Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 3, No. 2, pp. 245 – 255, May 2007

DUALITY IN SET OPTIMIZATION WITH SET-VALUED MAPS

Elvira Hernández*† and Luis Rodríguez-Marín

Abstract: A general optimization problem with set-valued maps is studied. The solution is taken in terms of the set optimization criterion due to D. Kuroiwa [11]. We introduce a dual problem by means a generalized Lagrangian, obtain weak and strong duality and establish a relation between the primal and the dual problem. Finally, a necessary optimality condition for minimality in set optimization is given through a multiplier rule of linear type.

Key words: *set-valued maps, set optimization, duality theory*

Mathematics Subject Classification: *90C29, 90C46, 49N15*

---

1 Introduction

In economics and optimization theory, to name a few, the set-valued maps receive great attention from more and more authors.

Consider the following set-valued optimization problem (P):

$$\text{Optimize } F(x) \quad \text{subject to} \quad x \in X$$

where X is a nonempty set, Y is a topological linear space ordered by a convex cone K and $F\colon X \longrightarrow 2^Y$ is a set-valued map.

We consider the criterion of solution called set optimization which is based on a set relation, see [1, 11, 12, 13]. This criterion is very new and has not been fully explored.

We point out that there exists another possible criterion of solution for problem (P). We recall that this criterion called vector criterion is the most known and investigated and consists in looking for efficient elements of the set $F(X) = \bigcup_{x\in X} F(x)$. See for example [5, 10, 15, 16].

Therefore, solving the set-valued optimization problem (P) by means the vector criterion is equivalent to a rather simple problem: find the solutions of the following vector problem

$$\text{Optimize } \Pi_Y(x,y) \quad \text{subject to} \quad (x,y) \in \text{Graph}(F)$$

where $\text{Graph}(F) = \{(x,y) \in X \times Y : x \in X,\, y \in F(x)\}$ and Π_Y is the projection on the second space. The before result is a peculiar characteristic of the vector criterion. Roughly

*This work for the first author was supported in part by Ministerio de Ciencia y Tecnología (Spain) Project MTM2006-02629 and by Junta de Castilla y León (Spain) Project VA027B06.

†Corresponding author.

speaking, the vector criterion establishes that to choose the best football team of a country it is sufficient to choose the one that has the best footballer.

One advantage of the set optimization criterion over the vector criterion is the possibility of considering preference relations on 2^Y. To introduce this criterion of solution, Kuroiwa [11] considered set relations that generalize the ordering defined by K on Y. We emphasize that such set relations are widely used in theoretical computer science, see for example Brink [3]. The first systematic treatment of such set relations in the framework of ordered vector spaces seems to be due to Kuroiwa, Tanaka and Ha [14].

Later, these set relations have been implicitly used to give scalar representations of a vector optimization problem in Truong [20], to obtain fixed point theorems in Dhage [6] and to present existence results for inclusion problems in Heikkilä [9].

On the other hand, it is worth pointing out that the set approach proposed by Kuroiwa for minimizing set-valued maps has been studied by Hamel and Löhne [8] and Ha [7] in order to establish Ekeland variational principles.

In our estimation, the set-valued optimization theory considering the set optimization criterion is a natural extension of vector optimization theory and seems to have the potential to become an important tool for many areas in optimization. In the same direction, Jahn [10] asserts that such set relations open a new and wide field of research and turn out to be promising in set optimization.

To provide an insight into the set optimization criterion we give the following example in economic terms.

In the second half of the 20th century, activity analysis introduced by Koopmans revolutionized traditional production analysis. It postulates the set P of production processes available in a given economy E (for example a firm, a collection of firms, the entire national economy, $\dots$). This set P is called production set. Let us give a briefly explanation of this concept.

Let n be the number of existing goods in E, some of which are only production factors or inputs (for example, labor), some are only products or outputs (for example, jewels) and some can be both (for example, corn). Each production process or activity describes one way of transforming inputs into outputs, that is, a technological relation of the input-output combination of a process of production. It is very common to represent a production process by an ordered n-tuple, $y \in \mathbb{R}^n$, where negative components mean quantities of various inputs absorbed by the activity and positive components mean quantities of various outputs. The production set P is the collection of these n-tuples. In addition, $P \subset \mathbb{R}^n$ must satisfy some general properties for example, P is a closed convex set, $0 \in P$ and $P \cap \mathbb{R}^n_+ \subset \{0\}$. For more details see Takayama [19].

The most important concept in activity analysis is the following:

Given a production set P, a point $y \in P$ is an efficient point of Y if there does not exist $y' \in Y$ such that $y \leq y'$ where $\leq$ denotes the ordering defined by $\mathbb{R}^n_+$ on $\mathbb{R}^n$. In other words, an efficient point is an input-output combination such that no output can be increased without decreasing other outputs or increasing inputs.

Set theory plays an important role in activity analysis. On the other hand, most of the results in activity analysis follow in an arbitrary normed linear space Y ordered by a closed convex cone K.

Now we consider a family $\mathcal{P} = \{P_1, \dots, P_r\}$ of production sets which are qualitatively homogeneous and are associated to a family of economies $\{E_1, \dots, E_r\}$. In $\mathcal{P}$ there are some production sets which, from the activity point of view, are "superior" to others. Thus, it seems natural to consider some criterion of preference on $2^{\mathbb{R}^n}$ that allows to compare two production sets. For example, if P_1, $P_2 \in \mathcal{P}$ then it makes sense to consider that P_1 is preferred to P_2 ($P_2 \prec P_1$) if for each activity $y \in P_1$ there exists an activity $z \in P_2$ such that $z \leq y$ and for each activity $z \in P_2$ there exists an activity $y \in P_1$ such that $z \leq y$. In terms of set relations, the before preference is defined as follows:

$$P_2 \prec P_1 \quad \text{if and only if} \quad P_2 \leq^l P_1 \quad \text{and} \quad P_2 \leq^u P_1$$

where $\leq^l$ and $\leq^u$ are partial orderings which have been used by Kuroiwa to introduce the set optimization criterion. See [11, 10, 14] for the definitions of $\leq^l$ and $\leq^u$.

So, the concept of efficient element of $\mathcal{P}$ associated to the set relation $\prec$ should be considered. Namely, an element $P_i \in \mathcal{P}$ is efficient of $\mathcal{P}$ if there does not exist $P_j \in \mathcal{P}$ such that $P_i \prec P_j$ where $i, j \in \{1, \ldots, r\}$.

In this framework, the investigation of the power set $2^{\mathbb{R}^n}$ plays an important role.

In the present work, our main purpose is to develop a new Lagrangian duality theory for a set-valued optimization problem whose solutions are defined by the set optimization criterion.

This paper is organized as follows: in Section 2 we present several definitions, notations and preliminaries. In Section 3 we introduce a Lagrangian of linear type in the framework of set optimization and derive some duality results. Lastly using generalized Slater constraint qualification a multiplier rule is obtained under convexity assumptions.

2 Notations

Throughout this work, unless otherwise stated, we will assume that Y is a real separated topological linear space ordered by a closed pointed ($K \cap -K = \{0\}$), solid ($\text{int}(K) \neq \emptyset$) convex cone $K \subset Y$. So, we write $y \leq y'$ if and only if $y' - y \in K$ and $y << y'$ if and only if $y' - y \in \text{int}(K)$.

Denote by Y^* the topological dual space of Y and by K^+ the positive dual cone of K, that is, $K^+ = \{\psi \in Y^* \colon \psi(k) \geq 0 \text{ for all } k \in K\}$.

Let A be a nonempty subset of Y. An element $y \in A$ is minimal (resp. weakly minimal) of A and we write $y \in \operatorname{Min} A$ (resp. $y \in \operatorname{WMin} A$) if $(y - K) \cap A = \{y\}$ (resp. $(y - \text{int}(K)) \cap A = \emptyset$). Analogously, an element $y \in A$ is maximal (resp. weakly maximal) of A and we write $y \in \operatorname{Max} A$ (resp. $y \in \operatorname{WMax} A$) if $(y + K) \cap A = \{y\}$ (resp. $(y + \text{int}(K)) \cap A = \emptyset$). Clearly, $\operatorname{Min} A \subset \operatorname{WMin} A$ and $\operatorname{Max} A \subset \operatorname{WMax} A$.

Given a set-valued map $F \colon X \longrightarrow 2^Y$ with $\text{dom}(F) = X$, that is, $X = \{x \in X \colon F(x) \neq \emptyset\}$. If X is a convex set, we say that F is K-convex on X if for all $y, y' \in X$, and for all $\alpha \in (0, 1)$ the following inclusion holds

$$\alpha F(y) + (1 - \alpha) F(y') \subset F(\alpha y + (1 - \alpha) y') + K.$$

In this work we will use the set relation denoted by $\leq^l$ and called lower set relation. This set relation is reflexive and transitive and is defined as follows: if A, B are nonempty subsets of Y, we denote by

$$A \leq^l B \quad \text{if and only if} \quad B \subset A + K.$$

Obviously, the set relation $\sim^l$ defined by $A \sim^l B$ if and only if $A \leq^l B$ and $B \leq^l A$ is an equivalent relation on all nonempty subsets of Y.

According to Jahn [10] the set relation $\leq^l$ has been independently introduced in a modified form by another authors. On the other hand, the lower set relation $\leq^l$ has been used in the framework of vector optimization in [14, 17].

Using this set relation$\leq^l$, Kuroiwa in [11] introduced, in a natural way, the following concept of l-minimal solution associated to problem (P). An element $x_0 \in X$ is an l-minimal solution of (P) if

$$x' \in X,\ F(x') \leq^l F(x_0) \quad \text{imply} \quad F(x_0) \leq^l F(x').$$

Analogously, $x_0 \in X$ is an l-maximal solution of (P) if

$$x' \in X,\ F(x_0) \leq^l F(x') \quad \text{imply} \quad F(x') \leq^l F(x_0).$$

We denote by $\mathcal{F}$ the family of all image sets under F, i.e., $\mathcal{F} = \{F(x) \colon x \in X\}$. If x_0 is an l-minimal (resp. l-maximal) solution then we write $F(x_0) \in l-\operatorname{Min}\mathcal{F}$ (resp. $F(x_0) \in l-\operatorname{Max}\mathcal{F}$).

Remark 2.1. Whenever F is a vector valued map the set relation $\leq^l$ coincides with the ordering defined by K on Y. So, the concept of l-minimal solution and minimal solution are the same.

It is easy to check that if $x' \in X$ and $F(x') \sim^l F(x_0)$ and x_0 is an l-minimal (resp. l-maximal) solution of (P) then x' is also an l-minimal (resp. l-maximal) solution of (P).

We define the following set relation denoted by $<<^l$. If A and B are nonempty subsets of Y, we write

$$A <<^l B \quad \text{if and only if} \quad B \subset A + \operatorname{int}(K).$$

The following lemma will be used in the next section.

Lemma 2.2. *If A, B, and D are nonempty subsets of Y, then the following statements hold:*

(i) *If $A <<^l B$ and $B <<^l A$ then $A \sim^l B$,*

(ii) *If $A <<^l B$ and $B \sim^l D$ then $A <<^l D$,*

(iii) *If $A <<^l B$ and $A \sim^l D$ then $D <<^l B$,*

(iv) *If $A <<^l B$ and $A \sim^l B$ then $B <<^l A$.*

Proof. (i) It is straightforward because $A <<^l B$ implies $A \leq^l B$.

(ii) Suppose that $A <<^l B$ and $B \sim^l D$. Since $B \subset A + \operatorname{int}(K)$ then

$$B + K \subset A + \operatorname{int}(K) + K = A + \operatorname{int}(K).$$

From $B \sim^l D$ we have $D \subset B + K$. Therefore, $D \subset A + \operatorname{int}(K)$, that is, $A <<^l D$.

(iii) Suppose that $A <<^l B$ and $A \sim^l D$. It is easy to see that $A \sim^l D$ is equivalent to $A + K = D + K$. Therefore, if $A \sim^l D$ we have $A + K + \operatorname{int}(K) = D + K + \operatorname{int}(K)$ or equivalently

$$A + \operatorname{int}(K) = D + \operatorname{int}(K).$$

From this and $B \subset A + \operatorname{int}(K)$ we conclude that $B \subset D + \operatorname{int}(K)$, that is, $D <<^l B$.

(iv) It follows from (ii) and (iii). $\square$

We introduce the efficient notions of weak type associated to the above set relation.

Definition 2.3. We say that $x_0 \in X$ is a

(i) weakly l-minimal solution of (P) if

$$x' \in X,\ F(x') <<^l F(x_0) \quad \text{imply} \quad F(x_0) <<^l F(x'),$$

(ii) weakly l-maximal solution of (P) if

$$x' \in X,\ F(x_0) <<^l F(x') \quad \text{imply} \quad F(x') <<^l F(x_0).$$

If $x_0 \in X$ is a weakly l-minimal (resp. weakly l-maximal) solution then we write $F(x_0) \in l-\mathrm{WMin}\,\mathcal{F}$ (resp. $F(x_0) \in l-\mathrm{WMax}\,\mathcal{F}$). In addition, applying Lemma 2.2(iv), we can check that

$$l-\mathrm{Min}\,\mathcal{F} \subset l-\mathrm{WMin}\,\mathcal{F} \quad \text{and} \quad l-\mathrm{Max}\,\mathcal{F} \subset l-\mathrm{WMax}\,\mathcal{F}.$$

So, the above concepts of weak type are well defined.

Similarly, taking into account Lemma 2.2, if $x' \in X$ and $F(x') \sim^l F(x_0)$ and x_0 is a weakly l-minimal (resp. weakly l-maximal) solution of (P) then x' is also a weakly l-minimal (resp. weakly l-maximal) solution of (P).

In the example below we show a set-valued optimization problem considering the set optimization criterion for the set relation $\leq^l$.

Example 2.4. We consider the problem (P) where $X = [1, 2]$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Let $F\colon X \longrightarrow 2^Y$ be a set-valued map defined as follows

$$F(x) = \begin{cases} [(1,0),(2,0)] & \text{if} \quad x = 1 \\ [(2,1),(x,2)] & \text{if} \quad x \in (1,2) \\ [(0,1),(0,2)] & \text{if} \quad x = 2 \end{cases}$$

We can see that $x = 1$ and $x = 2$ are l-minimal and weakly l-minimal solutions of (P). In addition, $x = 2$ is weakly l-maximal solution of (P) but $x = 2$ is not l-maximal solution of (P).

3 Duality in 2^Y

The Lagrangian map in the conventional theory for solving vector optimization problems and set-valued optimization problems using the vector criterion is a linear combination of the objective map and constraint maps, see [4, 5, 10, 16, 18]. This type of Lagrangian map has also been investigated by Kuroiwa [11] and [12] in the set relation case.

In this section, we introduce a new generalized Lagrangian map of linear type in the sense of set optimization and derive some duality results.

In the sequel, Y and Z are real separated topological linear spaces, $C \subset Z$ and $K \subset Y$ closed, pointed, solid and convex cones. We consider $F\colon X \longrightarrow 2^Y$ and $G\colon X \longrightarrow 2^Z$ set-valued maps with domain X and the following set optimization problem

$$(SOP) \qquad \begin{cases} l-\mathrm{WMin}\,\{F(x)\} \\ \text{subject to } x \in \Omega = \{x \in X \colon G(x) \cap -C \neq \emptyset\}. \end{cases}$$

Solving this problem means to find the family of sets

$$l-\mathrm{WMin}\,\{F(x) \colon x \in \Omega\}.$$

Denote by $\mathcal{L}(Z, Y)$ the set of all continuous linear maps from Z to Y and

$$\mathcal{L}_+(Z, Y) = \{h \in \mathcal{L}(Z, Y) \colon h(C) \subset K\}.$$

We consider the Lagrangian map L defined as follows

$$L(x,h) = F(x) + (h \circ G)(x) \quad \text{for} \quad (x,h) \in X \times \mathcal{L}_+(Z,Y) \tag{3.1}$$

where

$$(h \circ G)(x) = \bigcup_{z \in G(x)} h(z) \quad \text{for} \quad x \in X.$$

The dual set-valued map $\Phi \colon \mathcal{L}_+(Z,Y) \longrightarrow 2^Y$ associated to the generalized Lagrangian map L is the following one

$$\Phi(h) = l - \mathrm{WMin}\,\{L(x,h) \colon x \in X\} \quad \text{for} \quad h \in \mathcal{L}_+(Z,Y).$$

Then the dual problem associated to problem (SOP) is defined by

$$(SOP^*) \qquad \begin{cases} l - \mathrm{WMax}\,\{\Phi(h)\} \\ \text{subject to } h \in \mathcal{L}_+(Z,Y). \end{cases}$$

Remark 3.1. We must emphasize that Kuroiwa in [11] and [12] considers a Lagrangian map different from L. Roughly speaking, the image under the Lagrangian map is a sum of a set and a point. However, in (3.1) the image under the Lagrangian map L is a sum of two sets.

We will use the following concepts associated to problems (SOP) and (SOP*).

Definition 3.2. We say that

(a) $h_0 \in \mathcal{L}_+(Z,Y)$ is a weakly l-maximal solution of (SOP*) if there exists $x_0 \in X$ such that $L(x_0,h_0) \in \Phi(h_0)$ and $L(x_0,h_0) \in l - \mathrm{WMax}\,\{\Phi(h) \colon h \in \mathcal{L}_+(Z,Y)\}$.

(b) $(x_0,h_0) \in X \times \mathcal{L}_+(Z,Y)$ is a feasible pair of (SOP*) if $F(x_0) + (h_0 \circ G)(x_0) \in \Phi(h_0)$.

The next theorem is a generalization of the weak duality theorem in terms of set optimization. As an application, we establish a relationship between the primal problem and the dual problem.

Theorem 3.3 (Weak Duality). *Assume that $x_0 \in \Omega$ and and (x',h) is a feasible pair of (SOP*). If $F(x_0) \nsim^l F(x') + (h \circ G)(x')$ then $F(x_0) <<^l F(x') + (h \circ G)(x')$ does not hold.*

Proof. If $F(x_0) <<^l F(x') + (h \circ G)(x')$ then

$$F(x_0) \leq^l F(x') + (h \circ G)(x'). \tag{3.2}$$

On the other hand, since $x_0 \in \Omega$, we have $(h \circ G)(x_0) \cap (-K) \neq \emptyset$. Let $y \in Y$ be such that $y \in (h \circ G)(x_0) \cap (-K)$. We can see that

$$F(x_0) \subset F(x_0) + y + K \subset F(x_0) + (h \circ G)(x_0) + K.$$

Thus,

$$F(x_0) + (h \circ G)(x_0) \leq^l F(x_0). \tag{3.3}$$

By the hypothesis $F(x_0) <<^l F(x') + (h \circ G)(x')$ and the above condition we deduce that

$$F(x_0) + (h \circ G)(x_0) <<^l F(x') + (h \circ G)(x'). \tag{3.4}$$

From this, and taking into account that (x', h) is a feasible pair of (SOP*), we have $F(x') + (h \circ G)(x') \in \Phi(h)$. Thus,

$$F(x') + (h \circ G)(x') <<^l F(x_0) + (h \circ G)(x_0).$$

Therefore, by (3.4) and Lemma 2.2(i),

$$F(x') + (h \circ G)(x') \sim^l F(x_0) + (h \circ G)(x_0).$$

From this and (3.3),

$$F(x') + (h \circ G)(x') \leq^l F(x_0).$$

Hence, by (3.2), we obtain a contradiction with the assumption $F(x_0) \nsim^l F(x') + (h \circ G)(x')$ and the proof is concluded. □

Corollary 3.4. *Assume that $x_0 \in \Omega$ and $(x_1, h_1) \in X \times \mathcal{L}_+(Z, Y)$ is a feasible pair of (SOP*) such that $F(x_0) \sim^l F(x_1) + (h_1 \circ G)(x_1)$. Then the following statements hold:*

(i) *x_0 is a weakly l-minimal solution of (SOP) and*

(ii) *h_1 is a weakly l-maximal solution of (SOP*).*

Proof. Let us see (i). Suppose that there exists $x' \in \Omega$ such that $F(x') <<^l F(x_0)$. Since $F(x_0) \sim^l F(x_1) + (h_1 \circ G)(x_1)$, applying Lemma 2.2(ii), we deduce that

$$F(x') <<^l F(x_1) + (h_1 \circ G)(x_1).$$

Thus, according to Theorem 3.3, we deduce that

$$F(x') \sim^l F(x_1) + (h_1 \circ G)(x_1).$$

So, by the assumption $F(x_0) \sim^l F(x_1)+(h_1 \circ G)(x_1)$ we have $F(x') \sim^l F(x_0)$ and by Lemma 2.2(iv) we conclude that x_0 is a weakly l-minimal solution of (SOP).

To prove (ii), it is sufficient to see that $F(x_1) + (h_1 \circ G)(x_1) \in l-\mathrm{WMax}\,\{\Phi(h) \colon h \in \mathcal{L}_+(Z, Y)\}$ since (x_1, h_1) is a feasible pair of (SOP*).
Suppose that there exists a feasible pair $(x', h') \in X \times \mathcal{L}_+(Z, Y)$ such that

$$F(x_1) + (h_1 \circ G)(x_1) <<^l F(x') + (h' \circ G)(x').$$

From this and $F(x_0) \sim^l F(x_1) + (h_1 \circ G)(x_1)$, according to Lemma 2.2(iii), we obtain

$$F(x_0) <<^l F(x') + (h' \circ G)(x').$$

Again, by Theorem 3.3, we deduce that $F(x_0) \sim^l F(x') + (h' \circ G)(x')$. Consequently,

$$F(x_1) + (h_1 \circ G)(x_1) \sim^l F(x') + (h' \circ G)(x')$$

and by Lemma 2.2(iv) the proof is concluded. □

We note that Theorem 3.3 can be considered an extension of Theorem 3 in Corley [4] and Theorem 4.2 in Corley [5].

Theorem 3.5 (Strong Duality). *Let $x_0 \in \Omega$ be a weakly l-minimal solution of problem (SOP). Assume that there exists $h_0 \in \mathcal{L}_+(Z,Y)$ such that (x_0,h_0) is a feasible pair of (SOP*) and $(h_0 \circ G)(x_0) \subset K$. Then h_0 is a weakly l-maximal solution of (SOP*).*

Proof. Suppose that there exists a feasible pair (x_1,h_1) of (SOP*) such that

$$F(x_0) + (h_0 \circ G)(x_0) <<^l F(x_1) + (h_1 \circ G)(x_1).$$

Since $(h_0 \circ G)(x_0) \subset K$ then

$$F(x_0) <<^l F(x_1) + (h_1 \circ G)(x_1).$$

By Theorem 3.3, we obtain

$$F(x_0) \sim^l F(x_1) + (h_1 \circ G)(x_1). \tag{3.5}$$

On the other hand, since $x_0 \in \Omega$, $h_0 \in \mathcal{L}_+(Z,Y)$, $(h_0 \circ G)(x_0) \subset K$ and K is pointed we deduce that $0 \in (h_0 \circ G)(x_0)$.

From this, $(h_0 \circ G)(x_0) \subset K$ and (3.5) we deduce that

$$F(x_0) + (h_0 \circ G)(x_0) \sim^l F(x_1) + (h_1 \circ G)(x_1).$$

Consequently, by Lemma 2.2(iv), h_0 is a weakly l-maximal solution of (SOP*). □

4 A Lagrange Multiplier Rule

In this section, we show that a weakly l-minimal solution x_0 of problem (SOP) is exactly a weakly l-minimal solution of an unconstrained set optimization problem under some convexity assumption.

In order to obtain a Lagrange multiplier rule we will need the following generalized Slater constraint qualification.

Definition 4.1. We say that the set optimization problem (SOP) satisfies the generalized Slater constraint qualification if there exists $x \in X$ such that $G(x) \cap -\operatorname{int}(C) \neq \emptyset$.

Theorem 4.2. *Consider problem (SOP). Assume X is a convex set, $(F,G)\colon X \longrightarrow 2^{Y\times Z}$ is $K \times C$-convex on X and (SOP) satisfies the generalized Slater constraint qualification. If $x_0 \in \Omega$ is a weakly l-minimal solution of (SOP) such that $F(x_0) \subset y_0 + K$ for some $y_0 \in F(x_0)$. Then there exists $h \in \mathcal{L}_+(Z,Y)$ such that such that $0 \in (h \circ G)(x_0)$ and x_0 is a weakly l-minimal solution of the following unconstrained problem:*

$$\begin{cases} l - \mathrm{WMin}\,\{F(x) + (h \circ G)(x)\} \\ \textit{subject to } x \in X. \end{cases}$$

Proof. Let $Q\colon X \longrightarrow 2^{Y\times Z}$ be the set-valued map defined by $Q(x) = (F(x), G(x))$. Then $Q(X) + K \times C$ is convex since $(F,G)\colon X \longrightarrow 2^{Y\times Z}$ is $K \times C$-convex on X. Furthermore,

$$(Q(X) + K \times C) \cap \operatorname{int}((y_0 - K) \times -C) = \emptyset.$$

Indeed, if there exists $x \in X$ such that $(y,z) \in (Q(x) + K \times C) \cap \operatorname{int}((y_0 - K) \times -C)$ then $y \in (F(x) + K) \cap (y_0 - \operatorname{int}(K))$ and $z \in (G(x) + C) \cap -\operatorname{int}(C)$. Therefore $(F(x) + K) \cap (y_0 - \operatorname{int}(K)) \neq \emptyset$ and $G(x) \cap -C \neq \emptyset$ or equivalently

$$y_0 \in F(x) + \operatorname{int}(K) \tag{4.1}$$

and $x \in \Omega$. Since $F(x_0) \subset y_0 + K$ by (4.1) we have $F(x) <<^l F(x_0)$ with $x \in \Omega$. On the other hand, as x_0 is weakly l-minimal solution of (SOP) we have $F(x_0) <<^l F(x)$. Consequently,

$$F(x) \subset F(x_0) + \operatorname{int}(K) \subset y_0 + \operatorname{int}(K)$$

which contradicts (4.1).
Therefore, since $(Q(X) + K \times C)$ and $(y_0 - K) \times -C$ are convex, applying a standard separation result (see [2, Theorem 1.14]), we get a pair $(\varphi, \psi) \in Y^* \times Z^* \backslash \{(0,0)\}$ verifying

$$\varphi(\bar{y}) + \psi(\bar{z}) \geq \varphi(y_0 - k) + \psi(-c) \text{ for } (\bar{y}, \bar{z}) \in Q(X) + K \times C,\ (k, c) \in K \times C. \tag{4.2}$$

In particular, if $c = 0$ and $k = 0$ we obtain that

$$\varphi(\bar{y}) + \psi(\bar{z}) \geq \varphi(y_0) \quad \text{for} \quad (\bar{y}, \bar{z}) \in Q(X). \tag{4.3}$$

and if $y = y_0$ we have

$$\psi(z) \geq 0 \quad \text{for} \quad z \in G(x_0). \tag{4.4}$$

From (4.2), we can easily see that $(\varphi, \psi) \in K^+ \times C^+$. In addition, we note that $\varphi \neq 0$. Otherwise, if $\varphi = 0$ then by condition (4.3), it results

$$\psi(z) \geq 0 \quad \text{for } z \in G(x) \quad \text{and} \quad x \in X. \tag{4.5}$$

As a consequence of the generalized Slater constraint qualification, there exists $x \in X$ such that $z \in G(x) \cap - \operatorname{int}(C)$ and, as $\psi \in C^+$, we obtain a contradiction with (4.5).

Since $\varphi \neq 0$ and $\varphi \in K^+$, we can choose $k_0 \in \operatorname{int}(K)$ such that $\varphi(k_0) = 1$. We define the linear map $h \colon Z \longrightarrow Y$ by

$$h(\cdot) = k_0 \psi(\cdot).$$

Due to $\psi \in C^+$ then $h \in \mathcal{L}_+(Z, Y)$ and by (4.4) we obtain that

$$h(z) = 0 \quad \text{for} \quad z \in G(x_0) \cap -C.$$

Thus, as $x_0 \in \Omega$, we deduce that $0 \in (h \circ G)(x_0)$.

Let us see that x_0 is a weakly l-minimal solution of the unconstrained problem for $h(\cdot) = k_0 \psi(\cdot)$. Suppose that $F(x_0) + (h \circ G)(x_0) \notin l-\operatorname{WMin}\{F(x) + (h \circ G)(x) \colon x \in X\}$ then there exists $x' \in X$ such that

$$F(x') + (h \circ G)(x') <<^l F(x_0) + (h \circ G)(x_0)$$

i.e.,

$$F(x_0) + (h \circ G)(x_0) \subset F(x') + (h \circ G)(x') + \operatorname{int}(K). \tag{4.6}$$

Let $-c_0$ be an element of $G(x_0) \cap (-C)$ which exists because of $x_0 \in \Omega$, then $y_0 + k_0 \psi(-c_0) \in F(x_0) + (h \circ G)(x_0)$.

On the other hand, by (4.6), there exist $y' \in F(x')$, $z' \in G(x')$ and $k' \in \operatorname{int}(K)$ such that

$$y_0 + k_0 \psi(-c_0) = y' + k_0 \psi(z') + k'.$$

Therefore,

$$\varphi(y_0) + \varphi(k_0)\psi(-c_0) = \varphi(y') + \varphi(k_0)\psi(z') + \varphi(k')$$

and as $\varphi(k_0) = 1$ we deduce

$$\varphi(y_0) + \psi(-c_0) = \varphi(y') + \psi(z') + \varphi(k').$$

We rewrite the above expression as follows

$$\varphi(y_0) + \psi(-c_0) - \varphi(y') - \psi(z') = \varphi(k').$$

Thus,

$$\varphi(y_0) + \psi(-c_0) - \varphi(y') - \psi(z') > 0$$

which contradicts (4.2). □

Once again the above result extends Theorem 4.1 in Corley [5] and Corollary 2 in Corley [4] which are given in the framework of vector optimization.

In conclusion, we have introduced a new dual problem associated to problem (SOP) and as an application we have extended several vector results. We point out that further research on saddle points associated to the Lagrangian map L would be desirable.

References

[1] M. Alonso, L. Rodríguez-Marín, Set-relations and optimality conditions in set-valued maps, *Nonlinear Anal.* 63 (2005) 1167-1179.

[2] V. Barbu, Th. Precupanu, *Convexity and Optimization in Banach spaces*, Second edition. Mathematics and its Applications, 10. D. Reidel Publishing Co., Bucharest, 1986.

[3] C. Brink, Power structures, *Algebra Universalis*, 30 (1993) 177–216.

[4] H.W. Corley, Duality theory for maximizations with respect to cones, *J. Math. Anal. Appl.* 84 (1981) 560–568.

[5] H.W. Corley, Existence and Lagrangian duality for maximizations of set-valued functions, *J. Optim. Theory Appl.* 54 (1997) 489–501.

[6] B.C. Dhage, A functional integral inclusion involving Caratheodories, Electron. *J. Qual. Theory Differ. Equ.* 14 (2003) 1–18.

[7] T.X.D. Ha, Some variant of the Ekeland variational principle for a set-valued map, *J. Optim. Theory Appl.* 124 (2005) 187–206.

[8] A. Hamel, A. Löhne, Minimal set theorems, Report of the Institute of Optimization and Stochastics, Vol. 11, 2002.

[9] S. Heikkilä, Inclusion problems in ordered topological vector spaces and applications, *J. Math. Anal. Appl.* 298 (2004) 94–105.

[10] J. Jahn, *Vector Optimization. Theory, Applications, and Extensions*, Springer-Verlag, Berlin, 2004.

[11] D. Kuroiwa, Some duality theorems of set-valued optimization with natural criteria in *Nonlinear analysis and convex analysis (Niigata, 1998)* W. Takahashi and T. Tanaka (eds.), World Sci. Publishing, River Edge, NJ, 1999, pp. 221–228.

[12] D. Kuroiwa, On set-valued optimization, *Nonlinear Anal.* 47 (2001) 1395–1400.

[13] D. Kuroiwa, Existence of efficient points of set optimization with weighted criteria, *J. Nonlinear Convex Anal.* 4 (2003) 117–123.

[14] D. Kuroiwa, T. Tanaka, T.X. Ha, On cone convexity of set-valued maps, *Nonlinear Anal.* 30 (1997) 1487–1496.

[15] Z.F. Li, G.Y. Chen, Lagrangian multipliers, saddle points, and duality in vector optimization of set-valued maps, *J. Math. Anal. Appl.* 215 (1997) 297–316.

[16] D.T. Luc, *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems, Springer Verlag, Berlin, 319, 1989.

[17] S. Nishizawa, M. Onodsuka, T. Tanaka, Alternative theorems for set-valued maps based on a nonlinear scalarization, *Pacific J. Optim.* 1 (2005) 147–159.

[18] Y. Sawaragi, H. Nakayama, T. Tanino, *Theory of Multiobjective Optimization. Mathematics in Science and Engineering*, 176. Academic Press, Inc., Orlando, FL, 1985

[19] A. Takayama, *Mathematical Economics*, Second edition. Cambridge University Press, Cambridge, 1985.

[20] X.D.H. Truong, Cones admitting strictly positive functionals and scalarization of some vector optimization problems, *J. Optim. Theory Appl.* 93 (1997) 355–372.

Manuscript received 23 December 2005
revised 17 May 2006
accepted for publication 15 June 2006

Elvira Hernández
Dept. Matemática Aplicada, UNED, E.T.S.I. Industriales
Juan del Rosal 12, 28.040 Madrid, Spain
E-mail address: ehernandez@ind.uned.es

Luis Rodríguez-Marín
Dept. Matemática Aplicada, UNED, E.T.S.I. Industriales
Juan del Rosal 12, 28.040 Madrid, Spain
E-mail address: lromarin@ind.uned.es