Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 3, No. 2, pp. 301 – 307, May 2007

ON GENERALIZED VECTOR QUASI-EQUILIBRIUM PROBLEMS*

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Abstract: This paper deals with generalized vector quasi-equilibrium problems. Using a nonlinear scalarization function and Ky Fan minimax inequality, existence theorems for two classes of generalized vector quasi-equilibrium problems are established.

Key words: *generalized vector quasi-equilibrium problem, nonlinear scalarization function, Ky Fan minimax inequality*

Mathematics Subject Classification: *90C29, 90C46, 49J53*

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1 Introduction

Throughout this paper, let X and Z be two locally convex Hausdorff topological spaces and E be a nonempty, compact and convex subset of X. We also assume that $C : X \to 2^Z$ is a set-valued mapping such that $C(x)$ is a proper, closed and convex cone of Z with $\text{int}C(x) \neq \emptyset$, for each $x \in X$. A vector-valued mapping $e : X \to Z$ is said to be a selection from $\text{int}C(\cdot)$ if for any $x \in X, e(x) \in \text{int}C(x)$. Let $K : E \to 2^E$ be a set-valued mapping with closed values and $F : E \times E \to 2^Z$ be a set-valued mapping.

Consider two classes of generalized vector quasi-equilibrium problems of finding an $\bar{x} \in E$ such that

$$\text{(GVQEP1)} \qquad \bar{x} \in K(\bar{x}) \text{ and } F(\bar{x}, y) \not\subseteq -\text{int}C(\bar{x}),\ \forall y \in K(\bar{x}),$$

and of finding an $\tilde{x} \in E$ such that

$$\text{(GVQEP2)} \qquad \tilde{x} \in K(\tilde{x}) \text{ and } F(\tilde{x}, y) \subset -C(\tilde{x}),\ \forall y \in K(\tilde{x}).$$

It is well known that the vector equilibrium problem provides a unified model of several classes of problems, for example, vector variational inequality problems, vector complementarity problems, vector optimization problems and vector saddle point problems. See [3, 8, 10]. Many authors (see [1, 4, 5, 7, 9, 11]) have intensively studied different types of vector equilibrium problems. In [1], Ansari and Flores-Bazan first investigated the existence

*This research was partially supported by the Postdoctoral Fellowship Scheme of The Hong Kong Polytechnic University and the National Natural Science Foundations of China (Grant numbers: 60574073 and 10471142).

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of the problem (GVQEP1). In [7], Fu studied a special case of the problem (GVQEP2) when $C(x)$ and $K(x)$ are constant for any $x \in E$. In [11], we discussed the existences of solutions for the problems (GVQEP1) and (GVQEP2). Under the following assumptions:
(i) $C : X \to 2^Z$ and $W(\cdot) = Z \backslash \text{int}C(\cdot)$ are upper semi-continuous on X and $\text{int}C(\cdot)$ has a continuous selection $e(\cdot)$,
(ii) $F : E \times E \to 2^Z$ is a continuous mapping with compact values on $E \times E$,
and other assumptions, by using the Fan-Glicksber-Kakutani fixed point theorem and a nonlinear scalarization function, we obtained an existence result for (GVQEP1). We also obtained an existence result for (GVQEP2).

In this paper, we improve two results in [11]. Under the following assumptions:
(i) $W(\cdot) = Z \backslash \text{int}C(\cdot)$ is upper semi-continuous on X and $\text{int}C(\cdot)$ has a continuous selection $e(\cdot)$,
(ii) $F : E \times E \to 2^Z$ is an upper semicontinuous mapping with compact values on $E \times E$,
and other assumptions, we use the Ky Fan minimax inequality theorem and a nonlinear scalarization function to obtain an existence result for (GVQEP1). Simultaneously, we also obtain an existence result of (GVQEP2) under weaker conditions.

The rest of the paper is organized as follows. In Section 2, we recall Ky Fan minimax inequality theorem, the nonlinear scalarization function and its properties. In Sections 3, we show existence results for (GVQEP1) and (GVQEP2).

2 Preliminary Results and a Nonlinear Scalarization Function

From [12], we introduce a concept of the convex set-valued mapping.

Definition 2.1. Let $F : E \times E \to 2^Z$ be a set-valued mapping. $F(x, \cdot)$ is said to be $C(x)$-convex on E for a fixed $x \in E$ if, for any $y_1, y_2 \in E$ and $\lambda \in (0, 1)$,

$$F(x, \lambda y_1 + (1 - \lambda)y_2) \subset \lambda F(x, y_1) + (1 - \lambda)F(x, y_2) - C(x).$$

Let $e : X \to Z$ be a vector-valued mapping and, for any $x \in X$, $e(x) \in \text{int}C(x)$. Now we recall the definition of a nonlinear scalarization function [5] and its corresponding result.

Definition 2.2. The nonlinear scalarization function $\xi_e : X \times Z \to \mathcal{R}$ is defined by

$$\xi_e(x, y) = \inf\{\lambda \in \mathcal{R} \mid y \in \lambda e(x) - C(x)\}.$$

Theorem 2.3. *Let X and Z be two locally convex Hausdorff topological vector spaces, and let $C : X \to 2^Z$ be a set-valued mapping such that, for each $x \in X$, $C(x)$ is a proper, closed, convex cone in Z with $\text{int}C(x) \neq \emptyset$. Furthermore, let $e : X \to Z$ be a selection from the set-valued map $\text{int}C(\cdot)$. Define a set-valued mapping $W : X \to 2^Z$ by $W(x) = Z \backslash \text{int}C(x)$, for $x \in X$. Then, it holds that*

(i) *If $W(\cdot)$ is upper semi-continuous on X, then $\xi_e(\cdot, \cdot)$ is upper semicontinuous on $X \times X$,*

(ii) *If $C(\cdot)$ is upper semicontinuous on X, then $\xi_e(\cdot, \cdot)$ is lower semicontinuous on $X \times X$.*

Note that for the detailed definitions of lower and upper semicontinuities and continuity of set-valued mappings, see pp.108-110 of Aubin and Ekeland [2].

Remark 2.4. $\xi_e(\cdot, \cdot)$ was introduced in [5]. It was shown that Theorem 2.3 and some propositions hold when X and Z are the same locally convex Hausdorff topological vector space. In fact, it follows from the proofs of these results presented in [5] that, when X and Z are two different locally convex Hausdorff topological vector spaces, Theorem 2.3 and those propositions in [5] still hold.

From Ky Fan minimax inequality theorem [6], we have

Theorem 2.5. *Let E be a nonempty convex, compact subset of X and $g : E \times E \to \mathcal{R}$ a function satisfying*

(i) *for any $x \in E, g(x,x) \geq 0$;*

(ii) *for any $x \in E$, $g(x, \cdot)$ is quasiconvex on E;*

(iii) *for any $y \in E$, $g(\cdot, y)$ is upper semicontinuous on E.*

Then, there exists an element $\bar{x} \in E$ such that

$$g(\bar{x}, y) \geq 0, \ \forall y \in E.$$

3 Existences of Solutions for (GVQEP1) and (GVQEP2)

In this section, we shall use Ky Fan minimax inequality theorem and the nonlinear scalar function to prove the existences of solutions for (GVQEP1) and (GVQEP2).

Theorem 3.1. *Suppose that the following conditions hold:*

(i) *$W(\cdot) = Z \backslash intC(\cdot)$ is upper semicontinuous on X and $intC(\cdot)$ has a selection $e(\cdot)$;*

(ii) *$K : E \to 2^E$ is a continuous mapping with compact and convex values on E;*

(iii) *$F : E \times E \to 2^Z$ is an upper semi-continuous mapping with compact values on $E \times E$;*

(iv) *For any $x \in E$, $\quad F(x,x) \not\subseteq -intC(x)$;*

(v) *For every fixed $x \in E$, $F(x, \cdot)$ is $C(x)$-convex.*

Then, there exists an $x^* \in E$ such that

$$F(x^*, y) \not\subseteq -intC(x^*), \ \forall y \in K(x^*). \tag{3.1}$$

Proof. Suppose that

$$\phi(x,y) = \max \bigcup_{z \in F(x,y)} \xi_e(x,z).$$

Firstly, we prove that there is an $x^* \in E$ such that

$$x^* \in K(x^*) \quad \text{and} \quad \phi(x^*, y) \geq 0, \ \forall y \in K(x^*). \tag{3.2}$$

Suppose that the above result is false. Then, for any $x \in E$, we have that $x \notin K(x)$ or there is a point $y \in K(x)$ such that $\phi(x,y) < 0$. When $x \notin K(x)$, it follows from convex set separation theorem that there exists a $p \in X^*$such that

$$\langle p, x \rangle - \min_{y \in K(x)} \langle p, y \rangle < 0.$$

For every $p \in X^*$, set

$$V_p = \left\{ x \in E : \langle p, x \rangle - \min_{y \in K(x)} \langle p, y \rangle < 0 \right\}.$$

Since $K(\cdot)$ is an upper semicontinuous mapping with compact value on E and $\langle p, \cdot\rangle$ is continuous, by Proposition 19 in Section 1 of Chapter 3 [2], $-\min_{y\in K(x)}\langle p, y\rangle$ is upper semicontinuous in variable x. Then, V_p is open in E for all $p \in X^*$. Set

$$V_{p_0} = \left\{x \in E : \min_{y\in K(x)} \phi(x, y) < 0\right\}$$

Since $F(\cdot, \cdot)$ is the upper semicontinuous on $E \times E$, it follows from Theorem 2.3 (i) and Proposition 19 in Section 1 of Chapter 3 [2] that $\phi(\cdot, \cdot)$ is upper semicontinuous on $E \times E$. From the lower semicontinuity of $K(\cdot)$ on E, we have that $\min_{y\in K(x)} \phi(x, y)$ is upper semicontinuous on E. Thus, V_{p_0} is open in E. Obviously, we have

$$E = V_{p_0} \bigcup_{p\in X^*} V_p.$$

By the compactness of E, there exists a finite set $\{p_1, \cdots, p_n\} \subset X^*$ such that

$$E = V_{p_0} \bigcup_{i=1}^{n} V_{p_i}.$$

Then, there exists a continuous partition of unity subordinated to $\{V_{p_0}, V_{p_i} : i = 1, \cdots, p\}$, i.e., there is a family of continuous functions $\{\beta_0(x), \beta_1(x), \cdots, \beta_n(x)\}$ such that, for any $x \in E$,

$$\beta_j(x) \geq 0, j = 0, 1, \cdots, n, \quad \sum_{i=0}^{n} \beta_i(x) = 1,$$

and for each $x \notin V_{p_i}$ $(0 \leq i \leq n)$, $\beta_i(x) = 0$.

Define $\psi : E \times E \to \mathcal{R}$ by

$$\psi(x, y) = \beta_0(x)\phi(x, y) + \sum_{i=1}^{n} \beta_i(x)\langle p_i, x - y\rangle.$$

Now we prove that $\psi(x, y)$ satisfies all conditions of Theorem 2.5.

(i) For any $x \in E$, $\psi(x, x) \geq 0$.

In fact, it follows from assumption (iv) and Proposition 2.3 (iii) in [5] that (i) holds.

(ii) For any $x \in E$, $\psi(x, \cdot)$ is quasiconvex on E.

Since $\langle p_i, x-y\rangle, i = 1, \cdots, n$ is linear function, it is necessary to prove only that $\phi(x, \cdot)$ is convex on E. In fact, since $F(x, y)$ is a compact set for any $y \in E$ and $\xi_e(x, \cdot)$ is continuous on E, $\xi_e(x, F(x, y))$ is a compact set for any $y \in E$. Suppose that $y_1, y_2 \in E$ and $\lambda \in (0, 1)$. Then, there exists a $z \in F(x, \lambda y_1 + (1 - \lambda)y_2)$ such that

$$\xi_e(x, z) = \phi(x, \lambda y_1 + (1 - \lambda)y_2).$$

From the $C(x)$-convexity of $F(x, \cdot)$, there exist $z_1 \in F(x, y_1), z_2 \in F(x, y_2)$ and $c \in C(x)$ such that

$$z = \lambda z_1 + (1 - \lambda)z_2 - c.$$

It follows from Propositions 2.4 and 2.5 in [5] that

$$\begin{aligned} \phi(x, \lambda y_1 + (1 - \lambda)y_2) = \xi_e(x, z) &\leq \lambda\xi_e(x, z_1) + (1 - \lambda)\xi_e(x, y_2) \\ &\leq \lambda\phi(x, y_1) + (1 - \lambda)\phi(x, y_2). \end{aligned}$$

Thus, $\phi(x, \cdot)$ is convex.

(iii) For any $y \in E$, $\psi(\cdot, y)$ is upper semicontinuous on E.

Since $\langle p_i, x - y\rangle, i = 1, \cdots, n$ is linear continuous function, $\beta_i(x), i = 0, 1, \cdots, n$ is continuous and $\phi(\cdot, \cdot)$ is upper semcontinuous on $E \times E$, $\psi(\cdot, y)$ is upper semicontinuous on E.

Thus, by Theorem 2.5, there exists $\bar{x} \in E$ such that

$$\psi(\bar{x}, y) \geq 0, \ \forall y \in E. \tag{3.3}$$

We shall show that (3.3) leads to a contradiction.

If $\bar{x} \in V_{p_0} \backslash (\bigcup_{i=1}^{n} V_{p_i})$, then $\beta_0(\bar{x}) = 1, \beta_i(\bar{x}) = 0, i = 1, \cdots, n$, and there is a $y \in K(\bar{x})$ with $\phi(\bar{x}, y) < 0$. Then one gets

$$\psi(\bar{x}, y) = \phi(\bar{x}, y) < 0.$$

If $\bar{x} \in V_{p_i} \backslash V_{p_0}$ for some i $(1 \leq i \leq n)$, then $\beta_0(\bar{x}) = 0$ and

$$\langle p, x\rangle - \min_{y \in K(x)} \langle p, y\rangle < 0.$$

So,

$$\psi(\bar{x}, y) < 0, \ \forall y \in K(\bar{x}).$$

If $\bar{x} \in V_{p_i} \bigcap V_{p_0}$ for some $i (1 \leq i \leq n)$, then either $\beta_0(\bar{x}) > 0$ or there is $i_0 (1 \leq i_0 \leq n)$ with $\beta_{i_0}(\bar{x}) > 0$. Then, there exists $y \in K(\bar{x})$ with $\psi(\bar{x}, y) < 0$.

In a word, for the $\bar{x}$ satisfying (3.3), there is a $y \in K(\bar{x})$ with $\psi(\bar{x}, y) < 0$, which contradicts (3.3). Therefore, (3.2) holds, that is

$$x^* \in K(x^*) \ \text{ and } \ \max \xi_e(x^*, F(x^*, y)) \geq 0, \ \forall y \in K(x^*).$$

By the compactness of $F(x^*, y)$, there exists a $z \in F(x^*, y)$ such that

$$\xi_e(x^*, z) \geq 0.$$

By Proposition 2.3(iii) in [5], we have

$$z \notin -\text{int}C(x_0).$$

Thus, (3.1) holds and this completes the proof. □

Now we prove the existence of a solution for (GVQEP2).

Theorem 3.2. *Suppose that the following conditions hold:*

(i) $C : X \to 2^Z$ *is upper semi-continuous on* X *and* $intC(\cdot)$ *has a selection* $e(\cdot)$*;*

(ii) $K : E \to 2^E$ *is a continuous set-valued mapping with compact and convex values on E;*

(iii) $F : E \times E \to 2^Z$ *is a lower semicontinuous mapping with compact values on* $E \times E$*;*

(iv) *For any* $x \in E$, $F(x, x) \subset -C(x)$*;*

(v) *For every fixed* $x \in E$, $\max \xi_e(x, F(x, \cdot))$ *is quasiconcave.*

Then, there exists an $x^* \in E$ such that

$$x* \in K(x^*) \quad and \quad F(x^*, y) \subseteq -C(x^*), \ \forall y \in K(x^*). \tag{3.4}$$

Proof. Suppose

$$\varphi(x, y) = -\max \bigcup_{z \in F(x,y)} \xi_e(x, z).$$

Following the proof of Theorem 3.1, we have that there is an $x^* \in E$ such that

$$x^* \in K(x^*) \quad \text{and} \quad \varphi(x^*, y) \geq 0, \ \forall y \in K(x^*),$$

i.e.,

$$\xi_e(x^*, z) \leq 0, \ \forall y \in K(x^*) \text{ and } z \in F(x^*, y).$$

Then, by Proposition 2.3 (ii) in [5], we have

$$x^* \in K(x^*) \quad \text{and} \quad z \in -C(x^*), \ \forall y \in K(x^*) \text{ and } z \in F(x^*, y).$$

Thus, (3.4) holds and this completes the proof. □

Remark 3.3. If, for each fixed $x \in E$, $F(x, \cdot)$ is $C(x)$-properly quasiconvex [12] on E, then, $\max \xi_e(x, F(x, \cdot))$ is quasiconcave on E. However, the converse relation may not hold.

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Manuscript received 1 December 2005
revised 5 June 2006
accepted for publication 12 June 2006

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