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# AN OUTER APPROXIMATION METHOD FOR GENERALIZED EQUILIBRIUM PROBLEMS, FIXED POINT PROBLEMS AND VARIATIONAL INEQUALITY PROBLEMS*

Jian-Wen Peng and Jen-Chih Yao†

**Abstract:** In this paper, we introduce an iterative scheme for finding a common element of the set of solutions of a generalized equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of solutions to the variational inequality for a monotone, Lipschitz-continuous mapping in a Hilbert space. The iterative process is based on two well-known methods: hybrid and extragradient. Based on this result, we also get several new and interesting results which generalize and extend some well-known strong convergence theorems in the literature.

**Key words:** *generalized equilibrium problem, hybrid method, nonexpansive mapping, monotone mapping, variational inequality, strong convergence, fixed point*

**Mathematics Subject Classification:** *47J20, 47H10, 49J40*

## 1 Introduction

Let $H$ be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, respectively. Let $C$ be a nonempty closed convex subset of $H$. Let $F$ be a bifunction from $C \times C$ to $\mathbf{R}$ and let $B : C \to H$ be a nonlinear mapping, where $\mathbf{R}$ is the set of real numbers. Takahashi and Takahashi [25] considered the following generalized equilibrium problem:

$$\text{Find } \; x \in C \; \text{ such that } F(x,y) + \langle Bx, y-x \rangle \geq 0, \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GEP(F)$. If $B = 0$, the generalized equilibrium problem (1.1) becomes the equilibrium problem for $F : C \times C \to \mathbf{R}$, which is to find $x \in C$ such that

$$F(x,y) \geq 0, \; \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $EP(F)$.

The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others; see for instance, [3] and [8].

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†Corresponding author.

Recall that a mapping $S$ of a closed convex subset $C$ of $H$ is nonexpansive [10] if there holds that

$$\|Sx - Sy\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

We denote the set of fixed points of $S$ by $Fix(S)$. It is known (see [10]) that $Fix(S)$ is closed and convex, but possibly empty. A mapping $A$ of $C$ into $H$ is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0$$

for all $x, y \in C$. A mapping $A$ of $C$ into $H$ is called $\alpha$-inverse-strongly monotone if there exists a positive real number $\alpha$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$. It is noted that the $\alpha$-inverse-strong monotonicity of $A$ is also called cocoerciveness of $A$. A mapping $A : C \to H$ is called $k$-Lipschitz-continuous if there exists a positive real number $k$ such that

$$\|Ax - Ay\| \leq k\|x - y\|$$

for all $x, y \in C$. It is easy to see that the class of $\alpha$-inverse-strongly monotone mappings does not contain some important classes of mappings even in a finite-dimensional case. For example, if the matrix in the corresponding linear complementarity problem is positively semidefinite, but not positively definite, then the mapping $A$ will be monotone and Lipschitz-continuous, but not $\alpha$-inverse-strongly monotone.

Let $A : C \to H$. The variational inequality problem is to find a $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0$$

for all $y \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Takahashi and Takahashi [25] introduced the following iterative scheme for finding a common element of the set of solutions of (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Starting with an arbitrary $u \in C$ and $x_1 \in C$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S[\alpha_n u + (1 - \alpha_n)u_n], \forall n \in N. \end{cases} \tag{1.3}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ generated by (1.3) converges strongly to $z \in Fix(S) \cap GEP(F)$.

Some methods have been proposed to solve the problem (1.2); see, for instance, [3, 7, 8, 19, 24, 26]. Recently, Combettes and Hirstoaga [7] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. Takahashi and Takahashi [26] introduced the following iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of (1.2) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \forall n \in N. \end{cases} \tag{1.4}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.4) converge strongly to $z \in Fix(S) \cap EP(F)$, where $z = P_{Fix(S)\cap EP(F)}f(z)$ and $f$ is a contraction on $H$.

Tada and Takahashi [24] introduced the following iterative scheme by the hybrid method for finding a common element of the set of solutions of (1.2) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} u_n \in C, \quad F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ w_n = (1-\alpha_n)x_n + \alpha_n S u_n, \\ C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x. \end{cases} \tag{1.5}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.5) converge strongly to $P_{Fix(S)\cap EP(F)}x$. Generally speaking, the algorithm suggested by Tada and Takahashi is based on two well-known types of methods, namely, on the Mann iterative methods and so-called hybrid or "outer-approximation" for solving fixed point problem. The idea of "hybrid" or "outer-approximation" types of methods was originally introduced by Haugazeau in 1968 and was successfully generalized and extended in recent papers of Bauschke and Combettes [1], [2], Burachik, Lopes and Svaiter [5], Combettes [6], Nakajo and Takahashi [16], and Solodov and Svaiter [21], Kikkawa and Takahashi [12], Nadezhkina and Takahashi [14].

On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean $\mathbf{R}^n$, Korpelevich [13] introduced the following so-called extragradient method:

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases} \tag{1.6}$$

for every $n = 0, 1, 2, ...,$ where $\lambda \in (0, \frac{1}{k})$. She showed that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.6), converge to the same point $z \in VI(C, A)$. The idea of the extragradient iterative process introduced by Korpelevich was successfully generalized and extended not only in Euclidean but also in Hilbert and Banach spaces; see, e. g., the recent papers of He, Yang and Yuan [11], Gárciga Otero and Iuzem [9], Noor [17], Solodov and Svaiter [22], Solodov [23]. Moreover, Zeng and Yao [28] and Nadezhkina and Takahashi [14, 15] where some iterative processes based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequality problem for a monotone, Lipschitz-continuous mapping where introduced. Yao and Yao [27] introduced an iterative process based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequality problem for an $\alpha$-inverse strongly monotone mapping. Plubtieng and Punpaeng [19] introduced an iterative process based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings, the set solutions of an equilibrium problem and the set of solutions of variational inequality problem for $\alpha$-inverse strongly monotone mappings.

In the present paper, by combining the hybrid and extragradient methods, we introduce an iterative process for finding a common element of the set of solutions of a generalized equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for a monotone, Lipschitz-continuous mapping in a

Hilbert space. We derive a strong convergence theorem for four sequences generated by this process. Based on this result, we also get several new and interesting results which generalize and extend some well-known strong convergence theorems in the literature.

## 2 Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $C$ be a nonempty closed convex subset of $H$. Let symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively. In a real Hilbert space $H$, it is well known that

$$\|\lambda x+(1-\lambda)y\|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

For any $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C(x)$, such that $\|x-P_C(x)\| \leq \|x-y\|$ for all $y \in C$. The mapping $P_C$ is called the metric projection of $H$ onto $C$. We know that $P_C$ is a nonexpansive mapping from $H$ onto $C$. It is also known that $P_Cx \in C$ and

$$\langle x-P_C(x), P_C(x)-y\rangle \geq 0 \tag{2.1}$$

for all $x \in H$ and $y \in C$.

It is easy to see that (2.1) is equivalent to

$$\|x-y\|^2 \geq \|x-P_C(x)\|^2+\|y-P_C(x)\|^2 \tag{2.2}$$

for all $x \in H$ and $y \in C$. Let $A$ be a monotone mapping of $C$ into $H$. In the context of the variational inequality problem the characterization of projection (2.1) implies the following:

$$u \in VI(C,A) \ \Rightarrow \ \ u=P_C(u-\lambda Au),\ \lambda>0,$$

and

$$u=P_C(u-\lambda Au) \text{ for some } \lambda>0 \Rightarrow \ \ u \in VI(C,A).$$

It is also known that $H$ satisfies the Opial's condition [18], i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty}\|x_n-x\| < \liminf_{n\to\infty}\|x_n-y\|$$

holds for every $y \in H$ with $x \neq y$.

A set-valued mapping $T: H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ we have $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T: H \to 2^H$ is maximal if its graph $G(T)$ of $T$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $T$ is maximal if and only if for $(x,f) \in H\times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y,g) \in G(T)$ implies $f \in Tx$. Let A be a monotone, $k$-Lipschitz-continuous mapping of $C$ into $H$ and let $N_Cv$ be the normal cone to $C$ at $v \in C$, i.e, $N_Cv=\{w \in H : \langle v-u, w\rangle \geq 0, \forall u \in C\}$. Define

$$Tv=\begin{cases} Av+N_Cv & \text{if} \quad v \in C, \\ \emptyset & \text{if} \quad v \notin C. \end{cases}$$

Then $T$ is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C,A)$(see [20]).

For solving the equilibrium problem, let us assume that the bifunction $F$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) $F$ is monotone, i.e. $F(x,y) + F(y,x) \leq 0$ for any $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y);$$

(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.
We recall some lemmas which will be needed in the rest of this paper.

**Lemma 2.1** ([3]). *Let $C$ be a nonempty closed convex subset of $H$, let $F$ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \tfrac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \text{for} \quad \text{all} \;\; y \in C.$$

**Lemma 2.2** ([7]). *Let $C$ be a nonempty closed convex subset of $H$, let $F$ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows.*

$$T_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

*for all $x \in H$. Then, the following statements hold:*
*(1) $T_r$ is single-valued;*
*(2) $T_r$ is firmly nonexpansive, i.e, for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

*(3) $F(T_r) = EP(F)$;*
*(4) $EP(F)$ is closed and convex.*

## 3 Strong Convergence Theorem

In this section, we first show a strong convergence theorem which solves the problem of finding a common element of the set of solutions of a generalized equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for a monotone, Lipschitz-continuous mapping in a Hilbert space. Then, based on this result, we also get several new and interesting results which generalize and extend some well-known results in [26] and [6].

**Theorem 3.1.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F$ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)-(A4) and let $A$ be a monotone and $k$-Lipschitz-continuous mapping of $C$ into $H$, $B$ be an $\alpha$-inverse-strongly monotone mapping of $C$ into $H$. Let $S$ be a nonexpansive mapping of $C$ into itself such that $Fix(S) \cap VI(C,A) \cap GEP(F) \neq \emptyset$. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in H, \\ \quad F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = \alpha_n x_n + (1-\alpha_n) S P_C(u_n - \lambda_n A y_n), \\ C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

*for every $n = 1, 2, ....$ If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$ and $\{r_n\} \subset [d, e]$ for some $d, e \in (0, 2\alpha)$. Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $w = P_{Fix(S)\cap VI(C,A)\cap GEP(F)}(x)$.*

*Proof.* It is obvious that $C_n$ is closed and $Q_n$ is closed and convex for every $n = 1, 2, ....$ From [12], we know that

$$C_n = \{z \in H : \|z_n - x_n\|^2 + 2\langle z_n - x_n, x_n - z\rangle \leq 0\}.$$

Thus $C_n$ is convex for every $n = 1, 2, ....$ It is easy to see that $\langle x_n - z, x - x_n\rangle \geq 0$ for all $z \in Q_n$ and by (2.1), $x_n = P_{Q_n} x$. Put $t_n = P_C(u_n - \lambda_n Ay_n)$ for every $n = 1, 2, ....$ Let $u \in Fix(S) \cap VI(C, A) \cap GEP(F)$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.2. Then $u = P_C(u - \lambda_n Au) = T_{r_n}(u - r_n Bu)$. From $u_n = T_{r_n}(x_n - r_n Bx_n) \in C$ and the $\alpha$-inverse-strong monotonicity of $B$, we have

$$\begin{aligned} \|u_n - u\|^2 &= \|T_{r_n}(x_n - r_n Bx_n) - T_{r_n}(u - r_n Bu)\|^2 \\ &\leq \|x_n - r_n Bx_n - (u - r_n Bu)\|^2 \\ &\leq \|x_n - u\|^2 - 2r_n\langle x_n - u, Bx_n - Bu\rangle + r_n^2\|Bx_n - Bu\|^2 \\ &\leq \|x_n - u\|^2 - 2r_n\alpha\|Bx_n - Bu\|^2 + r_n^2\|Bx_n - Bu\|^2 \\ &= \|x_n - u\|^2 + r_n(r_n - 2\alpha)\|Bx_n - Bu\|^2 \\ &\leq \|x_n - u\|. \end{aligned} \tag{3.1}$$

From (2.2), the monotonicity of $A$ and $u \in VI(C, A)$, we have

$$\begin{aligned} \|t_n - u\|^2 &\leq \|u_n - \lambda_n Ay_n - u\|^2 - \|u_n - \lambda_n Ay_n - t_n\|^2 \\ &= \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n\langle Ay_n, u - t_n\rangle \\ &= \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n(\langle Ay_n - Au, u - y_n\rangle \\ &\quad + \langle Au, u - y_n\rangle + \langle Ay_n, y_n - t_n\rangle) \\ &\leq \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\ &\leq \|u_n - u\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\ &\quad + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\ &= \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\quad + 2\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle. \end{aligned}$$

Further, Since $y_n = P_C(u_n - \lambda_n Au_n)$ and $A$ is $k$-Lipschitz-continuous, we have

$$\begin{aligned} \langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle &= \langle u_n - \lambda_n Au_n - y_n, t_n - y_n\rangle + \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n\rangle \\ &\leq \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n\rangle \\ &\leq \lambda_n k\|u_n - y_n\|\|t_n - y_n\|. \end{aligned}$$

So, we have

$$\begin{aligned} \|t_n - u\|^2 &\leq \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k\|u_n - y_n\|\|t_n - y_n\| \\ &\leq \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + {\lambda_n}^2 k^2\|u_n - y_n\|^2 + \|t_n - y_n\|^2 \\ &= \|u_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 \\ &\leq \|u_n - u\|^2. \end{aligned} \tag{3.2}$$

Therefore from (3.1), (3.2), $z_n = \alpha_n x_n + (1-\alpha_n)St_n$ and $u = Su$, we have

$$\begin{aligned}
\|z_n - u\|^2 &= \|\alpha_n x_n + (1-\alpha_n)St_n - u\|^2 \\
&\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|St_n - u\|^2 \\
&\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|t_n - u\|^2 \\
&\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)[\|u_n - u\|^2 + ({\lambda_n}^2k^2 - 1)\|u_n - y_n\|^2] \\
&\leq \|x_n - u\|^2 + (1-\alpha_n)({\lambda_n}^2k^2 - 1)\|u_n - y_n\|^2 \\
&\leq \|x_n - u\|^2, \qquad (3.3)
\end{aligned}$$

for every $n = 1, 2, ...$ and hence $u \in C_n$. So, $Fix(S) \cap VI(C, A) \cap GEP(F) \subset C_n$ for every $n = 1, 2, ....$ Next, let us show by mathematical induction that $\{x_n\}$ is well defined and $Fix(S) \cap VI(C, A) \cap GEP(F) \subset C_n \cap Q_n$ for every $n = 1, 2, ....$ For $n = 1$ we have $x_1 = x \in C$ and $Q_1 = H$. Hence we obtain $Fix(S) \cap VI(C, A) \cap GEP(F) \subset C_1 \cap Q_1$. Suppose that $x_k$ is given and $Fix(S) \cap VI(C, A) \cap GEP(F) \subset C_k \cap Q_k$ for some $k \in N$. Since $Fix(S) \cap VI(C, A) \cap GEP(F)$ is nonempty, $C_k \cap Q_k$ is a nonempty closed convex subset of $H$. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k}x$. It is also obvious that there holds $\langle x_{k+1} - z, x - x_{k+1}\rangle \geq 0$ for every $z \in C_k \cap Q_k$. Since $Fix(S) \cap VI(C, A) \cap GEP(F) \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, x - x_{k+1}\rangle \geq 0$ for every $z \in Fix(S) \cap VI(C, A) \cap GEP(F)$ and hence $Fix(S) \cap VI(C, A) \cap GEP(F) \subset Q_{k+1}$. Therefore, we obtain $Fix(S) \cap VI(C, A) \cap GEP(F) \subset C_{k+1} \cap Q_{k+1}$.

Let $l_0 = P_{Fix(S)\cap VI(C,A)\cap GEP(F)}x$. From $x_{n+1} = P_{C_n \cap Q_n}x$ and $l_0 \in Fix(S) \cap VI(C, A) \cap GEP(F) \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \leq \|l_0 - x\| \qquad (3.4)$$

for every $n = 1, 2, ....$ Therefore, $\{x_n\}$ is bounded. From (3.1)-(3.3), we also obtain that $\{t_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded. Since $x_{n+1} \in C_n \cap Q_n \subset C_n$ and $x_n = P_{Q_n}x$, we have

$$\|x_n - x\| \leq \|x_{n+1} - x\|$$

for every $n = 1, 2, ....$ Therefore, $\lim_{n\to\infty} \|x_n - x\|$ exists.

Since $x_n = P_{Q_n}x$ and $x_{n+1} \in Q_n$, using (2.2), we have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2$$

for every $n = 1, 2, ....$ This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have $\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$ and hence

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \leq 2\|x_n - x_{n+1}\|$$

for every $n = 1, 2, ....$ From $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we have $\|x_n - z_n\| \to 0$.

For $u \in Fix(S) \cap VI(C, A) \cap GEP(F)$, from (3.3) we obtain

$$\|z_n - u\|^2 \leq \|x_n - u\|^2 + (1-\alpha_n)({\lambda_n}^2k^2 - 1)\|u_n - y_n\|^2.$$

Therefore, we have

$$\begin{aligned}
\|u_n - y_n\|^2 &\leq \frac{1}{(1-\alpha_n)(1-{\lambda_n}^2k^2)}\Big(\|x_n - u\|^2 - \|z_n - u\|^2\Big) \\
&= \frac{1}{(1-\alpha_n)(1-{\lambda_n}^2k^2)}(\|x_n - u\| + \|z_n - u\|)(\|x_n - u\| - \|z_n - u\|) \\
&= \frac{1}{(1-\alpha_n)(1-{\lambda_n}^2k^2)}(\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|. \qquad (3.5)
\end{aligned}$$

Since $\|x_n - z_n\| \to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|u_n - y_n\| \to 0$. By the same process as in (3.2), we also have

$$\begin{aligned}\|t_n - u\|^2 &\leq \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k\|u_n - y_n\|\|t_n - y_n\| \\ &\leq \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \|u_n - y_n\|^2 + {\lambda_n}^2 k^2\|t_n - y_n\|^2 \\ &= \|u_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|y_n - t_n\|^2.\end{aligned}$$

Then, we have by (3.5),

$$\begin{aligned}\|t_n - y_n\|^2 &\leq \lambda_n\|Ay_nAu_N\| \leq k\lambda_n\|y_n - u_n\| \\ &\leq \frac{k\lambda_n}{(1-\alpha_n)(1-{\lambda_n}^2k^2)}\Big(\|x_n - u\|^2 - \|z_n - u\|^2\Big) \\ &= \frac{k\lambda_n}{(1-\alpha_n)(1-{\lambda_n}^2k^2)}(\|x_n - u\| + \|z_n - u\|)(\|x_n - u\| - \|z_n - u\|) \\ &\leq \frac{k\lambda_n}{(1-\alpha_n)(1-{\lambda_n}^2k^2)}(\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.\end{aligned}$$

Since $\|x_n - z_n\| \to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|t_n - y_n\| \to 0$. As $A$ is $k$-Lipschitz-continuous, we have $\|Ay_n - At_n\| \to 0$. From $\|u_n - t_n\| \leq \|u_n - y_n\| + \|y_n - t_n\|$ we also have $\|u_n - t_n\| \to 0$.

From (3.3) and (3.1), we have

$$\begin{aligned}\|z_n - u\|^2 &\leq \alpha_n^2\|x_n - u\|^2 + (1-\alpha_n)[\|u_n - u\|^2 + (\lambda_n^2k^2 - 1)\|u_n - y_n\|^2] \\ &\leq \alpha_n^2\|x_n - u\|^2 + (1-\alpha_n)\|u_n - u\|^2 \\ &\leq \alpha_n^2\|x_n - u\|^2 + (1-\alpha_n)[\|x_n - u\|^2 + r_n(r_n - 2\alpha)\|Bx_n - Bu\|^2] \\ &= \|x_n - u\|^2 + (1-\alpha_n)r_n(r_n - 2\alpha)\|Bx_n - Bu\|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}(1-c)d(2\alpha - e)\|Bx_n - Bu\|^2 &\leq (1-\alpha_n)r_n(2\alpha - r_n)\|Bx_n - Bu\|^2 \\ &\leq \|x_n - u\|^2 - \|z_n - u\|^2 \\ &\leq (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.\end{aligned}$$

Since $\|x_n - z_n\| \to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|Bx_n - Bu\| \to 0$.

For $u \in Fix(S) \cap VI(C, A) \cap GEP(F)$, we have, from Lemma 2.2,

$$\begin{aligned}\|u_n - u\|^2 &= \|T_{r_n}(x_n - r_nBx_n) - T_{r_n}(u - r_nBu)\|^2 \\ &\leq \langle T_{r_n}(x_n - r_nBx_n) - T_{r_n}(u - r_nBu), x_n - r_nBx_n - (u - r_nBu)\rangle \\ &= \frac{1}{2}\{\|u_n - u\|^2 + \|x_n - r_nBx_n - (u - r_nBu)\|^2 \\ &\quad - \|x_n - r_nBx_n - (u - r_nBu) - (u_n - u)\|^2\} \\ &\leq \frac{1}{2}\{\|u_n - u\|^2 + \|x_n - u\|^2 - \|x_n - r_nBx_n - (u - r_nBu) - (u_n - u)\|^2\} \\ &= \frac{1}{2}\{\|u_n - u\|^2 + \|x_n - u\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle - r_n^2\|Bx_n - Bu\|^2.\end{aligned}$$

Hence,

$$\|u_n - u\|^2 \le \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle - r_n^2\|Bx_n - Bu\|^2.$$

Then, by (3.3) and (3.2),

$$\begin{aligned}\|z_n - u\|^2 &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|t_n - u\|^2\\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|u_n - u\|^2\\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)[\|x_n - u\|^2 - \|x_n - u_n\|^2\\ &\quad + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle - r_n^2\|Bx_n - Bu\|^2]\\ &\le \|x_n - u\|^2 - (1-\alpha_n)\|x_n - u_n\|^2 + (1-\alpha_n)2r_n\|Bx_n - Bu\|\|x_n - u_n\|.\end{aligned}$$

Hence,

$$\begin{aligned}(1-c)\|x_n - u_n\|^2 &\le (1-\alpha_n)\|x_n - u_n\|^2\\ &\le \|x_n - u\|^2 - \|z_n - u\|^2 + (1-\alpha_n)2r_n\|Bx_n - Bu\|\|x_n - u_n\|\\ &= (\|x_n - u\| + \|z_n - u\|)(\|x_n - u\| - \|z_n - u\|)\\ &\quad + (1-\alpha_n)2r_n\|Bx_n - Bu\|\|x_n - u_n\|\\ &\le (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\| + (1-\alpha_n)2r_n\|Bx_n - Bu\|\|x_n - u_n\|.\end{aligned}$$

Since $\|x_n - z_n\| \to 0$, $\|Bx_n - Bu\| \to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|x_n - u_n\| \to 0$. From $\|z_n - t_n\| \le \|z_n - x_n\| + \|x_n - u_n\| + \|u_n - t_n\|$ we have $\|z_n - t_n\| \to 0$. From $\|t_n - x_n\| \le \|t_n - u_n\| + \|x_n - u_n\|$ we also have $\|t_n - x_n\| \to 0$.

Since $z_n = \alpha_n x_n + (1-\alpha_n)St_n$, we have $(1-\alpha_n)(St_n - t_n) = \alpha_n(t_n - x_n) + (z_n - t_n)$. Then

$$(1-c)\|St_n - t_n\| \le (1-\alpha_n)\|St_n - t_n\| \le \alpha_n\|t_n - x_n\| + \|z_n - t_n\|$$

and hence $\|St_n - t_n\| \to 0$. As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{ni} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$, we obtain that $u_{ni} \rightharpoonup w$. From $\|u_n - t_n\| \to 0$, we also obtain that $t_{ni} \rightharpoonup w$. Since $\{u_{ni}\} \subset C$ and $C$ is closed and convex, we obtain $w \in C$.

Now, we first show $w \in GEP(F)$. By $u_n = T_{r_n}(x_n - r_n Bx_n)$, we know that

$$F(u_n, y) + \langle Bx_n, y - u_n\rangle + \tfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \forall y \in C.$$

It follows from (A2) that

$$\langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n), \forall y \in C.$$

Hence,

$$\langle Bx_{n_i}, y - u_{n_i}\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \ge F(y, u_{n_i}), \forall y \in C. \tag{3.6}$$

For $t$ with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we obtain $y_t \in C$. So, from (3.6) we have

$$\begin{aligned}\langle y_t - u_{n_i}, By_t\rangle &\ge \langle y_t - u_{n_i}, By_t\rangle - \langle y_t - u_{n_i}, Bx_{n_i}\rangle\\ &\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i})\\ &= \langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle + \langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle\\ &\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i}).\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Bu_{n_i} - Bx_{n_i}\| \to 0$. Further, from the inverse-strong monotonicity of $B$, we have $\langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle \geq 0$. So, from (A4) we have

$$\langle y_t - w, By_t\rangle \geq F(y_t, w), \tag{3.7}$$

as $i \to \infty$. From (A1), (A4) and (3.7), we also have

$$\begin{aligned} 0 = F(y_t, y_t) &\leq tF(y_t, y) + (1-t)F(y_t, w) \\ &\leq tF(y_t, y) + (1-t)\langle y_t - w, By_t\rangle \\ &= tF(y_t, y) + (1-t)t\langle y - w, By_t\rangle \end{aligned}$$

and hence

$$0 \leq F(y_t, y) + (1-t)\langle y - w, By_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F(w, y) + \langle y - w, Bw\rangle \geq 0. \tag{3.8}$$

This implies that $w \in GEP(F)$.

We next show that $w \in Fix(S)$. Assume $w \notin Fix(S)$. Since $t_{n_i} \rightharpoonup w$ and $w \neq Sw$, from the Opial theorem [18] we have

$$\begin{aligned} \liminf_{i\to\infty} \|t_{n_i} - w\| &< \liminf_{i\to\infty} \|t_{n_i} - Sw\| \\ &\leq \liminf_{i\to\infty} \{\|t_{n_i} - St_{n_i}\| + \|St_{n_i} - Sw\|\} \\ &\leq \liminf_{i\to\infty} \|t_{n_i} - w\|. \end{aligned}$$

This is a contradiction. So, we get that $w \in Fix(S)$.

Finally we show that $w \in VI(C, A)$. Let

$$Tv = \begin{cases} Av + N_C v & \text{if} \quad v \in C, \\ \emptyset & \text{if} \quad v \notin C \end{cases}$$

where $N_C v$ is the normal cone to $C$ at $v \in C$. We have already mentioned that in this case the mapping $T$ is maximal monotone, $0 \in Tv$ if and only if $v \in VI(C, A)$. Let $(v, g) \in G(T)$. Then $Tv = Av + N_C v$ and hence $g - Av \in N_C v$. So, we have $\langle v - t, g - Av\rangle \geq 0$ for all $t \in C$. On the other hand, from $t_n = P_C(u_n - \lambda_n Ay_n)$ and $v \in C$ we have

$$\langle u_n - \lambda_n Ay_n - t_n, t_n - v\rangle \geq 0$$

and hence

$$\langle v - t_n, \frac{t_n - u_n}{\lambda_n} + Ay_n\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
\langle v - t_{n_i}, g\rangle &\geq \langle v - t_{n_i}, Av\rangle \\
&\geq \langle v - t_{n_i}, Av\rangle - \langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\rangle \\
&= \langle v - t_{n_i}, Av - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\
&= \langle v - t_{n_i}, Av - At_{n_i} + At_{n_i} - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\
&= \langle v - t_{n_i}, Av - At_{n_i}\rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\
&\geq \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle.
\end{aligned}$$

Hence we obtain $\langle v - w, g\rangle \geq 0$ as $i \to \infty$. Since $T$ is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(C, A)$. This implies $w \in Fix(S) \cap VI(C, A) \cap GEP(F)$.

From $l_0 = P_{Fix(S)\cap VI(C,A)\cap GEP(F)}x$, $w \in Fix(S) \cap VI(C, A) \cap GEP(F)$ and (3.4), we have

$$\|l_0 - x\| \leq \|w - x\| \leq \liminf_{i\to\infty} \|x_{n_i} - x\| \leq \limsup_{i\to\infty} \|x_{n_i} - x\| \leq \|l_0 - x\|.$$

So, we obtain

$$\lim_{i\to\infty} \|x_{n_i} - x\| = \|w - x\|.$$

From $x_{n_i} - x \rightharpoonup w - x$ we have $x_{n_i} - x \to w - x$ and hence $x_{n_i} \to w$. Since $x_n = P_{Q_n}x$ and $l_0 \in Fix(S) \cap VI(C, A) \cap GEP(F) \subset C_n \cap Q_n \subset Q_n$, we have

$$-\|l_0 - x_{n_i}\|^2 = \langle l_0 - x_{n_i}, x_{n_i} - x\rangle + \langle l_0 - x_{n_i}, x - l_0\rangle \geq \langle l_0 - x_{n_i}, x - l_0\rangle.$$

As $i \to \infty$, we obtain $-\|l_0 - w\|^2 \geq \langle l_0 - w, x - l_0\rangle \geq 0$ by $l_0 = P_{Fix(S)\cap VI(C,A)\cap GEP(F)}x$ and $w \in Fix(S) \cap VI(C, A) \cap GEP(F)$. Hence we have $w = l_0$. This implies that $x_n \to l_0$. It is easy to see $u_n \to l_0$, $y_n \to l_0$ and $z_n \to l_0$. The proof is now complete. □

By Theorem 3.1, we can obtain the following new and interesting strong convergence theorems in a real Hilbert space.

**Corollary 3.2.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F$ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)-(A4) and let $B$ be an $\alpha$-inverse-strongly monotone mapping of $C$ into $H$. Let $S$ be a nonexpansive mapping of $C$ into itself such that $Fix(S) \cap GEP(F) \neq \emptyset$. Let $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases}
x_1 = x \in H, \\
\quad F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\
z_n = \alpha_n x_n + (1 - \alpha_n)Su_n, \\
C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\
x_{n+1} = P_{C_n \bigcap Q_n} x
\end{cases}$$

*for every $n = 1, 2, \ldots$. If $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$ and $\{r_n\} \subset [d, e]$ for some $d, e \in (0, 2\alpha)$. Then, $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ converge strongly to $w = P_{Fix(S)\cap GEP(F)}(x)$.*

*Proof.* Putting $A = 0$, by Theorem 3.1 we obtain the desired result. □

**Corollary 3.3.** *Let $H$ be a real Hilbert space. Let $S$ be a nonexpansive mapping of $C$ into itself such that $Fix(S) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in H, \\ z_n = \alpha_n x_n + (1-\alpha_n) S x_n, \\ C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

*for every $n = 1, 2, \ldots$. If $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$. Then, $\{x_n\}$ and $\{z_n\}$ converge strongly to $w = P_{Fix(S)}(x)$.*

*Proof.* Putting $C = H$, $F = 0$ and $A = B = 0$, by Theorem 3.1 we obtain the desired result. □

A mapping $T$ of a closed convex subset $C$ into itself is pseudocontractive if there holds that

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2$$

for all $x, y \in C$; see [4]. Obviously, the class of pseudocontractive mappings is more general than the class of nonexpansive mappings. Now we prove a strong convergence theorem of a new iterative process for finding a common element of the set of solutions of a generalized equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of fixed points of a Lipschitz pseudocontractive mapping.

**Corollary 3.4.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F$ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)-(A4) and let $B$ be an $\alpha$-inverse-strongly monotone mapping of $C$ into $H$. Let $T$ be a pseudocontrative and $m$-Lipschitz-continuous mapping of $C$ into itself. Let $S$ be a nonexpansive mapping of $C$ into itself such that $Fix(S) \cap Fix(T) \cap GEP(F) \neq \emptyset$. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in H, \\ \quad F(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \qquad \forall y \in C, \\ y_n = u_n - \lambda_n (u_n - T u_n), \\ z_n = \alpha_n x_n + (1-\alpha_n) S P_C (u_n - \lambda_n (y_n - T y_n)), \\ C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

*for every $n = 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{m+1})$, $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$ and $\{r_n\} \subset [d, e]$ for some $d, e \in (0, 2\alpha)$. Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $w = P_{Fix(S) \cap Fix(T) \cap GEP(F)}(x)$.*

*Proof.* Let $A = I - T$. We show that the mapping $A$ is monotone and $(m+1)$-Lipschitz-continuous. From the definition of the mapping $A$, we have

$$\langle Ax - Ay, x - y \rangle = \langle x - y - Tx + Ty, x - y \rangle$$
$$= \|x - y\|^2 - \langle Tx - Ty, x - y \rangle \geq \|x - y\|^2 - \|x - y\|^2 = 0.$$

So, $A$ is monotone. We also have

$$\begin{aligned} \|Ax - Ay\|^2 = \|(I-T)x - (I-T)y\| &= \|x - y\|^2 + \|Tx - Ty\|^2 - 2\langle x - y, Tx - Ty \rangle \\ &\leq \|x - y\|^2 + m\|x - y\|^2 + 2\|x - y\|\|Tx - Ty\| \\ &\leq \|x - y\|^2 + m\|x - y\|^2 + 2m\|x - y\|^2 \\ &= (m+1)^2 \|x - y\|^2. \end{aligned}$$

So, we have $\|Ax - Ay\| \leq (m+1)\|x - y\|$ and $A$ is $(m+1)$-Lipschitz-continuous. It is easy to check that $Fix(T) = VI(C, A)$. By Theorem 3.1 we obtain the desired result. $\square$

**Corollary 3.5.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $A$ be a monotone and $k$-Lipschitz-continuous mapping of $C$ into $H$, $B$ be an $\alpha$-inverse-strongly monotone mapping of $C$ into $H$. Let $S$ be a nonexpansive mapping of $C$ into itself such that $Fix(S) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in H, \\ u_n = P_C(x_n - r_n B x_n), \\ y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) S P_C(u_n - \lambda_n A y_n), \\ C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

*for every $n = 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$ and $\{r_n\} \subset [d, e]$ for some $d, e \in (0, 2\alpha)$. Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $w = P_{Fix(S) \cap VI(C,A) \cap VI(C,B)}(x)$.*

*Proof.* In Theorem 3.1, put $F = 0$. Then, we obtain that

$$\langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \qquad \forall y \in C, \forall n \in N.$$

This implies that

$$\langle y - u_n, u_n - (x_n - r_n B x_n) \rangle \geq 0, \qquad \forall y \in C, \forall n \in N.$$

So, we get that $u_n = P_C(x_n - r_n B x_n)$ for all $n \in N$. Then we obtain the desired result from Theorem 3.1. $\square$

**Corollary 3.6.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $B$ be an $\alpha$-inverse-strongly monotone mapping of $C$ into $H$. Let $S$ be a nonexpansive mapping of $C$ into itself such that $Fix(S) \cap VI(C, B) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in H, \\ z_n = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - r_n B x_n), \\ C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

*for every $n = 1, 2, \ldots$. If $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$ and $\{r_n\} \subset [d, e]$ for some $d, e \in (0, 2\alpha)$. Then, $\{x_n\}$ and $\{z_n\}$ converge strongly to $w = P_{Fix(S) \cap VI(C,B)}(x)$.*

*Proof.* In Corollary 3.5, put $A = 0$. Then we obtain the desired result from Corollary 3.5. $\square$

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Jian-Wen Peng
School of Mathematics, Chongqing Normal University
Chongqing 400047, P. R. China
E-mail address: `jwpeng2008@gmail.com`

Jen-Chih Yao
Department of Applied Mathematics, National Sun Yat-sen University
Kaohsiung, Taiwan 804 R. O. C.
E-mail address: `yaojc@math.nsysu.edu.tw`