without constraint, little difference can be seen for the consumption when CVaR constraint is applied, we will not put the figures here. And this will be explained in the next numerical example.

For a given wealth and time, we examine closer the optimal portfolios with CVaR constraint under the normal market and a bad economy. A risk constraint $R = 70$ is assumed. Let λ be 0.12 and y be -0.3, t =10, and x varies from 0 to 1000. From Figure 5.2.4, similar investment trends are observed for both cases. Once the constraint becomes active, the investor reduces the risky investment to decrease the risk exposure. The result is similar to that reported in [18], where the normal case with a VaR constraint is considered. Figure 5.2.5 depicts the consumption pattern at the time $t = 10$. Under the parameters, the investor will consume more in a bad economy. When the risk constraint is imposed, the investor would consume slightly more. However, the risk constraint has a rather limited effect on the consumption pattern, which reflects the fact that the risky investment has a heavier weight in the portfolio; it is therefore sufficient to decrease the risk by cutting on the risky investment. Figure 5.2.6 plots $h(t)$ over time t for two different values of wealth. When the risk constraint is imposed, $h(t)$ has shown very little variation in x with the approximation (5.4). In fact, the accuracy of the solution can be assessed by calculating the residual value of the HJB equation as in [21]. We find that the discrete error measure of the HJB equation is less than 8.35557×10^{-4}.

6 Conclusion

In this paper, we considered the CVaR-constrained optimal portfolios with the addition of a jump diffusion model for stress analysis. Among all the market factors, we focus on the risk arising from the risky assets. Different scenarios of large fluctuation in returns are constructed via adding a compound Poisson process to the dynamics of a stock price. The stress test results show a significantly larger CVaR for distressing scenarios. Since investors will adjust their portfolios according to the dynamics of stock price, we also explored the stabilization of the portfolio allocations by imposing an CVaR constraint. We showed that when the CVaR constraint is imposed, the investment in risky asset is decreased so that the given risk level is satisfied.

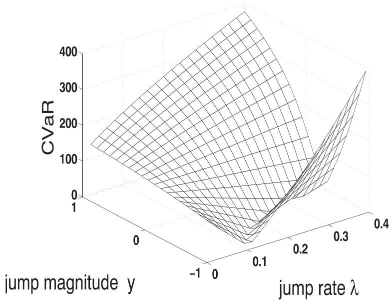

Figure 5.2.1: CVaR under different jump magnitudes and jump rates

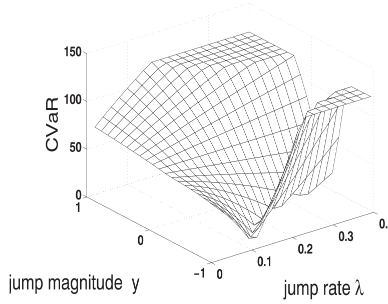

Figure 5.2.1′: x=400, t = 10, for different jump magnitudes and jump rates, CVaRs with stressed risk constraint

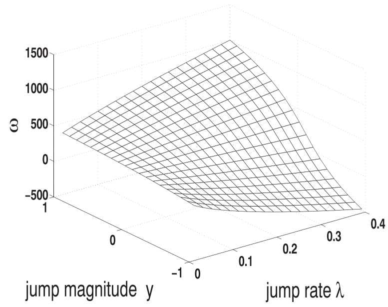

Figure 5.2.2: Risky investments under different jump magnitudes and jump rates

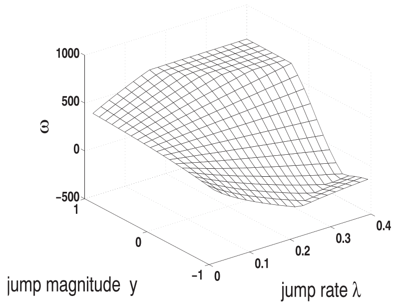

Figure 5.2.2′: x=400, t=10, for different jump magnitudes and jump rates, the optimal risky investment with stressed risk constraint

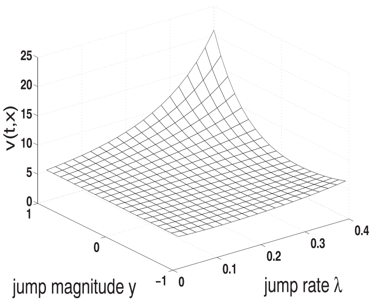

Figure 5.2.3: The value functions under different jump magnitudes and jump rates

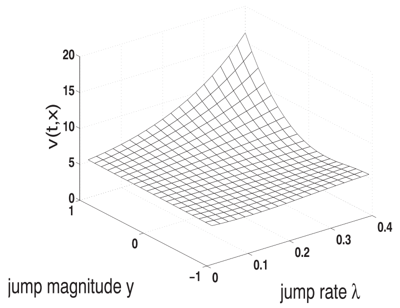

Figure 5.2.3′: x = 400, for different jump magnitudes and jump rates, v(x,t) with stressed risk constraint

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