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THE EOQ MODELS UNDER BOTH OPTIONS OF CASH DISCOUNT AND DELAY IN PAYMENT

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Abstract: In this paper, we propose a new EOQ model considering both options of cash discount and delay in payment and allowing two-stage payments. In business transactions, the retailer may obtain cash discount when the payment is done before cash discount due date designated by the supplier. On the other hand, the supplier may offer a fixed credit period to encourage the retailer's demand since the retailer may keep accumulating revenue and earning interest during the credit period before making its payment. The studies in the literature allow only one between either of the two options. Considering both options and allowing two-stage payments, this study proposes a new EOQ model and two effective algorithms that assist the retailer to obtain the optimal replenishment and payment strategy. Our numerical experiments show that the proposed EOQ model with two-stage payment strategy could be superior to the single-stage ones in the literature.

Key words: *EOQ, cash discount, permissible delay in payments*

Mathematics Subject Classification: *90B05, 90B50, 90C26, 90C46, 90C90*

1 Introduction and Literature Review

The traditional economic order quantity (EOQ) assumes that the retailer must pay for the item as soon as the items are received. But, there are usually two typical payments for the good in the real world. Generally, a supplier often offers his retailer some trade credit periods. One is "permissible delay period". If the retailer settles the payments within this permissible delay period, then the retailer will not be charged any interest. But if the payment is not paid in full by the end of this permissible delay period, then the interest of unpaid payment will be charged. Within this permissible delay period, the retailer may accumulate the money obtained from selling the goods, and depot it in the bank for obtaining the interest; hence the permissible delay in payments reduces the cost of the retailer. On the other hand, the suppliers may offer a cash discount rate to encourage the retailer to pay for its purchase before a given period. This period is called as "the period of cash discount". The permissible delay period is usually larger than the period of cash discount in practice.

Researchers had addressed their efforts to study the strategies based on "permissible delay period" and "the period of cash discount". They derived the EOQ models for the retailers under the conditions the supplier offers permissible delay in payment and/or cash discount, respectively.

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Before this study, Goyal [7] first developed the EOQ models under conditions of permissible delay in payments. Chand & Ward [2] added the concept of net present value (NPV) to classical EOQ models and compares their results with ones of [7]. Chu et al. [4] augmented the constraints of "the interest charged is greater than the interest earned" to Goyal's [7] model. He proved that the average total cost function is convex and proposed a simplified search scheme for solving the problem. Based on Goyal's [7] model, Aggarwal & Jaggi [1] developed the EOQ models of the deteriorating items under condition of permissible delay in payments. They proved that the objective function is piecewise-convex and showed some theoretical results. Later, Jamal et al. [12] further generalized Aggarwal & Jaggi's [1] model by allowing shortages. Chung [6] presented a theorem that determines the replenishment quantity under condition of permissible delay in payments. Huang [9] assumed that the supplier offers the retailer partially permissible delay in payments when the order quantity is smaller than a predetermined quantity. Huang & Hsu [11] investigated an extended case the retailer is offered with two levels of trade credit.

Ouyang et al. [13] added the conditions of cash discount to the models of Goyal [7] and developed inventory models under condition of permissible delay in payments and cash discount *simultaneously*. In their model, the retailers may simply compare these two models and pick the favorite one. Chang [3] extended the scenario by adding the constraint: the selling price is greater than purchasing price to the model of Ouyang et al. [13]. Huang & Chung [10] derived an EOQ model under the conditions of cash discount and trade credits. We may consider their model using so called "Single-Stage Payment Strategy (SSPS)" since the retailer makes its payment at only one stage but considering taking advantage of cash discount and trade credits. Note that they did not take into account the option of "permissible delay in payment" in their decision-making scenario. The theoretical analysis in their study could be simpler since they consider less number of possible cases and the derivation of the corresponding cost functions are more straightforward. However, since the decision-maker in the scenario of Huang & Chung [10] has no option of "permissible delay in payment", the proposed Two-Stage Payment Strategy in this study could obtain better solution than their optimal solution. (Please refer to the numerical experiments in Section 5.) Moreover, Huang [8] simplified the search of the optimal values of models in Chang [3] and offered an efficient algorithm.

In this study, we are interested in another possible practice (which is a kind of "hybrid" strategy, and will be called "Two-Stage Payment Strategy" later) in the real world, namely, the supplier allows some partial payment is paid within the period of cash discount and the remained unpaid payment is disbursed before the permissible delay period. We consider that the Two-Stage Payment Strategy may lead to a lower cost than the other two strategies in the literature. We are motivated to investigate the problem by formulating the corresponding mathematical model, gaining more insights through our theoretical analysis, and hopefully, to propose an efficient solution approach and the optimal strategy for the retailers based on our theoretical analysis.

The rest of the paper is organized as follows. Section 2 introduces the notations, the assumptions and the mathematical models under the Two-Stage Payment Strategy. We prove the convexity and monotonicity properties of all models in Section 3. These properties assure the existence of an optimal solution for all the models. Then, we conduct theoretical analysis and also present the proposed solution approach based on our theoretical results in Section 4. In Section 5, we first solve six examples in Huang & Chung [10] using the proposed solution approach under the Two-Stage Payment Strategy. Also, we conduct sensitivity analysis for obtaining more managerial insights as the second part of Section 5. Finally, Section 6 addresses our concluding remarks.

2 The Mathematical Models

Following the practice in real world, suppliers often offer a so-call "1/10, n/30" rule to retailers. We describe this rule as follows. The supplier would offer the retailer a 1%-cash discount, if the retailer is willing to pay for the purchased items within a period of 10 days. Otherwise, the retailer must pay in full before a permissible delay period of 30 days.

Before this study, Ouyang et al. [13] found the preferable one between the following two cases exclusively.

1. EOQ under conditions of cash discount.
2. EOQ under conditions of permissible delay in payments.

We also call them "Single-Stage Payment Strategy (SSPS)" since the retailer makes its payment at only one stage in either case. One may refer to Ouyang et al. [13] for the details of their formulation.

In this study, we are interested in another practice, namely, a "Two-Stage Payment Strategy (TSPS)", in which the retailer is able to take the advantage of cash discount for the first part of its payment, and pay the rest within 30 days. We also call the concerned problem as the "EOQ under both options of permissible delay in payments and cash discount."

Before presenting the proposed mathematical model, we first introduce the notation for the model formulation as follows.

D**:** the demand rate(unit/years)

h**:** the stock holding cost per year excluding interest charges(dollars/unit)

s**:** the selling price (dollars/unit)

p**:** the purchasing price (dollars/unit)

A**:** the ordering cost (dollars/order)

I_c**:** the interest charged in the stocks (dollars/year)

I_d**:** the interest earned (dollars/year)

M_1**:** the period of cash discount (years)

M_2**:** the permissible delay period (years)

T**:** the replenishment cycle time (years)

$ACT(T)$**:** the annual average total relevant cost (dollars/year)

$EOQ(T)$**:** the annual average total relevant cost of the traditional EOQ model, i.e. the sum of average holding cost and setup cost where $EOQ(T) = A/T + hDT/2$.

We define $ATC(T)$ as the average total cost for the decision maker (i.e., the retailer) as follows

$$ATC(T) = EOQ(T) - ID(T) - IR(T) + IC(T) \tag{2.1}$$

where $ID(T)$ indicates the earned discount, $IR(T)$ is the interest earned, and $IC(T)$ is the interest charged. The expressions of $ID(T)$, $IR(T)$ and $IC(T)$ may vary with different cases.

We also employ the following assumptions for our model formulation.

Table 1: The Payment Mechanism of the Six Cases

Cases	Payment at M_1	Payment at M_2	Other Payment time and amount
Case 1.1	pDM_1	$pD(T - M_1)$	No
Case 1.2	pDM_1	$pD(M_2 - M_1)$	Pay as sold
Case 2.1	pDT	No	No
Case 2.2	pDT	No	No
Case 2.3	No	pDT	No
Case 2.4	No	pDM_2	Pay as sold

1. The demand rate is known and constant.
2. Shortage is not allowed.
3. Replenishments are instantaneous.
4. The unit selling price is larger than the unit purchasing price.
5. The supplier offers a cash discount if it makes the payment with the discount rate of r within the cash-discount period M_1 (and enjoys the earned discount $ID(T)$ in the objective function). Otherwise, it shall pay the full payment within the permissible delay period M_2.
6. The permissible delay period M_2 is longer than the period M_1 of cash discount, i.e. $M_1 < M_2$.
7. The retailer deposits the sales revenue generated from the sold items in an interest-bearing account, and it receives interest income, corresponding to the term of $IR(T)$ in the objective function, from the deposited revenue before making the payment at M_2.
8. The supplier charges interest for the delayed payment beyond the permissible delay dead line (M_2), and the interest charged is the term $IC(T)$ in the objective function.
9. The interest earned is no larger than the interest charged, namely, $0 < I_d < I_c$.
10. We consider no extra financial options (e.g., investing additional cash) is allowed in the decision-making scenario.
11. The planning horizon is infinite.

First, we present the mathematical models for the Two-Stage Payment Strategy (TSPS), i.e., the EOQ under both options of permissible delay in payments and cash discount. There are two possible cases in TSPS, namely, Case 1.1: $T \in [M_1, M_2]$ and Case 1.2: $T \geq M_2$. We would formulate a mathematical model for the corresponding cases as follows.

Case 1.1: Given $T \in [M_1, M_2]$, the retailer made the payment before the permissible delay period M_2. Therefore, the interest charged is zero, i.e., $IC(T) = 0$. The retailer would make the first part of its payment at M_1 and earn the cash discount by $ID(T) = \dfrac{rpDM_1}{T}$. Also, it is best for the retailer to hold the second part of its payment until M_2.

In such a case, the interest earned is $IR(T) = sI_dD\left[\frac{M_1(M_1-M_2)}{T} + M_2 - \frac{T}{2}\right]$. In fact, the term $IR(T)$ includes the interest $\frac{sI_dDM_1^2}{T}$ from the unpaid amount of $sD(T-M_1)$ earned before M_1 and the interest earned $\frac{sI_dD(T-M_1)^2}{2T} + \frac{sI_dD(T-M_1)(M_2-T)}{T}$ between M_1 and M_2 in Case 1.1. Therefore, the average total cost function for Case 1.1, denoted as $ATC_{1.1}(T)$, is given as (2.2).

$$ATC_{1.1}(T) = EOQ(T) - \frac{rpDM_1}{T} - sI_dD\left[\frac{M_1(M_1-M_2)}{T} + M_2 - \frac{T}{2}\right] \tag{2.2}$$

Case 1.2: Given $T \geq M_2$. Similar to Case 1.1, it will be the best option for the retailer to make the first part of its payment at M_1 and receive a cash discount of $ID(T) = \frac{rpDM_1}{T}$. The retailer would make the second part of its payment at M_2 to avoid interest charge. Before making this payment, the retailer is able to get an interest income of $\frac{sI_dD[M_1^2+(M_2-M_1)^2]}{2T}$. Besides, since $T \geq M_2$, the retailer has to pay for the interest charge for those items sold after M_2, and the interest charged will be $\frac{pI_cD(T-M_2)^2}{2T}$. Then, the average total cost function for Case 1.2, denoted as $ATC_{1.2}(T)$, may be expressed as (2.3).

$$ATC_{1.2}(T) = EOQ(T) - \frac{rpDM_1}{T} - \frac{sI_dD[M_1^2+(M_2-M_1)^2]}{2T} + \frac{pI_cD(T-M_2)^2}{2T} \tag{2.3}$$

Recall that Ouyang et al. [13], Chang [3] and Huang [8] proposed two categories of Single-Stage Payment Strategies:

1. EOQ models under conditions of cash discount: There are two possible cases in this category as follows.

   Case 2.1: $T \leq M_1$;

   Case 2.2: $M_1 \leq T$;

2. EOQ models under conditions of permissible delay in payments: Two possible cases in this category are:

   Case 2.3: $T \leq M_2$;

   Case 2.4: $M_2 \leq T$.

We denote the average total cost function for Case 2.1, Case 2.2, Case 2.3 and Case 2.4 as $ATC_{2.1}(T)$, $ATC_{2.2}(T)$, $ATC_{2.3}(T)$ and $ATC_{2.4}(T)$ respectively. The expressions for these functions are given as follows.

$$ATC_{2.1}(T) = EOQ(T) - rpD - sI_dD\left(M_1 - \frac{T}{2}\right), \tag{2.4}$$

$$ATC_{2.2}(T) = EOQ(T) - rpD - \frac{sI_dDM_1^2}{2T} + \frac{p(1-r)I_cD(T-M_1)^2}{2T}, \tag{2.5}$$

$$ATC_{2.3}(T) = EOQ(T) - sI_dD\left(M_2 - \frac{T}{2}\right), \tag{2.6}$$

$$ATC_{2.4}(T) = EOQ(T) - \frac{sI_d DM_1^2}{2T} + \frac{pI_c D(T - M_2)^2}{2T} \tag{2.7}$$

The derivation of the function $ATC_{2.1}(T)$ can be referred to the equation (22) of Huang [8]. Moreover, the functions $ATC_{2.2}(T)$, $ATC_{2.3}(T)$ and $ATC_{2.4}(T)$ can be referred to the equation (21), (24) and (23) of Huang [8] respectively.

Obviously, the retailer would make the optimal decision by taking into accounts all the cases in both Single-Stage and Two-stage Payment Strategies. Therefore, we are interested in finding an optimal solution of (2.8) in this study.

$$\min\{ATC_{1.1}(T^*_{1.1}), ATC_{1.2}(T^*_{1.2}), ATC_{2.1}(T^*_{2.1}), ATC_{2.2}(T^*_{2.2}), ATC_{2.3}(T^*_{2.3}), ATC_{2.4}(T^*_{2.4})\} \tag{2.8}$$

where $T^*_{i,j}$ are satisfied

$$ATC_{1.1}(T^*_{1.1}) = \min_{T\in[M_1,M_2]} ATC_{1.1}(T) \tag{2.9}$$

$$ATC_{1.2}(T^*_{1.1}) = \inf_{T\in[M_2,\infty)} ATC_{1.2}(T) \tag{2.10}$$

$$ATC_{2.1}(T^*_{2.1}) = \inf_{T\in(0,M_1]} ATC_{2.1}(T) \tag{2.11}$$

$$ATC_{2.2}(T^*_{2.2}) = \inf_{T\in[M_1,\infty)} ATC_{2.2}(T) \tag{2.12}$$

$$ATC_{2.3}(T^*_{2.3}) = \inf_{T\in(0,M_2]} ATC_{2.3}(T) \tag{2.13}$$

$$ATC_{2.4}(T^*_{2.4}) = \inf_{T\in[M_2,\infty)} ATC_{2.4}(T). \tag{2.14}$$

3 Convexity and Monotonicity Analysis

Here, we would indicate that the cost functions (2.2), (2.3), (2.4), (2.5), (2.6) and (2.7) are convex under certain conditions. Also, we would assert that these cost functions are still non-decreasing functions, even if these conditions are not satisfied. Moreover, our theoretic results facilitate us in obtaining an optimal solution $T^*_{i,j}$ for each case.

Before presenting our theoretical analysis, we define a new symbol Δ as

$$\Delta = rp - sI_d(M_2 - M_1). \tag{3.1}$$

Note that the condition $\Delta > 0$ implies that the interest earned from the generated sales revenue between M_1 and M_2, is less than the profit made from cash discount. It is a crucial condition for deriving our theoretical results. On the other hand, to simplify our conditions in finding the optimal solution for all the cases, we further define k_1, k_2 and k_3 as follows.

$$k_1 = DM_1^2(h + sI_d) \tag{3.2}$$

$$k_2 = DM_2^2(h + sI_d) \tag{3.3}$$

$$k_3 = 2DM_1[rp - sI_d(M_2 - M_1)] \tag{3.4}$$

Next, we will first present our convexity and monotonicity analysis on the Two-Stage Payment Strategy, and do that for the Single-Stage Payment Strategy in the second part.

3.1 The Analysis on the Two-Stage Payment Strategy

Theorem 3.1 asserts the convexity and monotonicity of the function $ATC_{1.1}(T)$ for Case 1.1.

Theorem 3.1. *Let*

$$\lambda_1 = 2\{A - DM_1[rp - sI_d(M_2 - M_1)]\} \tag{3.5}$$

and

$$\overline{T_{1.1}} = \sqrt{\frac{2\{A - DM_1[rp - sI_d(M_2 - M_1)]\}}{D(h + sI_d)}} \tag{3.6}$$

1. *If* $\lambda_1 > 0$*, then* $ATC_{1.1}(T)$ *is convex and an optimal replenishment cycle of* $\min\limits_{T>0} ATC_{1.1}(T)$ *is* $\overline{T_{1.1}}$.
2. *If* $\lambda_1 = 0$*, then* $ATC_{1.1}(T)$ *is constant.*
3. *If* $\lambda_1 < 0$*, then* $ATC_{1.1}(T)$ *is increasing, concave and an optimal replenishment cycle of* $\min\limits_{T\in[M_1,M_2]} ATC_{1.1}(T)$ *is* M_1.

Theorem 3.1 assists to find an optimal solution of Case 1.1, i.e., $T^*_{1.1} = \operatorname*{argmin}\limits_{T\in[M_1,M_2]} ATC_{1.1}(T)$. We note that the fourth item in Corollary 3.2 also plays the same role, but it facilitates us in the organization of the possible situations when deriving the proposed solution approach for solving (2.8).

Corollary 3.2. *Assume* $\Delta > 0$.

1. *If* $2A > k_3$*, then* $\lambda_1 > 0$ *and* $\overline{T_{1.1}}$ *exists. If* $2A \le k_3$*, then* $\lambda_1 < 0$.
2. *If* $2A \ge k_1 + k_3$*, then* $\overline{T_{1.1}} \ge M_1$*. If* $2A \in (k_3, k_1 + k_3)$*, then* $\overline{T_{1.1}} < M_1$.
3. *If* $2A \in (k_3, k_2 + k_3]$*, then* $\overline{T_{1.1}} \le M_2$*. If* $2A > k_2 + k_3$*, then* $\overline{T_{1.1}} > M_2$.
4. *An optimal solution of* $\min\limits_{T\in[M_1,M_2]} ATC_{1.1}(T)$ *is given by*

$$T^*_{1.1} = \begin{cases} M_1 & \text{if } 2A < k_1 + k_3 \\ \overline{T_{1.1}} & \text{if } 2A \in [k_1 + k_3, k_2 + k_3] \\ M_2 & \text{if } 2A > k_2 + k_3 \end{cases}.$$

Though the analysis on the function $ATC_{1.2}(T)$ is more complicated, we may derive Theorem 3.3 and Corollary 3.4 by applying a similar approach.

Theorem 3.3. *Let*

$$\lambda_2 = 2\{A - DM_1[rp - I_d(M_2 - M_1)]\} + DM_2^2(pI_c - sI_d). \tag{3.7}$$

and

$$\overline{T_{1.2}} = \sqrt{\frac{2\{A - DM_1[rp - I_d(M_2 - M_1)]\} + DM_2^2(pI_c - sI_d)}{D(h + pI_c)}} \tag{3.8}$$

1. *If* $\lambda_2 > 0$*, then* $ATC_{1.2}(T)$ *is convex and an optimal replenishment cycle of* $\inf\limits_{T>0} ATC_{1.2}(T)$ *is* $\overline{T_{1.2}}$.

2. *If* $\lambda_2 = 0$*, then* $ATC_{1.2}(T)$ *is constant.*

3. *If* $\lambda_2 < 0$*, then* $ATC_{1.2}(T)$ *is increasing, concave and an optimal replenishment cycle* $\inf_{T\in[M_2,+\infty)} ATC_{1.2}(T)$ *is* M_2*.*

Also, Corollary 3.4 assists to find the optimal replenishment cycle of $\inf_{T\in[M_2,+\infty)} ATC_{1.2}(T)$.

Corollary 3.4. *Assume* $\Delta > 0$*.*

1. *If* $2A > k_2 + k_3$*, then* $\lambda_2 > 0$ *and* $\overline{T_{1.2}} > M_2$*.*

2. *If* $2A \le k_2 + k_3$ *and* $\lambda_2 > 0$*, then* $\overline{T_{1.2}} \le M_2$*.*

3. *An optimal solution of* $\inf_{T\in[M_2,+\infty)} ATC_{1.2}(T)$ *is*

$$T^*_{1.2} = \begin{cases} \overline{T_{1.2}} & \text{if } 2A > k_2 + k_3 \\ M_2 & \text{otherwise} \end{cases}.$$

3.2 The Analysis on the One-Stage Payment Strategy

We start the discussion in this section with our analysis on Cases 2.1 and 2.3, and proceed with those on Cases 2.2 and 2.4 later.

As analyzing Cases 2.1 and 2.3, we observe that the functions $ATC_{2.1}(T)$ and $ATC_{2.3}(T)$ have the same derivative, though they are different in their expressions. Consequently, Theorem 3.5 concludes that both functions locate the same optimal solution.

Theorem 3.5. *Let*

$$\overline{T_{2.1}} = \sqrt{\frac{2A}{D(h + sI_d)}} \tag{3.9}$$

Then the functions $ATC_{2.1}(T)$ *and* $ATC_{2.3}(T)$ *are convex. The replenishment cycle* $\overline{T_{2.1}}$ *is an optimal replenishment cycle of* $\inf_{T>0} ATC_{2.1}(T)$ *and* $\inf_{T>0} ATC_{2.3}(T)$*.*

From Corollary 3.6, one obtains $T^*_{2.1} = \operatorname*{arginf}_{T\in(0,M_1]} ATC_{2.1}(T)$ which is the optimal replenishment cycle of Case 2.1.

Corollary 3.6. *Assume* $\Delta > 0$*.*

1. *If* $2A < k_1$*, then* $\overline{T_{2.1}} < M_1$*. If* $2A \ge k_1$*, then* $\overline{T_{2.1}} \ge M_1$*.*

2. *An optimal solution of* $\inf_{T\in(0,M_1]} ATC_{2.1}(T)$ *is*

$$T^*_{2.1} = \begin{cases} \overline{T_{2.1}} & \text{if } 2A < k_1 \\ M_1 & \text{if } 2A \ge k_1 \end{cases}.$$

Corollary 3.7 gives the optimal replenishment cycle of Case 2.3, i.e., $T^*_{2.3} = \operatorname*{arginf}_{T\in(0,M_2]} ATC_{2.3}(T)$.

Corollary 3.7. *Assume* $\Delta \le 0$*.*

1. *If* $2A \le k_2$*, then* $\overline{T_{2.1}} \le M_2$*. If* $2A > k_2$*, then* $\overline{T_{2.1}} > M_2$*.*

2. *An optimal solution of* $\inf_{T\in(0,M_2]} ATC_{2.3}(T)$ *is*

$$T^*_{2.3} = \begin{cases} \overline{T_{2.1}} & \text{if } 2A \le k_2 \\ M_2 & \text{if } 2A > k_2 \end{cases}.$$

For Case 2.2, we declare the convexity and monotonicity of its total cost function $ATC_{2.2}(T)$ with Theorem 3.8.

Theorem 3.8. *Let*

$$\lambda_3 = 2A + DM_1^2[p(1-r)I_c - sI_d] \tag{3.10}$$

and

$$\overline{T_{2.2}} = \sqrt{\frac{2A + DM_1^2[p(1-r)I_c - sI_d]}{D[h + p(1-r)I_c]}} \tag{3.11}$$

1. *If* $\lambda_3 > 0$*, then* $ATC_{2.2}(T)$ *is convex and an optimal replenishment cycle of* $\inf_{T>0} ATC_{2.2}(T)$ *is* $\overline{T_{2.2}}$.

2. *If* $\lambda_3 = 0$*, then* $ATC_{2.2}(T)$ *is constant.*

3. *If* $\lambda_3 < 0$*, then* $ATC_{2.2}(T)$ *is increasing, concave and an optimal replenishment cycle of* $\inf_{T\in[M_1,\infty)} ATC_{2.2}(T)$ *is* M_1.

Also, Corollary 3.9 supports in finding the optimal replenishment cycle of Case 2.2, i.e., $T^*_{2.2} = \operatorname{arginf}_{T\in[M_1,\infty)} ATC_{2.2}(T)$.

Corollary 3.9. 1. *If* $2A \ge k_1$*, then* $\lambda_3 > 0$ *and* $\overline{T_{2.2}} \ge M_1$.

2. *If* $2A < k_1$ *and* $\lambda_3 > 0$*, then* $\overline{T_{2.2}} < M_1$.

3. *An optimal solution of* $\inf_{T\in[M_1,\infty)} ATC_{2.2}(T)$ *is*

$$T^*_{2.2} = \begin{cases} M_1 & \text{if } 2A < k_1 \\ \overline{T_{2.2}} & \text{if } 2A \ge k_1 \end{cases}.$$

Finally, for Case 2.4, Theorem 3.10 states the convexity and monotonicity of its cost function $ATC_{2.4}(T)$.

Theorem 3.10. *Let*

$$\lambda_4 = 2A + DM_2^2(pI_c - sI_d) \tag{3.12}$$

and

$$\overline{T_{2.4}} = \sqrt{\frac{2A + DM_2^2(pI_c - sI_d)}{D(h + pI_c)}} \tag{3.13}$$

1. *If* $\lambda_4 > 0$*, then* $ATC_{2.4}(T)$ *is convex and an optimal replenishment cycle of* $\inf_{T>0} ATC_{2.4}(T)$ *is* $\overline{T_{2.4}}$.

2. *If* $\lambda_4 = 0$*, then* $ATC_{2.4}(T)$ *is constant.*

3. *If* $\lambda_4 < 0$*, then* $ATC_{2.4}(T)$ *is increasing, concave and an optimal replenishment cycle of* $\inf_{T\in[M_2,\infty)} ATC_{2.4}(T)$ *is* M_2*.*

Also, Corollary 3.11 facilitates the task of locating the optimal replenishment cycle of Case 2.4, i.e., $T^*_{2.4} = \operatorname*{arginf}_{T\in[M_2,\infty)} ATC_{2.4}(T)$.

Corollary 3.11. 1. *If* $2A > k_2$*, then* $\lambda_4 > 0$ *and* $\overline{T_{2.4}} > M_2$*.*

2. *If* $2A \le k_2$ *and* $\lambda_4 > 0$*, then* $\overline{T_{2.4}} \le M_2$*.*

3. *An optimal solution of* $\inf_{T\in[M_2,\infty)} ATC_{2.4}(T)$ *is*

$$T^*_{2.4} = \begin{cases} M_2 & \text{if } 2A \le k_2 \\ \overline{T_{2.4}} & \text{if } 2A > k_2 \end{cases}.$$

4 The Proposed Solution Approach

Based on our theoretical results in Corollary 3.2 to 3.11, we are able to secure all $T^*_{i,j}$ and solve the problem in (2.8). In the first part of this section, we will conduct further analysis on the optimization of (2.8). Then, the second part summarizes the proposed solution approach.

4.1 Theoretical Foundation

Our theoretical analysis in this section lays important foundation for the proposed solution approach in Section 4.2.

Recall that the condition $\Delta > 0$ indicates that the interest earned from the generated sales revenue between M_1 and M_2, is less than the profit made from cash discount. Therefore, $\Delta > 0$ implies that we may exclude $\inf_{T\in(0,M_2]} ATC_{2.3}(T)$ and $\inf_{T\in[M_2,\infty)} ATC_{2.4}(T)$ from our consideration if $\Delta > 0$. Following such intuitive ideas, we summarize (and prove) them in Theorem 4.1.

Theorem 4.1. *Given* $\Delta = rp - sI_d(M_2 - M_1)$*, the following properties hold.*

1. *The condition* $\Delta > 0$ *is necessary and sufficient for the followings:*
   (a) *For all* $T \le M_1$, $ATC_{2.1}(T) < ATC_{2.3}(T)$ *holds.*
   (b) *For all* $T \in [M_1, M_2]$, $ATC_{1.1}(T) < ATC_{2.3}(T)$ *holds.*
   (c) *For all* $T \ge M_1$, $ATC_{1.2}(T) < ATC_{2.4}(T)$ *holds.*
2. *The condition* $\Delta < 0$ *is necessary and sufficient for the followings:*
   (a) *For all* $T \le M_1$, $ATC_{2.3}(T) < ATC_{2.2}(T)$ *holds.*
   (b) *For all* $T \in [M_1, M_2]$, $ATC_{2.3}(T) < ATC_{1.1}(T)$ *holds.*
   (c) *For all* $T \ge M_1$, $ATC_{2.4}(T) < ATC_{1.2}(T)$ *holds.*
3. *The condition* $\Delta = 0$ *is necessary and sufficient for the followings:*
   (a) *For all* $T \le M_1$, $ATC_{2.1}(T) = ATC_{2.3}(T)$ *holds.*

(b) *For all* $T \in [M_1, M_2]$, $ATC_{1.1}(T) = ATC_{2.3}(T)$ *holds.*

(c) *For all* $T \geq M_1$, $ATC_{1.2}(T) = ATC_{2.4}(T)$ *holds.*

The condition $\Delta > 0$ dichotomizes the possible situations, and Theorem 4.1 asserts that solving (2.8) is equivalent to

1. $\min\{ATC_{1.1}(T^*_{1.1}), ATC_{1.2}(T^*_{1.2}), ATC_{2.1}(T^*_{2.1}), ATC_{2.2}(T^*_{2.2})\}$, if $\Delta > 0$.
2. $\min\{ATC_{2.2}(T^*_{2.2}), ATC_{2.3}(T^*_{2.3}), ATC_{2.4}(T^*_{2.4}),\}$, otherwise.

Further, we would make use of the following two results so as to straight out the conditions for finding the optimal solution.

1. The assumption of $M_1 < M_2$ and the definition of k_1 and k_2 lead to $k_1 < k_2$.
2. The condition $\Delta > 0$ implies $k_1 < k_1 + k_3 < k_2 + k_3$ (since $\Delta > 0$ infers $k_3 > 0$ following the definition of Δ and k_3).

With the two results above, we are able to combine all the theoretical analysis in Theorem 3.1-3.10 and Corollary 3.2-3.11 into Theorem 4.2, that provides the backbone of the proposed algorithm for solving the optimal solution of (2.8).

Theorem 4.2. *Let* $u_1 = \overline{T_{2.1}} DM_1(h + sI_d)$.

1. *Suppose* $\Delta > 0$ *and* $2A < k_1$.

   (a) *The conditions* $u_1 \leq 2A < k_1$ *imply that*

$$ATC_{1.1}(M_1) = ATC_{2.2}(M_2)$$
$$= \min\{ATC_{1.1}(T^*_{1.1}), ATC_{1.2}(T^*_{1.2}), ATC_{2.1}(T^*_{2.1}), ATC_{2.2}(T^*_{2.2})\}.$$

   (b) *The conditions* $2A < u_1 \leq k_1$ *imply that*

$$ATC_{2.1}(\overline{T_{2.1}})$$
$$= \min\{ATC_{1.1}(T^*_{1.1}), ATC_{1.2}(T^*_{1.2}), ATC_{2.1}(T^*_{2.1}), ATC_{2.2}(T^*_{2.2})\}.$$

2. *Suppose* $\Delta > 0$.

   (a) *The conditions* $k_1 \leq 2A < k_1 + k_3$ *imply that*

$$ATC_{2.2}(\overline{T_{2.2}}) = \min\{ATC_{1.1}(T^*_{1.1}), ATC_{1.2}(T^*_{1.2}), ATC_{2.1}(T^*_{2.1}), ATC_{2.2}(T^*_{2.2})\}.$$

   (b) *The conditions* $k_1 + k_3 \leq 2A \leq k_2 + k_3$ *imply that*

$$ATC^* = \min\{ATC_{1.1}(\overline{T_{1.1}}), ATC_{2.2}(\overline{T_{2.2}})\}$$
$$= \min\{ATC_{1.1}(T^*_{1.1}), ATC_{1.2}(T^*_{1.2}), ATC_{2.1}(T^*_{2.1}), ATC_{2.2}(T^*_{2.2})\}.$$

   (c) *The condition* $2A > k_2 + k_3$ *implies that*

$$ATC^* = \min\{ATC_{1.2}(\overline{T_{1.2}}), ATC_{2.2}(\overline{T_{2.2}})\}$$
$$= \min\{ATC_{1.1}(T^*_{1.1}), ATC_{1.2}(T^*_{1.2}), ATC_{2.1}(T^*_{2.1}), ATC_{2.2}(T^*_{2.2})\}.$$

3. *Suppose* $\Delta \leq 0$.

(a) *The conditions* $2A < k_1$ *implies that*

$$ATC_{2.3}(\overline{T_{2.3}}) = \min\{ATC_{2.2}(T^*_{2.2}), ATC_{2.3}(T^*_{2.3}), ATC_{2.4}(T^*_{2.4})\}.$$

(b) *The conditions* $2A \in [k_1, k_2]$ *implies that*

$$\begin{aligned}&\min\{ATC_{2.2}(\overline{T_{2.2}}), ATC_{2.3}(\overline{T_{2.3}})\}\\ &= \min\{ATC_{2.2}(T^*_{2.2}), ATC_{2.3}(T^*_{2.3}), ATC_{2.4}(T^*_{2.4})\}.\end{aligned}$$

(c) *The conditions* $2A > k_2$ *implies that*

$$\begin{aligned}&\min\{ATC_{2.2}(\overline{T_{2.2}}), ATC_{2.4}(\overline{T_{2.4}})\}\\ &= \min\{ATC_{2.2}(T^*_{2.2}), ATC_{2.3}(T^*_{2.3}), ATC_{2.4}(T^*_{2.4})\}.\end{aligned}$$

From Theorem 4.2 (and the tree-structured conditions in Figure 1), we are able to make the following observations:

1. When the conditions $\Delta > 0$ and $u_1 \leq 2A < k_1$ hold, i.e., the conditions in (1a), the TSPS (more specifically, Case 1.1) solves the same optimal solution as the SSPS (Case 2.2).

2. The TSPS also possibly obtains an optimal solutions for the conditions in (2b) and (2c), but, the optimal solutions of Case 1.1 and Case 1.2 must be better than those from Case 2.2 (of the SSPS) for both cases, respectively.

3. The SSPS solves the optimal solution, for the following conditions: (i) Case 2.1 for the conditions in (2b), (ii) Case 2.2 for the conditions in (2a), (iii) Case 2.3 for the conditions in (2a), (iv) either Case 2.2 or Case 2.3 for the conditions in (3b), and (v) either Case 2.2 or Case 2.4 for the conditions in (3c).

Note that Theorem 4.2 lays important foundation for the proposed algorithm. The tree-structured conditions in Theorem 4.2 cover all the possible cases. Therefore, we shall make use of Theorem 4.2 to an algorithm that solves the optimal solution of (2.8) efficiently.

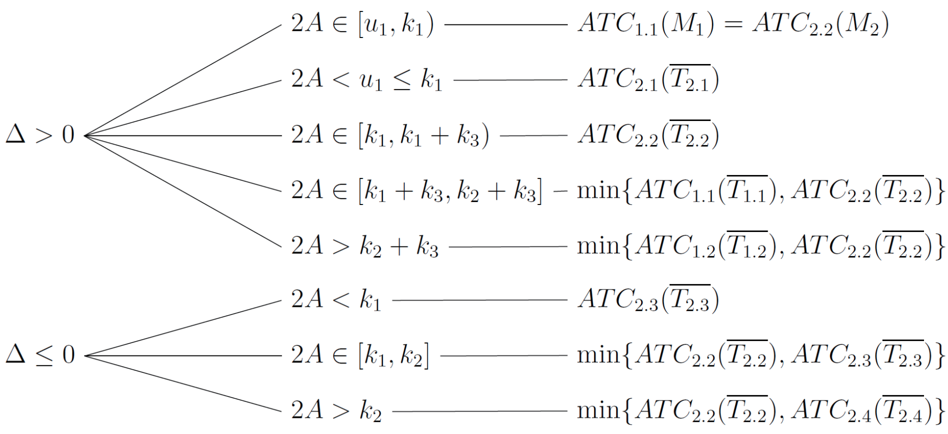

Figure 1: Tree Structure of Theorem 4.2

4.2 The Proposed Algorithm

Following the results in Theorem 4.2, we propose an effective algorithm for finding the optimal cycle time of (2.8) as follows.

Algorithm 1.

Step 1. If $\Delta > 0$, then go to Step 2. Otherwise, go to Step 3.

Step 2. (a) If $2A \in [u_1, k_1)$, then stop and report $ATC_{1.1}(M_1)$ as the optimal value.

(b) If $2A < u_1 \leq k_1$, then stop and report $ATC_{2.1}(\overline{T_{2.1}})$ as the optimal value.

(c) If $2A \in [k_1, k_1 + k_3)$, then stop and report $ATC_{2.2}(\overline{T_{2.2}})$ as the optimal value.

(d) If $2A \in [k_1 + k_3, k_2 + k_3]$, then stop and report

$$\min\{ATC_{1.1}(\overline{T_{1.1}}), ATC_{2.2}(\overline{T_{2.2}})\}$$

as the optimal value.

(e) If $2A > k_2 + k_3$, then stop and report

$$\min\{ATC_{1.2}(\overline{T_{1.2}}), ATC_{2.2}(\overline{T_{2.2}})\}$$

as the optimal value.

Step 3. (a) If $2A < k_1$, then stop and report $ATC_{2.3}(\overline{T_{2.3}})$ as the optimal value.

(b) If $2A \in [k_1, k_2]$, then stop and report

$$\min\{ATC_{2.2}(\overline{T_{2.2}}), ATC_{2.3}(\overline{T_{2.3}})\}$$

as the optimal value.

(c) If $2A > k_2$, then stop and report

$$\min\{ATC_{2.2}(\overline{T_{2.2}}), ATC_{2.4}(\overline{T_{2.4}})\}$$

as the optimal value.

Note that each step in Algorithm 1 corresponds to an exclusive case in Theorem 4.2. For example, Step 3(a) is corresponding to (3a) in Theorem 4.2. Therefore, Theorem 4.2 guarantees that Algorithm 1 obtains an optimal solution.

5 Numerical Experiments

We would like to make observation if the optimal solution comes from the Single-Stage Payment Strategy or the Two-Stage Payment Strategy in this section. Section 5.1 takes the six examples in Huang & Chung [10] for our numerical experiments. Then, we conduct sensitivity analysis to obtain further managerial insights as the second part of this section.

Table 2: The parameters of the six examples in Huang & Chung [10].

Example	1	2	3	4	5	6
A	10	100	100	100	100	100
h	10	10	10	10	10	10
r	0.1	0.005	0.001	0.01	0.001	0.001
M_1	0.08	0.06	0.02	0.02	0.05	0.02
M_2	0.1	0.1	0.1	0.1	0.1	0.1
D	5,000	2,600	1,000	4,000	5,000	1,000
I_c	0.15	0.15	0.15	0.15	0.15	0.15
I_d	0.14	0.12	0.12	0.12	0.12	0.10
p	150	100	60	50	50	100
s	150	100	60	50	50	100

Table 3: The optimal solutions of the six examples.

Example	1	2	3	4	5	6
The optimal Csae	Case2.1	Case2.1	Case2.4	Case1.1	Case2.3	Case2.3
Improvement	0	0	0	1.034%	0	0

5.1 The Six Examples in Huang & Chung (2003)

We take the six examples in Huang & Chung [10] as the benchmark instances for comparison here. Table 2 displays the sets of parameters for those six examples.

We solve these six examples using the proposed algorithm (i.e., Algorithm 1) under Two-Stage Payment Strategy (TSPS). Table 3 shows the optimal solutions from Algorithm 1. The TSPS solves the same optimal solution as the SSPS except one instance, namely, Example 4. We observe that the optimal solution commits Case 1.1 in which the retailer made the payment before the permissible delay period M_2. In this example, the TSPS obtains a better solution than the SSPS with an improvement of 1.034% in the annual average total relevant cost. The retailer would make the first part of its payment at M_1 and earn the cash discount.

5.2 Sensitivity Analysis

In this section, we would employ sensitivity analysis to gain more managerial insights for the Two-Stage Payment Strategy. To set up a base case for our sensitivity analysis, we would first introduce an example with its parameters showing in Table 4.

Before starting our sensitivity analysis, we first solve the base-case example. Follow-

Table 4: The parameters of the base-case example for our sensitivity analysis.

A=\$200 dollars / per order	D=3,000units / per year
h=\$10 dollars / per unit, per year	I_c is an annual interest of 15%
I_d is an annual interest of 7%	r=0.5%
s=\$25 dollars / per unit	p=\$20 dollars/per unit
M_1=10 days	M_2=30 days

Table 5: A summary of our sensitivity analysis.

Parameter	Range of parameter	TSPS-applied Range	TSPS-applied percentage	Best TSPS strategy	Average improvement
s	[20,40]	[20,26]	33.3%	Case1.2	0.54%
p	[12.5,25]	[19.2,21.4]	18.3%	Case1.2	0.38%
h	[1,135]	[1,112]	79.3%	Case1.x$^+$	0.63%
I_c	[0.071,0.9]	[0.107,0.9]	32.7%	Case1.2	85.17%
D	[1,20000]	[1,20000]	91.2%	Case1.x$^+$	1.59%
A	[5,4000]	[21,4000]	97.8%	Case1.x$^+$	7.09%
I_d	[0.0001,0.149]	[0.0001,0.0729]	48.9%	Case1.2	0.57%
r	[0.0001,0.05]	[0.0001,0.0053]	1.2%	Case1.2	0.42%

$^+$We mark "Case 1.x" since the best TSPS strategy could alternate between Case 1.1 and Case 1.2 for different instances.

ing the $\Delta = 0.0041096 > 0$ and $2A = 400 > 238.8065303 = k_2 + k_3$, it belongs to the conditions in (2c) in Theorem 4.2 (and, also Step 2(e) in Algorithm 1). The algorithm reports that the optimal replenishment cycle of Case 1.2 is $T^*_{1.2} = \overline{T_{1.2}} = 0.0867$ with $ATC^* = ATC_{1.2}(\overline{T_{1.2}}) = \$3399.1922$. Therefore, Algorithms 1 solves an optimal solution for the base-case example under the TSPS.

We would like to investigate is that if the TSPS strategy will sustain as the parameter changes and how significant will the improvement vary with the change of parameters via our sensitivity analysis. Therefore, we conduct our sensitivity analysis by changing only one parameter, but keeping the others fixed. But, we would fix the values of M_1 and M_2 in our sensitivity analysis since they apply to most of the accounting practice in the real-world.

Table 5 summarizes the settings and the results in our sensitivity analysis. The second column shows the test-range of the parameters. Note that we set the range by referring to the six examples in Huang & Chung [10], but attempt to test a much wider range than those appearing in the six examples for the interested parameters. The third column indicates the percentage of the test-range where the TSPS applied and obtained an optimal solution for the tested instances. Obviously, the larger the TSPS-applied percentage, the more robust the parameter. For this base-case example, we observe that the TSPS applied to almost all the tested range for the parameters D (demand rate) and A (ordering cost), and 79.3% of the range for h (holding cost). Also, the TSPS is very sensitive to the parameter r (the discount rate of cash discount), and the TSPS no longer obtains an optimal solution as the parameter r leaves the 1.2%-range, i.e., $[0.0001, 0.0053]$. This observation may result from the case that the decision maker would take full advantage of cash discount as the discount rate is good enough (and applies the EOQ with cash discount only), and would totally ignore cash discount as the discount rate becomes unattractive.

On the other hand, we may observe that Case 1.2 sustains for the changes of the parameters s, p, I_c, I_d, and r, from the fifth column of Table 5. (Recall that the base-case obtains its optimal solution from Case 1.2.) For those who are interested in the sensitivity on $(s-p)$, one may refer to the results on s since when we conduct the sensitivity analysis on s, we did it by fixing the value of p and varying the value of s, which is equivalent to do the sensitivity analysis on $(s-p)$. The best TSPS strategy alternates between Case 1.1.and 1.2 as the values of h, D and A changed. The last column of Table 5 shows that the parameters I_c (the interest charged) facilitates the TSPS strategy to gain the most significant average improvement of 85.17% comparing to the SSPS strategy. For the parameter I_d, we consider cash discount could be even more attractive and EOQ with cash discount will be preferable as the interest earned is low. It is reasonable that the earning from the interest may surpass the cash discount as the interest earned increases to some level, and the decision maker would turn to use the permissible delay in payment. Also, note that the parameter p (the purchasing price) has a strong link with cash discount, namely, the lower the purchasing price, the less the amount of cash discount. Therefore, the purchasing price directly impacts the advantage of cash discount, and the decision maker will switch to take the permissible delay in payment as the purchasing price is relatively low.

6 Conclusion and Future Research

In this paper, we introduce a new possible practice, called "Two-Stage Payment Strategy"(TSPS), in which both options of "the permissible delay period" and "the period of cash discount" are available to the retailer. The supplier often offers "the period of cash discount" to encourage the retailer to pay for his purchase to boost the turnover of cash flow. Also, as "the permissible delay period" is available, the retailer will be charged the interest of unpaid payment if the payment is not paid in full by the end of this permissible delay period. In the real world, the supplier may offer both options to the retailer to bridge the strategic alliance and collaboration relationship between each other. Or, the retailer may aggressively ask the supplier to have both options available if the retailer is in favor of dominance power in the supply chain. Obviously, the TSPS could be very practical in the real world. However, we found no EOQ model was proposed to assist the retailer's decision-making in such scenario.

In this study, we investigate the EOQ model with the TSPS available to the retailer. The proposed model is different from the EOQ models for the Single-Stage Payment Strategy (SSPS) in which the models were derived either for the condition of cash discount or the condition of permissible delay in payment exclusively. We would like to verify that the EOQ with the TSPS may lead to a lower cost than the latters under the SSPS. We formulate the EOQ model under both options of cash discount and permissible delay in payment, and conduct thorough theoretical analysis. Based on our theoretical results, we propose an effective solution approach (viz., Algorithm 1) that assists the retailer not only in the determination of taking the TSPS or the SSPS, but also determining the optimal replenishment cycle time respectively.

We solve the six examples in Huang & Chung [10], and demonstrate that the TSPS does obtain better solution than the SSPS for some case in our numerical experiments. Our sensitivity analysis indicates that the TSPS could be very sensitive to the discount rate of cash discount and the interest earned. Also, the purchasing price facilitates the TSPS strategy to gain the most significant average improvement comparing to the SSPS strategy.

We observe that there still exists some room for further studies of the lot sizing problems under the conditions of cash discount or the condition of permissible delay in payment. The interested researchers may take this study as the reference for their extensions.

References

[1] S.P. Aggarwal and C.K. Jaggi, Ordering policies of deteriorating items under permissible delay in payments, *J. Oper. Res. Soc.* 46 (1995) 658–662.

[2] S. Chand and J. Ward, A note on economic order quantity under conditions of permissible delay in payments, *J. Oper. Res. Soc.* 38 (1987) 83–84.

[3] C.-T. Chang, Extended economic order quantity model under cash discount and payment delay, *Internat. J. Inform. Management Sci.* 13 (2002) 57–69.

[4] P. Chu, K.-J. Chung, and S.-P. Lan, Economic order quantity of deteriorating items under permissible delay in payments, *Comput. Oper. Res.* 25 (1998) 817–824.

[5] K.-J. Chung and Y.-F. Huang, The optimal cycle time for EPQ inventory model under permissible delay in payments, *Internat. J Prod. Econ.* 84 (2003) 307–318.

[6] K.-J. Chung, A theorem on the determination of economic order quantity under conditions of permissible delay in payments, *Comput. Oper. Res.* 25 (1998) 49–52.

[7] S.K. Goyal, Economic order quantity under conditions of permissible delay in payments, *J. Oper. Res. Soc.* 36 (1985) 335–338.

[8] Y.-F. Huang, A note on EOQ model under cash discount and payment delay, *Internat. J. Inform. Management Sci.* 16 (2005) 97–107.

[9] Y.-F. Huang, Optimal Retailer's Replenishment Decisions in the EPQ Model under Two Levels of Trade Credit Policy, *European J. Oper. Res.* 176 (2007) 1577–1591.

[10] Y.-F. Huang and K.-J. Chung, Optimal replenishment and payment policies in the EOQ model under cash discount and trade credit, *Asia-Pac. J. Oper. Res.* 20 (2003) 177–190.

[11] Y.-F. Huang and K.-H. Hsu, An EOQ model under retailer partial trade credit policy in supply chain, *Internat. J. Prod. Econ.* 112 (2008) 655–664.

[12] A.M.M. Jamal, B.R. Sarker and S. Wang, An ordering policy for deteriorating items with allowable shortage and permissible delay in payment, *J. Oper. Res. Soc.* 48 (1997) 826–833.

[13] L.-Y. Ouyang, M.-S. Chen, and K.-W. Chuang, Economic order quantity model under cash discount and payment delay, *Internat. J. Inform. Management Sci.* 13 (2002) 1–10.

Appendix: The Derivation of the Annual Total Cost Function of Case 1.1

The inventory level of Case 1.1 is shown in Figure 2. From Figure 2, the number of sold products in Case 1.1 is shown in Figure 3. From Figure 3, we have

- the cost of traditional EOQ is given by $A + \frac{DT^2h}{2}$.
- the revenue from cash discount is $rpDM_1$.
- the Interest earned is

$$sI_dD\left[M_1\left(M_1 - M_2\right) + T\left(M_2 - \frac{T}{2}\right)\right]$$

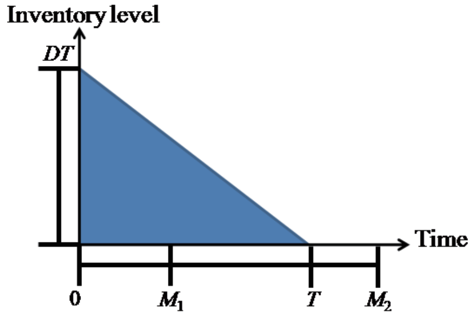

Figure 2: Inventory level of Case 1.1

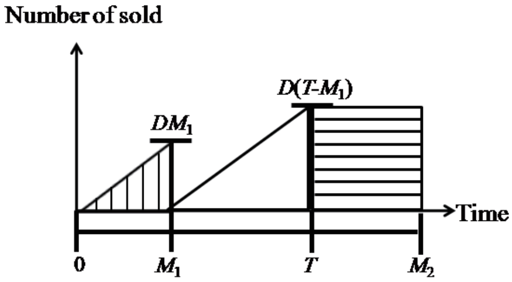

Figure 3: Number of sold products in Case 1.1

because

$$\begin{aligned}&\text{Interest earned}\\&=\left\{\tfrac{1}{2}\times DM_1\times M_1+\tfrac{1}{2}D\left(T-M_1\right)\times\left(T-M_1\right)+\left(M_2-T\right)\times\left[D\left(T-M_1\right)\right]\right\}\\&\quad\times s\times I_d\\&=sI_d\left[\tfrac{1}{2}DM_1^2+\tfrac{1}{2}D\left(T^2-2TM_1+M_1^2\right)+D\left(M_2T-T^2-M_2M_1+TM_1\right)\right]\\&=sI_d\left[DM_1^2+D\left(M_2T-\tfrac{1}{2}T^2-M_2M_1\right)\right]\\&=sI_dD\left[M_1\left(M_1-M_2\right)+T\left(M_2-\tfrac{T}{2}\right)\right]\end{aligned}$$

- the interest charged is 0 because $T \leq M_2$

Hence the total cost function is given by

$$A+\frac{DT^2h}{2}-rpDM_1-sI_dD\left[M_1\left(M_1-M_2\right)+T\left(M_2-\frac{T}{2}\right)\right]$$

and the average total cost function is given by

$$ATC_{1.1}\left(T\right)=\frac{A}{T}+\frac{DTh}{2}-\frac{rpDM_1}{T}-sI_dD\left[\frac{M_1\left(M_1-M_2\right)}{T}+\left(M_2-\frac{T}{2}\right)\right]$$

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