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A NEW CONSTRAINT QUALIFICATION AND A SECOND-ORDER NECESSARY OPTIMALITY CONDITION FOR MATHEMATICAL PROGRAMMING PROBLEMS*

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Abstract: We present a new condition of constraint qualification that involves a second-order tangent set. Based on this constraint qualification, we establish a second-order necessary optimality condition for mathematical programming problems. The new qualification condition is weaker than the popular constraint qualification of Robinson. An example is presented to show the usefulness of the result. The results can be applied to problems with abstract feasible sets as well as to problems with feasible regions defined explicitly by nonconvex equalities and inequalities.

Key words: *optimality conditions, nonlinear programming, tangent cones*

Mathematics Subject Classification: *90C26, 90C30*

1 Introduction

We derive a new second-order necessary optimality condition for mathematical programming problems. As is well known, a necessary optimality condition is valid only if a certain constraint qualification (CQ) is satisfied, or as is often called in the literature, only if the problem is "nondegenerate". Examples of CQ include the Mangasarian-Fromovitz constraint qualification, the Robinson constraint qualification, and the linear independence constraint qualification, etc. However, many practical problems are "degenerate" problems in some sense. It is therefore natural to look for new second-order optimality conditions under weaker constraint qualifications.

We start from considering problems where the feasible region is an abstract closed set, then by using second-order tangency formulae, we deduce a more explicit result for the case where the feasible region is defined by nonlinear equalities and inequalities. This approach can help to understand the roles of various constraint qualifications in the establishment of second-order necessary optimality conditions.

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2 A Second-Order Necessary Condition for Optimality

Let $f : \Re^n \to \Re$ be twice continuously differentiable and let K be a closed, not necessarily convex, subset of $\Re^n$. Consider the problem

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & x \in K. \end{array} \tag{2.1}$$

We adopt the notations in Bonnans and Shapiro [4]. For a linear operator A, A^* denotes the adjoint operator (transpose) of A. D and D^2 denote respectively the first and second-order derivatives of a function. For a nonempty set $S \subset \Re^n$, $\overline{S}$ denotes its closure, $\mathrm{int}(S)$ denotes its interior, $\sigma_S(x) := \sup_{y\in S}\langle x, y\rangle$ denotes its support function, and

$$\delta_S(x) := \begin{cases} 0 & x \in S, \\ \infty & x \notin S \end{cases}$$

denotes its indicator function, respectively. In addition, the (negative) dual cone of S is designated by

$$S^- := \{\xi : \langle \xi, x\rangle \le 0, \text{ for all } x \in S\}.$$

It is known that if $x_0 \in K$ is a locally optimal solution to (2.1), then the first order necessary optimality conditions can be written as $Df(x_0)h \ge 0$ for all $h \in T_K(x_0)$, where $T_K(x_0)$ stands for the (Bouligand-) *contingent cone* of K at x_0:

$$T_K(x_0) := \{h \in \Re^n : \mathrm{dist}(x_0 + t_n h, K) = o(t_n) \text{ for some } t_n \downarrow 0\},$$

where $\mathrm{dist}(x, K) := \inf\{\|x - y\| : y \in K\}$. For a closed convex set K and $x \in K$, $T_K(x)^-$ is a closed convex cone, called the *normal cone* of K at x, and is written as $N_K(x)$.

For $x_0 \in K$ and $h \in \Re^n$, let $T^2_K(x_0, h)$ denote the *outer second-order tangent set* at x_0 in the direction h, defined as

$$T^2_K(x_0, h) := \left\{w \in \Re^n : \mathrm{dist}(x_0 + t_n h + (t_n^2/2)w, K) = o(t_n^2) \text{ for some } t_n \downarrow 0\right\}.$$

Clearly, $T^2_K(x_0, h) = T_K(x_0)$ if $h = 0$. It is also known that $T^2_K(x_0, h) \ne \emptyset$ only if $h \in T_K(x_0)$, see Section 3.2.1 in [4] for an explanation. If K is convex, then $T_K(x_0)$ is a closed convex cone; however, $T^2_K(x_0, h)$ is not necessarily convex, though closed. Let

$$S(x_0) := T_K(x_0) \cap \{h : Df(x_0)h \le 0\}.$$

We next derive a second-order optimality condition for problem (2.1), involving the second-order tangent set $T^2_K(x_0, h)$.

Proposition 2.1. *If $x_0 \in K$ is a locally optimal solution to* (2.1)*, then for every $h \in S(x_0)$,*

$$Df(x_0)w + D^2 f(x_0)(h, h) \ge 0, \quad \textit{for every } w \in T^2_K(x_0, h).$$

Proof. Let $h \in S(x_0)$ and $w \in T^2_K(x_0, h)$. By definition of the outer second-order tangent set, there exists $t_n \downarrow 0$ such that $\mathrm{dist}(x_0 + t_n h + (1/2)t_n^2 w, K) = o(t_n^2)$. Therefore there exists $r(t_n) = o(t_n^2)$ such that $x_n := x_0 + t_n h + (1/2)t_n^2 w + r(t_n) \in K$. Since f is twice continuously differentiable, the second-order Taylor expansion yields that

$$\begin{aligned} f(x_n) &= f(x_0) + t_n Df(x_0)h + (1/2)t_n^2[Df(x_0)w + D^2 f(x_0)(h, h)] + o(t_n^2) \\ &\le f(x_0) + (1/2)t_n^2[Df(x_0)w + D^2 f(x_0)(h, h)] + o(t_n^2), \end{aligned}$$

where the inequality follows from that $h \in S(x_0)$ and hence $Df(x_0)h \le 0$. Since x_0 is a locally optimal solution and $x_n \in K$, we have $f(x_0) \le f(x_n)$ for sufficiently large n and hence $Df(x_0)w + D^2 f(x_0)(h, h) \ge 0$. □

Corollary 2.2. *Let x_0 be a locally optimal solution to* (2.1)*. Then for every $h \in S(x_0)$ and any convex set $T \subset T^2_K(x_0, h)$*

$$D^2 f(x_0)(h,h) - \sigma_T(-Df(x_0)) \geq 0;$$

and hence $\sigma_T(-Df(x_0)) < \infty$.

Proof. By Proposition 2.1,

$$\begin{aligned} 0 \leq & D^2 f(x_0)(h,h) + \inf\{Df(x_0)w : w \in T^2_K(x_0,h)\} \\ = & D^2 f(x_0)(h,h) - \sup\{-Df(x_0)w : w \in T^2_K(x_0,h)\} \\ = & D^2 f(x_0)(h,h) - \sigma_{T^2_K(x_0,h)}(-Df(x_0)) \\ \leq & D^2 f(x_0)(h,h) - \sigma_T(-Df(x_0)). \end{aligned}$$

This completes the proof. □

Remark. Note that we have not used any assumptions on constraint qualification so far.

Now we consider minimization problems with explicit constraints:

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & G(x) \in C, \end{array} \tag{2.2}$$

where $f : \Re^n \to \Re$ and $G : \Re^n \to \Re^m$ are twice continuously differentiable and C is a closed convex set in $\Re^m$. Let $K = G^{-1}(C)$. Then K is a closed set and problem (2.2) is a special case of problem (2.1).

Proposition 2.3. *Let $K = G^{-1}(C)$. For every $x_0 \in K$ and every $h \in \Re^n$ with $DG(x_0)h \in T_C(G(x_0))$,*

$$T^2_K(x_0,h) \subset \{w : D^2G(x_0)(h,h) + DG(x_0)w \in T^2_C(G(x_0), DG(x_0)h)\}. \tag{2.3}$$

Proof. Let $w \in T^2_K(x_0,h)$. Then there exist sequences $\{t_n\} \downarrow 0$ and $\{y_n\} \subset K$ such that the sequence $x_n := x_0 + t_n h + (1/2)t_n^2 w$ satisfies $y_n = x_n + o(t_n^2)$. This implies that $G(y_n) = G(x_n) + o(t_n^2)$ as the mapping G is continuously differentiable and hence locally Lipschitz at x_0. By the second-order Taylor expansion,

$$G(x_n) = G(x_0) + t_n DG(x_0)h + (1/2)t_n^2[DG(x_0)w + D^2G(x_0)(h,h)] + o(t_n^2).$$

Since $G(x_n) + +o(t_n^2) \in C$, it follows that

$$\begin{aligned} & \operatorname{dist}(G(x_0) + t_n DG(x_0)h + (1/2)t_n^2[DG(x_0)w + D^2G(x_0)(h,h)], C) \\ & \leq \|G(x_0) + t_n DG(x_0)h + (1/2)t_n^2[DG(x_0)w + D^2G(x_0)(h,h)] - (G(x_n) + o(t_n^2))\| = o(t_n^2), \end{aligned}$$

which implies $DG(x_0)w + D^2G(x_0)(h,h) \in T^2_C(G(x_0), DG(x_0)h)$ by definition. □

By virtue of Proposition 2.1, if x_0 is a locally optimal solution to the problem (2.2), then for $K = G^{-1}(C)$ and for every $h \in S(x_0)$,

$$Df(x_0)w + D^2 f(x_0)(h,h) \geq 0, \quad \text{for all } w \in T^2_K(x_0,h). \tag{2.4}$$

Although (2.3) provides an upper estimate (in the sense of inclusion relation) of $T^2_K(x_0,h)$, this does not help us to get further significant reformulation of the condition (2.4) in terms of the second-order derivative of the mapping G and the second-order tangent set of the set C. In order to obtain a significant second-order optimality condition, it is enough to require that (2.3) become an equality, resulting in a new constraint qualification as follows:

$$T_K^2(x_0, h) = \{w : D^2G(x_0)(h, h) + DG(x_0)w \in T_C^2(G(x_0), DG(x_0)h)\}, \tag{2.5}$$

which we call the *second-order Abadie constraint qualification* at x_0 in the direction h. Throughout this paper, when mentioning the second-order Abadie CQ, we always assume that the sets in both sides of (2.5) are nonempty. Thus x_0 must satisfy $G(x_0) \in C$ and h must satisfy $DG(x_0)h \in T_C(G(x_0))$. We use the terminology "second-order Abadie CQ" because it is somewhat analogous to the first-order Abadie constraint qualification at x_0, which says

$$T_K(x_0) = \{h : DG(x_0)h \in T_C(G(x_0))\}; \tag{2.6}$$

see [2, Section 5.1] for a special version stated for nonlinear programming. In his original work [1], Abadie considered the case of $C = \Re^m_-$ and called the right-hand side of (2.6) the *linearized cone* to the system $G(x) \in C$; and his CQ simply requires that the contingent cone at x_0 be identical to the linearized cone at x_0. Actually, it can be seen that the first-order Abadie constraint qualification at x_0 is just the second-order Abadie constraint qualification at x_0 in the direction $h = 0$. Moreover, if the second-order Abadie constraint qualification at x_0 holds in the direction $h = 0$, then the set $S(x_0)$ becomes

$$S(x_0) = \{v : DG(x_0)v \in T_C(G(x_0))\} \cap \{v : Df(x_0)v \leq 0\},$$

which is exactly the critical cone $C(x_0)$ of the problem (2.2) defined in [4, Section 3.1]. In general, if $h \neq 0$, we only have $S(x_0) \subset C(x_0)$.

Another important reason that makes us assume the equality (2.5) as a constraint qualification is the following fact: When *Robinson's constraint qualification* holds at x_0, that is,

$$0 \in \operatorname{int}(G(x_0) + DG(x_0)\Re^n - C), \tag{2.7}$$

the equality (2.5) holds naturally for all h; see the remark after Proposition 3.33 in [4].

In particular, if the constraint set K is of the following form

$$K := \{x \in \Re^n : g_i(x) \leq 0, i = 1, \dots, m; h_j(x) = 0, j = 1, \dots, \ell\},$$

then the second-order Abadie CQ at x_0 in the direction h becomes

$$T_K(x_0, h) = \left\{ w \in \Re^n : \begin{array}{l} Dg_i(x_0)w + D^2g_i(x_0)(h, h) \leq 0, i \in I(x_0, h); \\ Dh_j(x_0)w + D^2h_j(x_0)(h, h) = 0, j = 1, \dots, \ell \end{array} \right\},$$

where

$$I(x_0, h) := \{i : g_i(x_0) = 0, Dg_i(x_0)h = 0\}. \tag{2.8}$$

In the following, we let

$$\Lambda(x_0) := \{\lambda \in \Re^m : Df(x_0) + DG(x_0)^*\lambda = 0, \lambda \in N_C(G(x_0))\}$$

denote the set of Lagrange multipliers of the problem (2.2), where $DG(x_0)^*$ denotes the adjoint operator (transpose) of the linear operator $DG(x_0)$.

Theorem 2.4. *Let x_0 be a local optimal solution to* (2.2) *and $h \in C(x_0)$. Assume that $\Lambda(x_0)$ is nonempty and the second-order Abadie CQ holds at x_0 in the direction h. Let*

$\mathcal{T} \subset T_C^2(G(x_0), DG(x_0)h)$ be a convex set and let $M := \mathcal{T} + T_C(G(x_0)) - D^2G(x_0)(h,h)$. If $DG(x_0)^{-1}(\overline{M}) \neq \emptyset$, then

$$\limsup_{y \to Df(x_0)} \sup_{\lambda \in \Lambda(x_0,y)} \left\{ D^2 f(x_0)(h,h) + \langle \lambda, D^2G(x_0)(h,h) \rangle - \sigma_{\mathcal{T}}(\lambda) \right\} \geq 0, \tag{2.9}$$

where $\Lambda(x_0, y) := \{\lambda \in N_C(G(x_0)) : DG(x_0)^* \lambda = -y\}$.

Proof. Since $\mathcal{T} \subset T_C^2(G(x_0), DG(x_0)h)$ is a convex set and

$$M = \mathcal{T} + T_C(G(x_0)) - D^2G(x_0)(h,h), \tag{2.10}$$

by [4, (3.63)], $\mathcal{T} + T_C(G(x_0))$ is contained in $T_C^2(G(x_0), DG(x_0)h)$. Since the latter is closed, it follows that

$$\overline{M} \subset T_C^2(G(x_0), DG(x_0)h) - D^2G(x_0)(h,h).$$

Since the second-order Abadie CQ holds at x_0 in the direction h, it follows that

$$DG(x_0)^{-1}(\overline{M}) \subset T_K^2(x_0, h).$$

Since $T_K^2(x_0, h)$ is assumed nonempty whenever second-order Abadie CQ holds at x_0 in the direction h, we have $h \in T_K(x_0)$. Therefore $h \in S(x_0)$. In view of Corollary 2.2, there holds

$$D^2 f(x_0)(h,h) - \sigma_{DG(x_0)^{-1}(\overline{M})}(-Df(x_0)) \geq 0. \tag{2.11}$$

Define

$$g(x) := \inf \{\sigma_M(\lambda) : DG(x_0)^* \lambda = x\}. \tag{2.12}$$

It follows from (2.10) that

$$\begin{aligned} g(x) &= \inf \left\{ \sigma_{\mathcal{T}}(\lambda) + \sigma_{T_C(G(x_0))}(\lambda) - \langle \lambda, D^2G(x_0)(h,h) \rangle : DG(x_0)^* \lambda = x \right\} \\ &= \inf \left\{ \sigma_{\mathcal{T}}(\lambda) - \langle \lambda, D^2G(x_0)(h,h) \rangle : DG(x_0)^* \lambda = x, \lambda \in N_C(G(x_0)) \right\} \\ &= \inf \left\{ \sigma_{\mathcal{T}}(\lambda) - \langle \lambda, D^2G(x_0)(h,h) \rangle : \lambda \in \Lambda(x_0, -x) \right\}. \end{aligned}$$

By virtue of [7, Theorem 16.3] and [7, Corollary 16.3.1], $\sigma_{DG(x_0)^{-1}(M)}(\cdot)$ is the closure of the function $g(\cdot)$ and we have

$$g^*(x^*) = \delta_{DG(x_0)^{-1}(\overline{M})}(x^*).$$

Since $DG(x_0)^{-1}(\overline{M}) \neq \emptyset$, g^* is a proper convex function. It follows from [7, Theorem 12.2] that the convex function g is proper. Since $\sigma_{DG(x_0)^{-1}(\overline{M})}$ is the closure of the function g and since the closure of a proper convex function g is defined to be the function $x \mapsto \liminf_{y \to x} g(y)$ (see [7, p.52]), we have

$$\begin{aligned} \sigma_{DG(x_0)^{-1}(\overline{M})}(-Df(x_0)) &= \liminf_{y \to -Df(x_0)} g(y) \\ &= \liminf_{y \to -Df(x_0)} \inf_{\lambda \in \Lambda(x_0, -y)} \left\{ \sigma_{\mathcal{T}}(\lambda) - \langle \lambda, D^2G(x_0)(h,h) \rangle \right\} \\ &= \liminf_{y \to Df(x_0)} \inf_{\lambda \in \Lambda(x_0, y)} \left\{ \sigma_{\mathcal{T}}(\lambda) - \langle \lambda, D^2G(x_0)(h,h) \rangle \right\}. \end{aligned}$$

This together with (2.11) yields the inequality (2.9). □

Throughout the rest of this paper, we assume that the set C in the problem (2.2) is polyhedral. Thus the indicator function δ_C of C is a piecewise linear-quadratic function. Since C is polyhedral, it follows from [4, (3.63)] that

$$T_C^2(G(x_0), DG(x_0)h) = T_{T_C(G(x_0))}(DG(x_0)h), \tag{2.13}$$

which we write as $\mathcal{O}(h)$ in the sequel.

Theorem 2.5. *Let x_0 be a locally optimal solution to* (2.2) *and let $h \in C(x_0)$. If the second-order Abadie CQ holds at x_0 in the direction h, then $\Lambda(x_0)$ is nonempty and*

$$\sup_{\lambda\in\Lambda(x_0)} \{D^2 f(x_0)(h,h) + \langle \lambda, D^2G(x_0)(h,h)\rangle\} \geq 0.$$

Proof. Let $h \in C(x_0)$. The formula (2.13) implies that

$$\mathcal{O}(h)^- \subset N_C(G(x_0)) \text{ and } \sigma_{\mathcal{O}(h)}(\lambda) = \begin{cases} 0 & \lambda \in \mathcal{O}(h)^- \\ \infty & \text{otherwise.} \end{cases} \tag{2.14}$$

Let $T := DG(x_0)^{-1}(M)$, where $M := \mathcal{O}(h) - D^2G(x_0)(h,h)$. Then T is a nonempty convex subset of the second-order tangent set $T_K^2(x_0,h)$. Since $T_K^2(x_0,h)$ is assumed nonempty, we have $h \in T_K(x_0)$, and hence $h \in S(x_0)$. Applying Corollary 2.2, we obtain that

$$\sigma_T(-Df(x_0)) \leq D^2 f(x_0)(h,h). \tag{2.15}$$

Since C is a polyhedral set, we have the sets $\mathcal{O}(h)$, M, and hence T are also polyhedral. Since the indicator function $\delta_T(x)$ is equal to $(\delta_M \circ DG(x_0))(x)$ and since δ_M is a proper convex piecewise linear-quadratic as M is a polyhedral, it follows from [8, Corollary 11.33] that

$$\sigma_T(-Df(x_0)) = \inf\{\sigma_M(\lambda) : DG(x_0)^*\lambda = -Df(x_0)\}.$$

This together with (2.15) implies that

$$\begin{aligned}
&D^2 f(x_0)(h,h) \geq \sigma_T(-Df(x_0)) && (2.16)\\
&= \inf\{\sigma_M(\lambda) : DG(x_0)^*\lambda = -Df(x_0)\}\\
&= \inf\left\{\sigma_{\mathcal{O}(h)}(\lambda) - \langle\lambda, D^2G(x_0)(h,h)\rangle : DG(x_0)^*\lambda = -Df(x_0)\right\}\\
&= \inf\left\{-\langle\lambda, D^2G(x_0)(h,h)\rangle : DG(x_0)^*\lambda = -Df(x_0), \lambda \in \mathcal{O}(h)^-\right\}\\
&\geq \inf\left\{-\langle\lambda, D^2G(x_0)(h,h)\rangle : DG(x_0)^*\lambda = -Df(x_0), \lambda \in N_C(G(x_0))\right\}\\
&= \inf\left\{-\langle\lambda, D^2G(x_0)(h,h)\rangle : \lambda \in \Lambda(x_0)\right\}\\
&= -\sup\left\{\langle\lambda, D^2G(x_0)(h,h)\rangle : \lambda \in \Lambda(x_0)\right\}, && (2.17)
\end{aligned}$$

where the second equality and the second inequality follow from (2.14). The assertion of $\Lambda(x_0) \neq \emptyset$ is an immediate consequence of (2.17) as by convention supremum over empty set is $-\infty$. By virtue of (2.17), we obtain that

$$\begin{aligned}
0 &\leq D^2 f(x_0)(h,h) - \sigma_T(-Df(x_0))\\
&\leq \sup\nolimits_{\lambda\in\Lambda(x_0)}\{D^2 f(x_0)(h,h) + \langle\lambda, D^2G(x_0)(h,h)\rangle\}.
\end{aligned}$$

The proof is complete. ☐

Theorem 2.5 provides a new second-order necessary condition for optimality. We present an example to which Theorem 2.5 is applicable but the classical results in the literature is not.

Example. For $x \in \mathbb{R}^2$, define $f(x) = x_1^2 - x_2$

$$g_i(x) = \begin{cases} -x_1 & i = 1, \\ -x_2 & i = 2, \\ x_2 - x_1^2 & i = 3. \end{cases} \tag{2.18}$$

Let $K := \{x : g_i(x) \le 0, i = 1, 2, 3\}$ and let $x_0 = (0, 0)$. Then x_0 minimizes f over K. It can be seen that the active indices $I(x_0) = \{1, 2, 3\}$,

$$T_K(x_0) = \{h : h_1 \ge 0, h_2 = 0\} = \{h : Dg_i(x_0)h \le 0 \text{ for all } i \in I(x_0)\}, \tag{2.19}$$

and $I(x_0, h) = \{2, 3\}$ for $h \in T_K(x_0) \setminus \{0\}$ ($I(x_0, h)$ was defined in (2.8)). Since $Df(x_0) = (0, -1)$, we have

$$C(x_0) = \{h : h_1 > 0, h_2 = 0\}.$$

We will show that for every $h \in C(x_0)$,

$$\begin{aligned} T_K^2(x_0, h) &= \{\omega : Dg_i(x_0)\omega + D^2 g_i(x_0)(h, h) \le 0 \text{ for all } i \in \mathcal{I}(x_0, h)\} \\ &\equiv \{\omega : 0 \le \omega_2 \le 2h_1^2\}, \end{aligned} \tag{2.20}$$

and hence the second-order Abadie CQ holds at x_0 in every direction h in $C(x_0)$. By Proposition 2.3, it suffices to prove that the right-hand side of the first equality is contained in $T_K^2(x_0, h)$. Assume that $\omega \in \mathbb{R}^2$ be such that $0 \le \omega_2 \le 2h_1^2$ and $h = (h_1, 0)$ with $h_1 > 0$. We consider the following limit

$$\lim_{t \downarrow 0} \frac{\operatorname{dist}(th + \frac{1}{2}t^2\omega, K)}{t^2}.$$

It suffices to consider this limit for the case when $th + \frac{1}{2}t^2\omega \notin K$. In this case, for sufficiently small $t > 0$, $th + \frac{1}{2}t^2\omega$ belongs to $\Re^2_{++}$ and is above the curve $g_3(x) = 0$. Let $(u(t), v(t))$ be the projection of $th + \frac{1}{2}t^2\omega$ onto K, which always exists as K is closed. That is to say, $(u(t), v(t)) \in K$ solves the following minimization problem:

$$\begin{array}{rl} \min & \frac{1}{2}\|x - (th + \frac{1}{2}t^2\omega)\|^2 \\ \text{subject to} & x \in K \equiv \{x : g_i(x) \le 0, i = 1, 2, 3\}. \end{array} \tag{2.21}$$

It can be seen that for sufficiently small $t > 0$, $u(t) > 0$ and $(u(t), v(t))$ is on the curve $g_3(x) = 0$: $v(t) = u(t)^2$. In other words, $i = 3$ is the unique active index at $(u(t), v(t))$. Since the gradient of $g_3(\cdot)$ at the point $(u(t), v(t))$ is equal to $(-2u(t), 1)^T$, which is nonsingular, applying Lagrange multiplier theorem (cf. [3, Proposition 3.3.1]) to the problem (2.21) yields that there exists unique $\lambda(t) > 0$ such that

$$th + \frac{1}{2}t^2\omega - (u(t), u(t)^2) = \lambda(t)Dg_3(u(t), u(t)^2) = \lambda(t)(-2u(t), 1);$$

that is,

$$\begin{cases} (1 - 2\lambda(t))u(t) = th_1 + \frac{1}{2}t^2\omega_1 \\ u(t)^2 + \lambda(t) = \frac{1}{2}t^2\omega_2, \end{cases} \tag{2.22}$$

which implies that $u(t)$ is the real root of the cubic equation $u^3 + p(t)u + q(t) = 0$ for sufficiently small $t > 0$, where $p(t) = \frac{1}{2}(1 - t^2\omega_2)$ and $q(t) = -\frac{1}{2}(th_1 + \frac{1}{2}t^2\omega_1)$. It can be seen that this cubic equation has a unique real root

$$u(t) = \frac{3z(t)^2 - p(t)}{3z(t)}, \quad \text{where } z(t) = \sqrt[3]{-\frac{q(t)}{2} + \sqrt{\frac{q(t)^2}{4} + \frac{p(t)^3}{27}}}.$$

Note that when $t \downarrow 0$, $p(t) \to \frac{1}{2}$, $q(t) \to 0$, $p'(t) \to 0$, $q'(t) \to -\frac{1}{2}h_1$ and $z(t) \to \frac{1}{\sqrt{6}}$, and hence

$$z'(t) = \frac{1}{3}z(t)^{-2}\left\{-\frac{1}{2}q'(t) + \frac{\frac{1}{2}q(t)q'(t) + \frac{1}{9}p(t)^2p'(t)}{2\sqrt{\frac{q(t)^2}{4} + \frac{p(t)^3}{27}}}\right\} \to \frac{1}{2}h_1.$$

Applying the L'Hôspital rule, we have

$$\lim_{t\downarrow 0} \frac{3z(t)^2 - p(t)}{t} = \lim_{t\downarrow 0} 6z(t)z'(t) - p'(t) = \frac{\sqrt{6}}{2}h_1,$$

and hence it follows that

$$\lim_{t\downarrow 0} \frac{u(t)}{t} = \lim_{t\downarrow 0} \frac{3z(t)^2 - p(t)}{3z(t)t} = h_1.$$

Thus for sufficiently small $t > 0$,

$$\begin{aligned} 0 \le \frac{\text{dist}(th + \frac{1}{2}t^2\omega, K)}{t^2} &\le \frac{\|th + \frac{1}{2}t^2\omega - (u(t), u(t)^2)\|}{t^2} \\ &= \frac{\lambda(t)\sqrt{4u(t)^2 + 1}}{t^2} = \frac{u(t)^2}{t^2} - \frac{1}{2}\omega_2 \\ &\to h_1^2 - \frac{1}{2}\omega_2 \le 0, \end{aligned}$$

where the two equalities follow from (2.22), and the last inequality follows from our assumption that $\omega_2 \le 2h_1^2$. Therefore for any sequence $t_n \downarrow 0$,

$$\lim_{n\to\infty} \frac{\text{dist}(t_nh + \frac{1}{2}t_n^2\omega, K)}{t_n^2} = 0,$$

which implies that $\omega \in T_K^2(x_0, h)$. Thus (2.20) is verified.

In view of (2.19) and (2.20), the second-order Abadie CQ holds at x_0 in every direction $h \in C(x_0)$ and hence the assumption in Theorem 2.5 is satisfied. By Theorem 2.5, we conclude that the second-order necessary optimality condition holds at the optimal solution $(0, 0)$.

On the other hand, the inequalities system $g_i(x) \le 0$ $(i = 1, 2, 3)$ do not satisfy Robinson's constraint qualification at x_0. Otherwise, one would have

$$0 \in \text{int}(G(x_0) + DG(x_0)\Re^2 - \Re^3_-). \tag{2.23}$$

Noting that

$$\begin{aligned} G(x_0) &= (g_1(x_0), g_2(x_0), g_3(x_0)) = (0, 0, 0) \quad \text{and} \\ DG(x_0) &= (g_1'(x_0), g_2'(x_0), g_3'(x_0)) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}; \end{aligned}$$

by (2.23), we obtain the existence of $h = (h_1, h_2) \in \Re^2$ such that $DG(x_0)h$ belongs to $\mathrm{int}(\Re^3_-)$: $g_i'(x_0)h < 0$ for each $i = 1, 2, 3$. This is impossible as $g_2'(x_0)h = -h_2$ and $g_3'(x_0)h = h_2$.

Before ending this paper, we give a weaker sufficient condition than Robinson's CQ for the Abadie CQ to hold. This sufficient condition is essentially a local error bound for the system $G(x) \in C$. It is said that a *local error bound* holds for the system $K = \{x : G(x) \in C\}$ at $x_0 \in K$ if there are positive scalars γ and η such that

$$\mathrm{dist}(x, K) \leq \gamma \,\mathrm{dist}(G(x), C), \quad \text{for all } x \in B(x_0; \eta), \tag{2.24}$$

where dist is the distance function and $B(x_0; \eta)$ is the open ball centered at x_0 with radius η. A systematic discussion on the theory of error bound can be found in the survey paper [6] and Chapter 6 of the book [5].

It is known that Robinson's CQ at x_0 is equivalent to the so-called metrical regularity which says that there exist $\gamma > 0$ and a neighborhood V of $(x_0, 0)$ such that

$$\mathrm{dist}(x, G^{-1}(C - y)) \leq \gamma \,\mathrm{dist}(G(x) + y, C), \quad \text{for all } (x, y) \in V; \tag{2.25}$$

see [4, Proposition 2.89].

Comparing the metric regularity (2.25) with local error bound (2.24), it is clear that (2.24) is weaker than (2.25) and hence (2.24) is weaker than Robinson's CQ at x_0 (Fixing $y = 0$ in (2.25), we obtain the local error bound (2.24)).

Proposition 2.6. *Let K be the feasible set of the problem* (2.2). *If the local error bounds* (2.24) *holds at $x_0 \in K$ then the second-order Abadie CQ holds at x_0 in every direction h.*

Proof. In view of (2.3), it suffices to prove that

$$\{w : D^2G(x_0)(h, h) + DG(x_0)w \in T_C^2(G(x_0), DG(x_0)h)\} \subset T_K^2(x_0, h).$$

Let w belong to the set in the left-hand side of the above expression. Then there exists a sequence $t_n \downarrow 0$ such that

$$\mathrm{dist}(G(x_0) + t_n DG(x_0)h + (1/2)[D^2G(x_0)(h, h) + DG(x_0)w], C) = o(t_n^2).$$

Set $x_n := x_0 + t_n h + (1/2)t_n^2 w$ and

$$y_n := G(x_0) + t_n DG(x_0)h + (1/2)[D^2G(x_0)(h, h) + DG(x_0)w].$$

Then $\mathrm{dist}(y_n, C) = o(t_n^2)$. Since G is twice continuously differentiable, by the second-order Taylor expansion, $G(x_n) = y_n + o(t_n^2)$. Since local error bound (2.24) holds, it follows that for sufficiently large n,

$$\mathrm{dist}(x_n, K) \leq \gamma \,\mathrm{dist}(G(x_n), C) \leq \gamma \|G(x_n) - y_n\| + \gamma \,\mathrm{dist}(y_n, C) = o(t_n^2).$$

This verifies that $w \in T_K^2(x_0, h)$ and hence the conclusion. □

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