Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL
Pacific Journal of Optimization
Vol. 10, No. 2, pp. 305–319, April 2014

MIXED TYPE DUALITY FOR A NONDIFFERENTIABLE MINIMAX FRACTIONAL COMPLEX PROGRAMMING*

Hang-Chin Lai and Tone-Yau Huang†

Abstract: We consider a nondifferentiable minimax complex fractional programming problem (P) in this paper. We will construct a mixed type dual problem (MD) for problem (P). Therefore problem (MD) can include the Wolfe type dual problem (WD) and the Mond-Weir type dual problem (MWD) of problem (P). Finally, we prove the duality theorems of (MD). This means that there are no duality gaps between problem (P) and problem (MD).

Key words: *fractional programming, duality, generalized convexity*

Mathematics Subject Classification: *90C32, 90C46, 26A51*

---

1 Introduction

The concept of duality problem is a useful tool in mathematical programming, and there are many researchers interested the dual problems. Various types of the duality problems are considered (cf. [1–6, 11–14]). In 2009, Lai and Huang [10] considered a minimax fractional complex programming problem (P). In this paper, we want to construct a mixed type dual problem of problem (P).

Now, we focus on our primary problem: the minimax fractional complex programming problem (P) as follows:

$$\text{(P)} \qquad \min_{\zeta \in X} \max_{\eta \in Y} \frac{Re\,[f(\zeta,\eta) + (z^H Az)^{1/2}]}{Re\,[g(\zeta,\eta) - (z^H Bz)^{1/2}]}$$
$$\text{s.t.} \quad X = \left\{\zeta = (z,\overline{z}) \in \mathbb{C}^{2n} \mid -h(\zeta) \in S\right\}$$

where Y is a compact subset of $\{\eta = (w,\overline{w}) \mid w \in \mathbb{C}^m\} \subset \mathbb{C}^{2m}$; A and $B \in \mathbb{C}^{n\times n}$ are positive semidefinite Hermitian matrices; S is a polyhedral cone in $\mathbb{C}^p$; $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are continuous functions, and for each $\eta \in Y$, $f(\cdot,\eta)$ and $g(\cdot,\eta) : \mathbb{C}^{2n} \to \mathbb{C}$ are analytic. We assume further that $h(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^p$ is an analytic map defined on $Q \subset \mathbb{C}^{2n}$.
This set $Q = \{(z,\overline{z}) \mid z \in \mathbb{C}^n\}$ is a linear manifold over real field. Without loss of generality, it is assumed that $Re\,[f(\zeta,\eta) + (z^H Az)^{1/2}] \geq 0$ and $Re\,[g(\zeta,\eta) - (z^H Bz)^{1/2}] > 0$ for each $(\zeta,\eta) \in X \times Y$. We know that this problem will be nondifferentiable if there is a point $\zeta_0 = (z_0,\overline{z_0})$ such that $z_0^H Az_0 = 0$ or $z_0^H Bz_0 = 0$.

*This research was partly supported by NSC 100-2115-M-035 -004 -, Taiwan. & NSC 101-2115-M-035 -004 -, Taiwan.

†Corresponding author

We are interesting in problem (P), because this programming problem is a wide problem. Indeed, there are some special cases for problem (P). For details, please refer to [6–9] and [11]. The minimax fractional complex programming problem has several applications in such as electrical engineering, filter theory, etc. We can find an application in the field of filter theory, that is a problem to evaluate the eigenvalues $\lambda_1, \ldots, \lambda_m$ of the correlation matrix A as the following:

$$\lambda_k = \min_{dim(S)=k} \max_{z\in S} \frac{z^H A z}{z^H z}, \quad k = 1, \ldots, m$$

where S is a subspace of $\mathbb{C}^m$, $dim(S)$ denotes the dimension of subspace $S \in \mathbb{C}^m$, A is a positive semidefine Hermitian matrix, and the maximum is taken the nonzero complex vector z over the subspace S.

The main purpose of this article extend paper [11]. We will establish the mixed type duality (MD) w.r.t. problem (P). We will know that this new dual problem (MD) in this paper contains dual problems (WD) and (MWD) in [11], and then we will prove its duality theorems (cf. [1], [2]). In 2009, Lai and Huang [10] already found the optimality conditions of problem (P). We can employ to establish a mixed type duality of (P) as follows.

The constraint function in (P) is $h(\zeta) = \big(h_1(\zeta), h_2(\zeta), \ldots, h_p(\zeta)\big) \in \mathbb{C}^p$. By optimality conditions of (P), there is a vector multiplier $\mu = (\mu_1, \ldots, \mu_p) \in S^* \subset \mathbb{C}^p$ on $h(\zeta) \subset \mathbb{C}^p$, where S^* is the dual cone of S in $\mathbb{C}^p$. Now, we partition $P = \{1, \ldots, p\}$, the index set of the constraint function $h(\zeta)$ to be $P = P_0 \cup P_1 \cup \cdots \cup P_t$ with $Re \ \langle \ h_{P_r}(\zeta), \mu_{P_r} \ \rangle \leq 0$ for $r = 0, 1, \ldots, t$, where $h_{P_r}(\zeta) \equiv \big(h_i(\zeta)\big)_{i\in P_r}$, $\mu_{P_r} \equiv \big(\mu_i\big)_{i\in P_r}$.

We define the mixed type dual problem of (P) as the form:

$$\text{(MD)} \qquad \max_{(k,\widetilde{\lambda},\widetilde{\eta})\in K(\xi)} \ \max_{(\xi,\mu,w_1,w_2)\in X_3(k,\widetilde{\lambda},\widetilde{\eta})} \frac{\displaystyle\sum_{i=1}^{k} \lambda_i Re\big[f(\xi,\eta_i) + (\alpha^H A\alpha)^{1/2} + \langle h_{P_0}(\xi), \mu_{P_0}\rangle\big]}{\displaystyle\sum_{i=1}^{k} \lambda_i Re\big[g(\xi,\eta_i) - (\alpha^H B\alpha)^{1/2}\big]}$$

with the constraint satisfing some conditions (it will be described in section 4).

In 2010, Lai and Huang [11] constructed the Wolfe type dual (WD) and Mond-Weir type dual (MWD) problems. By later section 4 of this paper, we can know that dual problem (MD) of problem (P) contains dual problems (WD) and (MWD) as the special cases. Finally, we will prove the duality theorems of (MD) in section 5. This means that there are no duality gaps between problem (P) and problem (MD).

2 Some Definitions and Notations

In order to get the duality theorems, we need some generalizations of convexity as follows. (cf. Lai and Huang [9, 10]).

Definition 2.1. The real part of an analytic function $f(\cdot)$ from $\mathbb{C}^{2n}$ to $\mathbb{R}$ is called, respectively,

(i) **convex (strictly)** at $\zeta_0 \in Q \subset \mathbb{C}^{2n}$ if
$$Re\big[f(\zeta) - f(\zeta_0)\big] \underset{(>)}{\geq} Re\big[f'_\zeta(\zeta_0)(\zeta - \zeta_0)\big],$$

(ii) **pseudoconvex (strictly)** at $\zeta_0 \in Q$ if
$$Re\big[f'_\zeta(\zeta_0)(\zeta - \zeta_0)\big] \geq 0 \Rightarrow Re\big[f(\zeta) - f(\zeta_0)\big] \underset{(>0)}{\geq 0},$$

(iii) **quasiconvex** at $\zeta_0 \in Q$ if
$$Re\big[f(\zeta) - f(\zeta_0)\big] \le 0 \Rightarrow Re\big[f'_\zeta(\zeta_0)(\zeta - \zeta_0)\big] \le 0.$$

Definition 2.2. An analytic mapping $h(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^p$ is called, respectively,

(i) **convex** at $\zeta_0 \in Q$ with respect to (w.r.t.) a polyhedral cone S in $\mathbb{C}^p$ if there is a nonzero $\mu \in S^*\big(\subset \mathbb{C}^p\big)$, the dual cone of S, such that
$$Re\langle h(\zeta) - h(\zeta_0), \mu\rangle \ge Re\langle h'(\zeta_0)(\zeta - \zeta_0), \mu\rangle.$$
Here $\langle \cdot, \cdot \rangle$ stands for the inner product in complex spaces.

(ii) **pseudoconvex (strictly)** at $\zeta_0 \in Q$ w.r.t. S if there is a nonzero $\mu \in S^*\big(\subset \mathbb{C}^p\big)$ the dual cone of S, such that
$$Re\langle h'(\zeta_0)(\zeta - \zeta_0), \mu\rangle \ge 0 \Rightarrow Re\langle h(\zeta) - h(\zeta_0), \mu\rangle \quad \begin{matrix} \ge 0, \\ (> 0) \end{matrix}$$

(iii) **quasiconvex** at $\zeta_0 \in Q$ w.r.t. S if there is a nonzero $\mu \in S^*\big(\subset \mathbb{C}^p\big)$ such that
$$Re\langle h(\zeta) - h(\zeta_0), \mu\rangle \le 0 \Rightarrow Re\langle h'(\zeta_0)(\zeta - \zeta_0), \mu\rangle \le 0.$$

Definition 2.3. The problem (P) satisfies the **constraint qualification** at a point $\zeta_0 = (z_0, \overline{z_0})$ if for any nonzero $\mu \in S^* \subset \mathbb{C}^p$,

$$Re\ \langle\ h'(\zeta_0)(\zeta - \zeta_0), \mu\ \rangle \ne 0 \ \text{ for } \zeta \ne \zeta_0. \tag{2.1}$$

From the next section, we often use the differential property. In order to employ the behavior, the differential of a complex function is often replaced by the gradient expressions ∇_z and $\nabla_{\overline{z}}$ which we introduce as follows.

Lemma 2.4. *(Lai and Huang [9, Lemma 2])*
For $\eta \in Y \subset \mathbb{C}^{2m}$, $w \in \mathbb{C}^n$ and $\zeta = (z, \overline{z}) \in Q \subset \mathbb{C}^{2n}$, we denote the function

$$\Phi(\zeta) = f(\zeta, \eta) + z^H Aw + \langle h(\zeta), \mu\rangle.$$

Then $\Phi(\zeta)$ is differentiable at $\zeta_0 = (z_0, \overline{z_0})$, and

$$Re[\Phi'(\zeta_0)(\zeta - \zeta_0)] = Re\Big[\Big\langle\ z - z_0,\ \overline{\nabla_z f(\zeta_0, \eta)} + \nabla_{\overline{z}} f(\zeta_0, \eta) + Aw + \mu^T \overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\overline{z}} h(\zeta_0)\ \Big\rangle\Big].$$

The generalized Schwarz inequality in complex space can be as the form:

$$Re(z^H Au) \le (z^H Az)^{1/2}(u^H Au)^{1/2}.$$

3 Necessary and Sufficient Optimality Conditions

Throughout this paper, let $S = \{\xi \in \mathbb{C}^p \mid Re(K\xi) \ge 0\}$ be a polyhedral cone where $K \in \mathbb{C}^{k\times p}$ is a $k \times p$ matrix; the dual cone S^* of S is defined by

$$S^* = \{\mu \in \mathbb{C}^p \mid Re\langle \xi, \mu\rangle \ge 0, \text{ for } \xi \in S\}.$$

For $z_p \in S$, define the set $S(z_p)$ as the intersection of those closed half spaces which include z_p in their boundaries. That is, given $K = (a_1, \dots, a_k)^T \in \mathbb{C}^{k\times p}$ (of polyhedral cone S) for $a_i \in \mathbb{C}^p$, $i = 1, \dots, k$, and let $I(z_p) \equiv \{i \mid Re\langle z_p, a_i\rangle = 0\}$, define the set

$$S(z_p) \equiv \cap_{i\in I(z_p)}\{\xi \in \mathbb{C}^p \mid Re\langle \xi, a_i\rangle = 0\}.$$

Let X be a subset of $\mathbb{C}^{2n}$, and for $\zeta = (z, \overline{z}) \in X$, $f(\zeta, \cdot)$ and $g(\zeta, \cdot)$ are continuous on the compact set Y. Thus we can denote

$$Y(\zeta) = \left\{ \eta \in Y \;\Big|\; \frac{Re[f(\zeta,\eta) + (z^H Az)^{1/2}]}{Re[g(\zeta,\eta) - (z^H Bz)^{1/2}]} = \max_{\nu \in Y} \frac{Re[f(\zeta,\nu) + (z^H Az)^{1/2}]}{Re[g(\zeta,\nu) - (z^H Bz)^{1/2}]} \right\}$$

since Y is compact, the supremum in the above $v \in Y$ is attained. This set $Y(\zeta)$ is also a compact subset of Y.

In [10], Lai et al. have established the optimality conditions. We restate the necessary optimality conditions as follows.

Theorem 3.1 (Necessary Optimality Conditions [10, Theorem 2])**.** *Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be a (P)-optimal with optimal value v^*. Suppose that the problem (P) satisfies the constraint qualification at ζ_0 with assumptions $z_0^H Az_0 = \langle Az_0, z_0\rangle > 0$ and $z_0^H Bz_0 = \langle Bz_0, z_0\rangle > 0$. Then there exist $0 \neq \mu \in S^* \subset \mathbb{C}^p$, $u_1, u_2 \in \mathbb{C}^n$ and positive integer k with the following properties (as $Y(\zeta_0) \subset Y$ is provided a compact subset in $\mathbb{C}^{2m}$):*

(i) *finite points $\eta_i \in Y(\zeta_0)$ for $i = 1, \ldots, k$;*

(ii) *for $i = 1, \ldots, k$, multipliers $\lambda_i > 0$ and $\sum_{i=1}^k \lambda_i = 1$*

such that $\displaystyle\sum_{i=1}^k \lambda_i[f(\zeta,\eta_i) - v^* g(\zeta,\eta_i)] + \langle h(\zeta), \mu\rangle + \langle Az, z\rangle^{1/2} + v^*\langle Bz, z\rangle^{1/2}$ *satisfies the following conditions*

$$\sum_{i=1}^k \lambda_i \Big\{ \Big[\overline{\nabla_z f(\zeta_0,\eta_i)} + \nabla_{\overline{z}} f(\zeta_0,\eta_i)\Big] - v^*\Big[\overline{\nabla_z g(\zeta_0,\eta_i)} + \nabla_{\overline{z}} g(\zeta_0,\eta_i)\Big]\Big\} + \Big(\mu^T \overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\overline{z}} h(\zeta_0)\Big) + \big(Au_1 + v^* Bu_2\big) = 0; \tag{3.1}$$

$$Re\langle\, h(\zeta_0), \mu\,\rangle = 0; \tag{3.2}$$

$$u_1^H Au_1 \le 1,\ (z_0^H Az_0)^{1/2} = Re(z_0^H Au_1); \tag{3.3}$$

$$u_2^H Bu_2 \le 1,\ (z_0^H Bz_0)^{1/2} = Re(z_0^H Bu_2). \tag{3.4}$$

Theorem 3.1 holds under the conditions $z_0^H Az_0 = \langle Az_0, z_0\rangle > 0$ and $z_0^H Bz_0 = \langle Bz_0, z_0\rangle > 0$. In fact, we may show that this theorem will be true with the assumption either $\langle Az_0, z_0\rangle = 0$ or $\langle Bz_0, z_0\rangle = 0$. In order to prove it, we need the following notations.

$$Z_{\widetilde{\eta}}(\zeta_0) = \Big\{ \zeta \in \mathbb{C}^{2n} \;\Big|\; -h'_\zeta(\zeta_0)\zeta \in S(-h(\zeta_0)),\ \zeta = (z, \overline{z}) \in Q \text{ with any one of the next conditions (i), (ii) and (iii) holds} \Big\}.$$

(i) $Re\Big\{ \displaystyle\sum_{i=1}^k \lambda_i \Big[f'_\zeta(\zeta_0,\eta_i) - v^* g'_\zeta(\zeta_0,\eta_i)\Big]\zeta + \frac{\langle Az_0, z\rangle}{\langle Az_0, z_0\rangle^{1/2}} + \langle (v^*)^2 Bz, z\rangle^{1/2} \Big\} < 0,$
if $z_0^H Az_0 > 0$ and $z_0^H Bz_0 = 0$;

(ii) $Re\Big\{ \displaystyle\sum_{i=1}^k \lambda_i \Big[f'_\zeta(\zeta_0,\eta_i) - v^* g'_\zeta(\zeta_0,\eta_i)\Big]\zeta + \langle Az, z\rangle^{1/2} + \frac{\langle v^* Bz_0, z\rangle}{\langle Bz_0, z_0\rangle^{1/2}} \Big\} < 0,$
if $z_0^H Az_0 = 0$ and $z_0^H Bz_0 > 0$;

(iii) $Re\Big\{ \sum_{i=1}^{k} \lambda_i \Big[f'_\zeta(\zeta_0, \eta_i) - v^* g'_\zeta(\zeta_0, \eta_i) \Big] \zeta + \langle [A + (v^*)^2 B] z, z \rangle^{1/2} \Big\} < 0,$
if $z_0^H A z_0 = 0$ and $z_0^H B z_0 = 0$.

Here, $S(-h(\zeta_0))$ is the intersection of those closed half spaces which include $-h(\zeta_0)(\in \mathbb{C}^p)$ in their boundaries.

Theorem 3.2 (Necessary Optimality Conditions [10, Theorem 3]). *Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be (P)-optimal with optimal value v^*. Suppose that problem (P) possesses constraint qualification at ζ_0 and $Z_{\widetilde{\eta}}(\zeta_0) = \emptyset$. Then there exist a nonzero $\mu \in S^* \subset \mathbb{C}^p$ and vectors $u_1, u_2 \in \mathbb{C}^n$ such that the conditions (3.1)∼(3.4) in Theorem 3.1 hold.*

We state the sufficient optimality conditions of (P) as follows.

Theorem 3.3 (Sufficient Optimality Conditions [10, Theorem 4]). *Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be a feasible solution of (P). Suppose that there exist a positive integer $k > 0$, $v^* \in \mathbb{R}^+$, for $i = 1, \dots, k$, $\lambda_i > 0$, $\eta_i \in Y(\zeta_0)$ with $\sum_{i=1}^k \lambda_i = 1$, and that $0 \neq \mu \in S^* \subset \mathbb{C}^p$, u_1, $u_2 \in \mathbb{C}^n$ satisfying conditions (3.1)∼(3.4) of Theorem 3.1 for $Z_{\widetilde{\eta}}(\zeta_0) = \emptyset$. Assume that any one of the following conditions (i), (ii) and (iii) holds:*

(i) $Re\left\{ \sum_{i=1}^{k} \lambda_i \Big[\big(f(\zeta, \eta_i) + z^H A u_1 \big) - v^* \big(g(\zeta, \eta_i) - z^H B u_2 \big) \Big] \right\}$ *is pseudoconvex at* $\zeta = (z, \overline{z}) \in Q$, *and* $h(\zeta)$ *is quasiconvex at* $\zeta \in Q$ *w.r.t. the polyhedral cone* $S \subset \mathbb{C}^p$;

(ii) $Re\left\{ \sum_{i=1}^{k} \lambda_i \Big[\big(f(\zeta, \eta_i) + z^H A u_1 \big) - v^* \big(g(\zeta, \eta_i) - z^H B u_2 \big) \Big] \right\}$ *is quasiconvex at* $\zeta = (z, \overline{z}) \in Q$, *and* $h(\zeta)$ *is strictly pseudoconvex at* $\zeta \in Q$ *w.r.t.* $S \subset \mathbb{C}^p$;

(iii) $Re\left\{ \sum_{i=1}^{k} \lambda_i \Big[\big(f(\zeta, \eta_i) + z^H A u_1 \big) - v^* \big(g(\zeta, \eta_i) - z^H B u_2 \big) \Big] + \langle h(\zeta), \mu \rangle \right\}$ *is pseudoconvex at* $\zeta \in Q$.

Then $\zeta_0 = (z_0, \overline{z_0})$ is an optimal solution of (P).

4 Construction for a Mixed Type Duality Model

To perform a mixed type dual problem to the complex programming problem (P), we need the following preparation. Let $\zeta = (z, \overline{z}) \in Q \subset \mathbb{C}^{2n}$ be any feasible solution of problem (P). By the compactness of Y in (P), the closed subset $Y(\zeta)$ is also compact in which the constraints fractional function in η has finite points attained to its maximum, that is, to maximizing the fractional function

$$\varphi(\zeta) = \max_{\eta \in Y} \frac{Re\big[f(\zeta, \eta) + (z^H A z)^{1/2}\big]}{Re\big[g(\zeta, \eta) - (z^H B z)^{1/2}\big]} \quad \text{at } \eta_1,\ \eta_2, \dots,\ \eta_k \text{ for some } k \in \mathbb{N},$$

becomes the objective of problem (P).

Since for each $\zeta = (z, \overline{z}) \in Q$, for $i = 1, \dots, k$, $\eta_i \in Y(\zeta)$, $\lambda_i > 0$ with $\sum_{i=1}^k \lambda_i = 1$, and functions $f(\zeta,\cdot)$ and $g(\zeta,\cdot)$ are continuous on $Y(\zeta)$, then the objective fractional functional of problem (P) has the form:

$$\varphi(\zeta) \equiv \max_{\eta\in Y} \frac{Re\big[f(\zeta,\eta)+(z^HAz)^{1/2}\big]}{Re\big[g(\zeta,\eta)-(z^HBz)^{1/2}\big]} = \frac{\sum_{i=1}^{k}\lambda_i Re\big[f(\zeta,\eta_i)+(z^HAz)^{1/2}\big]}{\sum_{i=1}^{k}\lambda_i Re\big[g(\zeta,\eta_i)-(z^HBz)^{1/2}\big]} \tag{4.1}$$

and the problem (P) becomes

$$\text{(P)} \quad \min_{\zeta\in X}\varphi(\zeta). \tag{4.2}$$

Usually, we use the objective functional of expression (4.1) to construct the duality problems w.r.t. (P).

In 2010, Lai and Huang have established the duality models of Wolfe type duality (WD) and Mond-Weir type duality (MWD). We restate them as follows. (For detail, please refer to [11, Sections 5, 6].)

The Wolfe type dual in fractional programming problem is considered by the objective of fractional functional added the constraints of (P) with a multiplier $\mu \in S^*$ into the numerator of the fractional functional in (P), precisely, it is performed by:

$$\text{(WD)} \qquad \max_{(k,\widetilde{\lambda},\widetilde{\eta})\in K(\xi)} \ \max_{(\xi,\mu,w_1,w_2)\in X_1(k,\widetilde{\lambda},\widetilde{\eta})} \frac{\sum_{i=1}^{k}\lambda_i Re\big[f(\xi,\eta_i)+(\alpha^HA\alpha)^{1/2}+\langle h(\xi),\mu\rangle\big]}{\sum_{i=1}^{k}\lambda_i Re\big[g(\xi,\eta_i)-(\alpha^HB\alpha)^{1/2}\big]}.$$

Here,

(i) $K(\xi)$ stands for a set of points $(k,\widetilde{\lambda},\widetilde{\eta})$ (where $\widetilde{\lambda}=(\lambda_1,\dots,\lambda_k)$ and $\widetilde{\eta}=(\eta_1,\dots,\eta_k)$) satisfying the optimality conditions of problem (P) for any given feasible solution $\xi=(\alpha,\overline{\alpha})\in Q$, then there exists a nonzero multiplier $\mu\in S^*\subset\mathbb{C}^p$ such that $Re\langle v,\mu\rangle\ge 0$ for $v\in S$. Thus $Re\langle h(\xi),\mu\rangle\le 0$ as $-h(\xi)\in S\subset\mathbb{C}^p$.

(ii) The constraint set $X_1(k,\widetilde{\lambda},\widetilde{\eta})$ is the set of all feasible solutions (ξ,μ,w_1,w_2) of (WD), which satisfy the following expressions:

For $\xi=(\alpha,\overline{\alpha})\in Q\subset\mathbb{C}^{2n}$,

$$\Big\{\sum_{i=1}^{k}\lambda_i\big[\overline{\nabla_z f(\xi,\eta_i)}+\nabla_{\overline{z}}f(\xi,\eta_i)\big]+Aw_1+\mu^T\,\overline{\nabla_z h(\xi)}+\mu^H\nabla_{\overline{z}}h(\xi)\Big\}\times$$
$$\Big(\sum_{i=1}^{k}\lambda_i\ \big[g(\xi,\eta_i)-(\alpha^HB\alpha)^{1/2}\big]\Big)-\Big(\sum_{i=1}^{k}\lambda_i\ \big[f(\xi,\eta_i)+(\alpha^HA\alpha)^{1/2}+\langle h(\xi),\mu\rangle\big]\Big)\times$$
$$\Big\{\sum_{i=1}^{k}\lambda_i\big[\overline{\nabla_z g(\xi,\eta_i)}+\nabla_{\overline{z}}g(\xi,\eta_i)\big]-Bw_2\Big\}=0, \tag{4.3}$$

$$Re\langle h(\xi),\mu\rangle\ge 0,\quad \mu\ne 0 \text{ in } S^*, \tag{4.4}$$
$$w_1^HAw_1\le 1,\ (\alpha^HA\alpha)^{1/2}=Re(\alpha^HAw_1), \tag{4.5}$$
$$w_2^HBw_2\le 1,\ (\alpha^HB\alpha)^{1/2}=Re(\alpha^HBw_2), \tag{4.6}$$

The Mond-Weir type dual contains no constraints of problem (P) in the objective fractional functional of (MWD) as the following form:

$$\text{(MWD)} \qquad \max_{(k,\widetilde{\lambda},\widetilde{\eta})\in K(\xi)} \ \max_{(\xi,\mu,w_1,w_2)\in X_2(k,\widetilde{\lambda},\widetilde{\eta})} \frac{\sum_{i=1}^{k}\lambda_i Re\big[f(\xi,\eta_i)+(\alpha^H A\alpha)^{1/2}\big]}{\sum_{i=1}^{k}\lambda_i Re\big[g(\xi,\eta_i)-(\alpha^H B\alpha)^{1/2}\big]}$$

where $K(\xi)$ is the set of points $(k,\widetilde{\lambda},\widetilde{\eta})$ (where $\widetilde{\lambda}=(\lambda_1,\dots,\lambda_k)$ and $\widetilde{\eta}=(\eta_1,\dots,\eta_k)$) which are satisfying to the optimality conditions of (P) for any given feasible solution $\xi=(\alpha,\overline{\alpha})\in Q$, and $X_2(k,\widetilde{\lambda},\widetilde{\eta})$ denotes the set of all $(\xi,\mu,w_1,w_2)\in\mathbb{C}^{2n}\times\mathbb{C}^{p}\times\mathbb{C}^{n}\times\mathbb{C}^{n}$ to satisfy the following conditions:

$$\Big\{\sum_{i=1}^{k}\lambda_i\big[\overline{\nabla_z f(\xi,\eta_i)}+\nabla_{\overline{z}}f(\xi,\eta_i)\big]+Aw_1\Big\}\times\Big(\sum_{i=1}^{k}\lambda_i\ Re\ \big[g(\xi,\eta_i)-(\alpha^H B\alpha)^{1/2}\big]\Big)$$
$$-\Big(\sum_{i=1}^{k}\lambda_i Re\ \big[f(\xi,\eta_i)+(\alpha^H A\alpha)^{1/2}\big]\Big)\times\Big\{\sum_{i=1}^{k}\lambda_i\big[\overline{\nabla_z g(\xi,\eta_i)}+\nabla_{\overline{z}}g(\xi,\eta_i)\big]-Bw_2\Big\}$$
$$+\mu^T\ \overline{\nabla_z h(\xi)}+\mu^H\nabla_{\overline{z}}h(\xi)=0, \tag{4.7}$$
$$Re\langle h(\xi),\mu\rangle\ge 0, \tag{4.8}$$
$$w_1^H Aw_1\le 1,\ w_2^H Bw_2\le 1, \tag{4.9}$$
$$(\alpha^H A\alpha)^{1/2}=Re(\alpha^H Aw_1),\ (\alpha^H B\alpha)^{1/2}=Re(\alpha^H Bw_2), \tag{4.10}$$
$$0\ne\mu\in S^*. \tag{4.11}$$

The main purpose of this paper is to construct a new duality model w.r.t. (P) which include dual problems (WD) and (MWD). In order to construct this new dual problem, we take some notations as follows. The constraint function in (P) is $h(\zeta)=\big(h_1(\zeta),h_2(\zeta),\dots,h_p(\zeta)\big)\in(-S)\subset\mathbb{C}^p$, and the multiplier $\mu=(\mu_1,\dots,\mu_p)\in S^*\subset\mathbb{C}^p$. Now, we partition the index set $P=\{1,\dots,p\}$ of the constraint function $h(\zeta)$ to be $P=P_0\cup P_1\cup\cdots\cup P_t$ such that

$$Re\ \langle\ h_{P_r}(\zeta),\mu_{P_r}\ \rangle\le 0 \text{ for } r=0,1,\dots,t,$$

where $h_{P_r}(\zeta)\equiv\big(h_i(\zeta)\big)_{i\in P_r}$ and $\mu_{P_r}\equiv\big(\mu_i\big)_{i\in P_r}$.

Thus, $Re\ \langle h(\zeta),\mu\ \rangle=Re\ \langle\ h_{P_0}(\zeta),\mu_{P_0}\ \rangle+\sum_{r=1}^{t}Re\ \langle\ h_{P_r}(\zeta),\mu_{P_r}\ \rangle\le 0.$
And for $r=0,\ 1,\dots,\ t$,

$$\langle h_{P_r},\mu_{P_r}\rangle=\sum_{i\in P_r}\mu_i h_i(\zeta),$$
$$Re\langle h'_{P_r}(\zeta_0)(\zeta-\zeta_0),\mu_{P_r}\rangle=Re\big\langle\ z-z_0\ ,\ \mu_{P_r}^T\overline{\nabla_z h_{P_r}(\zeta_0)}+\mu_{P_r}^H\nabla_{\overline{z}}h_{P_r}(\zeta_0)\ \big\rangle, \tag{4.12}$$

where $\mu_{P_r}^T$ stands for transpose of μ_{P_r} and $\mu_{P_r}^H=\overline{\mu_{P_r}^T}$ is the Hermitian of μ_{P_r}.

For equality (4.12), the following is an easy explanation. Suppose that $P_r=\{1,2,3,4\}\subset P$ for some r, and $\zeta=(z,\overline{z})=(z_1,\dots,z_n,\overline{z_1},\dots,\overline{z_n})\in\mathbb{C}^{2n}$, $\zeta_0=(z_0,\overline{z_0})=(z_{0;1},\dots,z_{0;n},\ \overline{z_{0;1}},\dots,\overline{z_{0;n}})\in\mathbb{C}^{2n}$. Thus,

$$h_{P_r}=(h_1(\zeta),h_2(\zeta),h_3(\zeta),h_4(\zeta)),\quad \mu_{P_r}=(\mu_1,\mu_2,\mu_3,\mu_4)$$

and

$$\langle h_{P_r},\mu_{P_r}\rangle=\mu_1h_1(\zeta)+\mu_2h_2(\zeta)+\mu_3h_3(\zeta)+\mu_4h_4(\zeta).$$

$$\langle h'_{P_r}(\zeta_0)(\zeta-\zeta_0),\mu_{P_r}\rangle = \left\langle \Big(\nabla_z h_{P_r}(\zeta_0), \nabla_{\overline{Z}} h_{P_r}(\zeta_0)\Big) \begin{pmatrix} z-z_0 \\ \overline{z-z_0} \end{pmatrix}, \mu_{P_r} \right\rangle$$

$$= \left\langle \begin{pmatrix} \nabla_z h_1(\zeta_0), \nabla_{\overline{Z}} h_1(\zeta_0) \\ \nabla_z h_2(\zeta_0), \nabla_{\overline{Z}} h_2(\zeta_0) \\ \nabla_z h_3(\zeta_0), \nabla_{\overline{Z}} h_3(\zeta_0) \\ \nabla_z h_4(\zeta_0), \nabla_{\overline{Z}} h_4(\zeta_0) \end{pmatrix} \begin{pmatrix} z_1 - z_{0;1} \\ \vdots \\ z_n - z_{0;n} \\ \overline{z_1 - z_{0;1}} \\ \vdots \\ \overline{z_n - z_{0;n}} \end{pmatrix} , \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} \frac{\partial}{\partial z_1} h_1(\zeta_0), \dots, \frac{\partial}{\partial z_n} h_1(\zeta_0), \frac{\partial}{\partial \overline{z}_1} h_1(\zeta_0), \dots, \frac{\partial}{\partial \overline{z}_n} h_1(\zeta_0) \\ \frac{\partial}{\partial z_1} h_2(\zeta_0), \dots, \frac{\partial}{\partial z_n} h_2(\zeta_0), \frac{\partial}{\partial \overline{z}_1} h_2(\zeta_0), \dots, \frac{\partial}{\partial \overline{z}_n} h_2(\zeta_0) \\ \frac{\partial}{\partial z_1} h_3(\zeta_0), \dots, \frac{\partial}{\partial z_n} h_3(\zeta_0), \frac{\partial}{\partial \overline{z}_1} h_3(\zeta_0), \dots, \frac{\partial}{\partial \overline{z}_n} h_3(\zeta_0) \\ \frac{\partial}{\partial z_1} h_4(\zeta_0), \dots, \frac{\partial}{\partial z_n} h_4(\zeta_0), \frac{\partial}{\partial \overline{z}_1} h_4(\zeta_0), \dots, \frac{\partial}{\partial \overline{z}_n} h_4(\zeta_0) \end{pmatrix} \begin{pmatrix} z_1 - z_{0;1} \\ \vdots \\ z_n - z_{0;n} \\ \overline{z_1 - z_{0;1}} \\ \vdots \\ \overline{z_n - z_{0;n}} \end{pmatrix} , \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} \sum_{i=1}^n \frac{\partial}{\partial z_i} h_1(\zeta_0)(z_i - z_{0;i}) + \sum_{i=1}^n \frac{\partial}{\partial \overline{z}_i} h_1(\zeta_0)(\overline{z_i - z_{0;i}}) \\ \sum_{i=1}^n \frac{\partial}{\partial z_i} h_2(\zeta_0)(z_i - z_{0;i}) + \sum_{i=1}^n \frac{\partial}{\partial \overline{z}_i} h_2(\zeta_0)(\overline{z_i - z_{0;i}}) \\ \sum_{i=1}^n \frac{\partial}{\partial z_i} h_3(\zeta_0)(z_i - z_{0;i}) + \sum_{i=1}^n \frac{\partial}{\partial \overline{z}_i} h_3(\zeta_0)(\overline{z_i - z_{0;i}}) \\ \sum_{i=1}^n \frac{\partial}{\partial z_i} h_4(\zeta_0)(z_i - z_{0;i}) + \sum_{i=1}^n \frac{\partial}{\partial \overline{z}_i} h_4(\zeta_0)(\overline{z_i - z_{0;i}}) \end{pmatrix} , \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} \right\rangle$$

Hence,

$$\begin{aligned} \langle h'_{P_r}(\zeta_0)(\zeta-\zeta_0),\mu_{P_r}\rangle &= \textstyle\sum_{i=1}^4 \mu_i \Big[\nabla_z h_i(\zeta_0)(z-z_0) + \nabla_{\overline{z}} h_i(\zeta_0)(\overline{z-z_0})\Big] \\ &= \langle \nabla_z h_{P_r}(\zeta_0)(z-z_0) + \nabla_{\overline{z}} h_{P_r}(\zeta_0)(\overline{z-z_0}), \mu_{P_r}\rangle. \end{aligned}$$

By Lemma 2.4, we have

$$Re\langle h'_{P_r}(\zeta_0)(\zeta-\zeta_0),\mu_{P_r}\rangle = Re\Big\langle\ z - z_0\ ,\ \mu_{P_r}^T \overline{\nabla_z h_{P_r}(\zeta_0)} + \mu_{P_r}^H \nabla_{\overline{z}} h_{P_r}(\zeta_0)\ \Big\rangle.$$

Now, we can construct a mixed type dual (MD) to fractional programming problem (P) by considering the objective of fractional functional added a part of the constraints of (P) with a part of multiplier $\mu \in S^*$ into the numerator of the fractional functional in (P), it means that

$$\text{(MD)} \qquad \max_{(k,\widetilde{\lambda},\widetilde{\eta})\in K(\xi)} \ \max_{(\xi,\mu,w_1,w_2)\in X_3(k,\widetilde{\lambda},\widetilde{\eta})} \frac{\displaystyle\sum_{i=1}^k \lambda_i Re\big[f(\xi,\eta_i) + (\alpha^H A\alpha)^{1/2} + \langle h_{P_0}(\xi), \mu_{P_0}\rangle\big]}{\displaystyle\sum_{i=1}^k \lambda_i Re\big[g(\xi,\eta_i) - (\alpha^H B\alpha)^{1/2}\big]},$$

$$\Big(\equiv \max_{(k,\widetilde{\lambda},\widetilde{\eta})\in K(\xi)} \varphi_{MD}(\widetilde{\xi})\Big)$$

where $\widetilde{\xi} = (\xi,\mu,w_1,w_2) \in X_3(k,\widetilde{\lambda},\widetilde{\eta})$ is the feasible solution of (MD).
Here,

(i) $K(\xi)$ stands for a set of points $(k, \widetilde{\lambda}, \widetilde{\eta})$ (where $\widetilde{\lambda} = (\lambda_1, \dots, \lambda_k)$ and $\widetilde{\eta} = (\eta_1, \dots, \eta_k)$) satisfying the necessary optimality conditions of problem (P) for any given feasible solution $\xi = (\alpha, \overline{\alpha}) \in Q$. Then there exists a nonzero vector multiplier $\mu \equiv (\mu_1, \dots, \mu_p) \in S^*(\subset \mathbb{C}^p)$, the dual cone of the polyhedral cone S in $\mathbb{C}^p$, such that $Re\langle v, \mu\rangle \geq 0$ for $v \in S$. Thus the constraint function $h(\xi) \equiv \big(h_1(\xi), \dots h_p(\xi)\big)$ satisfies $Re\langle h(\xi), \mu\rangle \leq 0$ as $-h(\xi) \in S \subset \mathbb{C}^p$ and $\mu \in S^*$.

(ii) The new constraint set $X_3(k, \widetilde{\lambda}, \widetilde{\eta})$ is the set of all feasible solutions (ξ, μ, w_1, w_2) of (MD) satisfying the following expressions (4.13) $\sim$ (4.16).

That is, the constraints of (MD) are as the following expressions:
For $\xi = (\alpha, \overline{\alpha}) \in Q \subset \mathbb{C}^{2n}$,

$$\Big\{\sum_{i=1}^{k} \lambda_i \big[\overline{\nabla_z f(\xi, \eta_i)} + \nabla_{\overline{z}} f(\xi, \eta_i)\big] + Aw_1 + \mu_{P_0}^T\ \overline{\nabla_z h_{P_0}(\xi)} + \mu_{P_0}^H \nabla_{\overline{z}} h_{P_0}(\xi)\Big\} \times$$
$$\Big(\sum_{i=1}^{k} \lambda_i\ [g(\xi, \eta_i) - (\alpha^H B\alpha)^{1/2}]\Big) - \Big(\sum_{i=1}^{k} \lambda_i\ [f(\xi, \eta_i) + (\alpha^H A\alpha)^{1/2} + \langle h_{P_0}(\xi), \mu_{P_0}\rangle]\Big) \times$$
$$\Big\{\sum_{i=1}^{k} \lambda_i \big[\overline{\nabla_z g(\xi, \eta_i)} + \nabla_{\overline{z}} g(\xi, \eta_i)\big] - Bw_2\Big\} + \sum_{r=1}^{t} \Big(\mu_{P_r}^T\ \overline{\nabla_z h_{P_r}(\xi)} + \mu_{P_r}^H \nabla_{\overline{z}} h_{P_r}(\xi)\Big) = 0, \tag{4.13}$$

$$Re\langle h_{P_r}(\xi), \mu_{P_r}\rangle \geq 0, \quad r = 1, \dots, t, \tag{4.14}$$

$$w_1^H A w_1 \leq 1,\ (\alpha^H A\alpha)^{1/2} = Re(\alpha^H A w_1), \tag{4.15}$$

$$w_2^H B w_2 \leq 1,\ (\alpha^H B\alpha)^{1/2} = Re(\alpha^H B w_2). \tag{4.16}$$

In problem (MD), if the index set P of the constraints in (P) is separated by two parts P_0 and P_1, that is, $P = P_0 \cup P_1$, ($P_r = \emptyset$ for $r = 2, \dots, t$), then
(MD $\equiv$ (WD), when $P_0 = P$ and $P_1 = \emptyset$ and
(MD) $\equiv$ (MWD), when $P_0 = \emptyset$ and $P_1 = P$.

This shows that the Wolfe type dual (WD) and the Mond-Weir type dual (MWD) are the special cases of the mixed type dual (MD).

5 Duality Theorems

For convenient to establish the duality theorems of (MD), we define a function

$$\begin{aligned}\Phi_{MD}(\bullet) \quad &= \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[f(\bullet, \eta_i) + (\cdot)^H Aw_1 + \langle h_{P_0}(\bullet), \mu_{P_0}\rangle\big]\Big) \\ &\times \Big(\textstyle\sum_{i=1}^{k} \lambda_i Re\ \big[g(\xi, \eta_i) - \alpha^H Bw_2\big]\Big) \\ &- \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[f(\xi, \eta_i) + \alpha^H Aw_1 + \langle h_{P_0}(\xi), \mu_{P_0}\rangle\big]\Big) \\ &\times \Big(\textstyle\sum_{i=1}^{k} \lambda_i Re\ \big[g(\bullet, \eta_i) - (\cdot)^H Bw_2\big]\Big)\end{aligned}$$

where $\bullet = (\cdot, \bar{\cdot}) \in \mathbb{C}^{2n}$.

If we get the weak, strong and strict converse duality theorems, then we can know that there are no duality gaps between problems (P) and (MD).

The weak duality theorem means that: under some conditions, the objective value of the primary problem (P) is not less than the objective value of its dual problem (D), we state the theorem as in the following.

Theorem 5.1 (Weak Duality Theorem). *Let $\zeta = (z, \overline{z})$ be (P)-feasible, and $(k, \widetilde{\lambda}, \widetilde{\eta}, \xi, \mu, w_1, w_2)$ be (MD)-feasible. Suppose that any one of the following conditions (i) and (ii) holds:*

(i) $\Phi_{MD}(\bullet)$ *is pseudoconvex at $\xi \in Q$, and $\langle h_{P_r}(\bullet), \mu_{P_r}\rangle$ for $r = 1, \dots, t$ are quasiconvex at $\xi \in Q$,*

(ii) $\Phi_{MD}(\bullet)$ *is quasiconvex at $\xi \in Q$, and $\langle h_{P_r}(\bullet), \mu_{P_r}\rangle$ for $r = 1, \dots, t$ are strictly pseudoconvex at $\xi \in Q$.*

Then

(minimal) objective value (P) $\geq$ (maximal) objective value (MD).

That is,

$$\varphi(\zeta) = \max_{\eta \in Y} \frac{Re\big[f(\zeta, \eta) + (z^H A z)^{1/2}\big]}{Re\big[g(\zeta, \eta) - (z^H B z)^{1/2}\big]} \geq \varphi_{MD}(\widetilde{\xi}).$$

Proof. Suppose on the contrary that there is a $\widetilde{\xi} = (\xi, \mu, w_1, w_2)$ such that

$$\max_{\eta \in Y} \frac{Re\big[f(\zeta, \eta) + (z^H A z)^{1/2}\big]}{Re\big[g(\zeta, \eta) - (z^H B z)^{1/2}\big]} < \varphi_{MD}(\widetilde{\xi}) = \frac{\sum_{i=1}^{k} \lambda_i Re\big[f(\xi, \eta_i) + (\alpha^H A \alpha)^{1/2} + \langle h_{P_0}(\xi), \mu_{P_0}\rangle\big]}{\sum_{i=1}^{k} \lambda_i Re\big[g(\xi, \eta_i) - (\alpha^H B \alpha)^{1/2}\big]}.$$

Then for any $\eta \in Y$,

$$\begin{aligned} &Re\big[f(\zeta, \eta) + (z^H A z)^{1/2}\big] \times \sum_{i=1}^{k} \lambda_i Re\big[g(\xi, \eta_i) - (\alpha^H B \alpha)^{1/2}\big] \\ &\quad - \sum_{i=1}^{k} \lambda_i Re\big[f(\xi, \eta_i) + (\alpha^H A \alpha)^{1/2} + \langle h_{P_0}(\xi), \mu_{P_0}\rangle\big] \times Re\big[g(\zeta, \eta) - (z^H B z)^{1/2}\big] < 0. \end{aligned}$$

We replace η by η_i and multiply λ_i (with $\sum_{i=1}^{k} \lambda_i = 1$). Then the above inequality deduces to

$$\begin{aligned} &\Big\{\sum_{i=1}^{k} \lambda_i Re\big[f(\zeta, \eta_i) + (z^H A z)^{1/2}\big]\Big\} \times \Big\{\sum_{i=1}^{k} \lambda_i Re\big[g(\xi, \eta_i) - (\alpha^H B \alpha)^{1/2}\big]\Big\} \\ &- \Big\{\sum_{i=1}^{k} \lambda_i Re\big[f(\xi, \eta_i) + (\alpha^H A \alpha)^{1/2} + \langle h_{P_0}(\xi), \mu_{P_0}\rangle\big]\Big\} \times \Big\{\sum_{i=1}^{k} \lambda_i Re\big[g(\zeta, \eta_i) - (z^H B z)^{1/2}\big]\Big\} \\ &< 0. \qquad (5.1) \end{aligned}$$

From inequalities (4.15), (4.16) and generalized Schwarz inequality, we obtain

$$Re(z^H A w_1) \leq (z^H A z)^{1/2} (w_1^H A w_1)^{1/2} \leq (z^H A z)^{1/2} \quad \text{and} \qquad (5.2)$$

$$Re(z^H B w_2) \leq (z^H B z)^{1/2} (w_2^H B w_2)^{1/2} \leq (z^H B z)^{1/2}, \qquad (5.3)$$

since $w_1^H Aw_1 \leq 1$ and $w_2^H Bw_2 \leq 1$.

By (4.15), (4.16), (5.2), (5.3) and (5.1), we have

$$\begin{aligned}
&\Phi_{MD}(\zeta)\\
&= \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[f(\zeta,\eta_i) + z^H Aw_1 + \langle h_{P_0}(\zeta), \mu_{P_0}\rangle\big]\Big) \times \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[g(\xi,\eta_i) - \alpha^H Bw_2\big]\Big)\\
&- \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[f(\xi,\eta_i) + \alpha^H Aw_1 + \langle h_{P_0}(\xi), \mu_{P_0}\rangle\big]\Big) \times \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[g(\zeta,\eta_i) - z^H Bw_2\big]\Big)\\
&\leq \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[f(\zeta,\eta_i) + (z^H Az)^{1/2} + \langle h_{P_0}(\zeta), \mu_{P_0}\rangle\big]\Big) \times \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[g(\xi,\eta_i) - \alpha^H Bw_2\big]\Big)\\
&- \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[f(\xi,\eta_i) + \alpha^H Aw_1 + \langle h_{P_0}(\xi), \mu_{P_0}\rangle\big]\Big) \times \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[g(\zeta,\eta_i) - (z^H Bz)^{1/2}\big]\Big)\\
&= \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[f(\zeta,\eta_i) + (z^H Az)^{1/2} + \langle h_{P_0}(\zeta), \mu_{P_0}\rangle\big]\Big) \times \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[g(\xi,\eta_i) - \alpha^H Bw_2\big]\Big)\\
&- \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[f(\xi,\eta_i) + (\alpha^H A\alpha)^{1/2} + \langle h_{P_0}(\xi), \mu_{P_0}\rangle\big]\Big) \times \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[g(\zeta,\eta_i) - (z^H Bz)^{1/2}\big]\Big)\\
&< 0 + \langle h_{P_0}(\zeta), \mu_{P_0}\rangle \times \Big(\sum_{i=1}^{k} \lambda_i Re\ \big[g(\xi,\eta_i) - \alpha^H Bw_2\big]\Big). \qquad (5.4)
\end{aligned}$$

Since $Re\ \langle h_{P_0}(\zeta), \mu_{P_0}\rangle \leq 0$ and $\Big(\sum_{i=1}^{k} \lambda_i Re\ \big[g(\xi,\eta_i) - \alpha^H Bw_2\big]\Big) > 0$, from (5.4), we get

$$\Phi_{MD}(\zeta) < 0 = \Phi_{MD}(\xi). \qquad (5.5)$$

Since $\zeta = (z,\overline{z})$ and $\xi = (\alpha,\overline{\alpha})$ are feasible solutions of (P) and (MD), we have

$$Re\ \langle h_{P_r}(\zeta), \mu_{P_r}\rangle \leq 0 \leq Re\ \langle h_{P_r}(\xi), \mu_{P_r}\rangle \quad r = 1,\ldots,t. \qquad (5.6)$$

If hypothesis (i) holds, $\Phi_{MD}(\bullet)$ is pseudoconvex at ξ and $\langle h_{P_r}(\bullet), \mu_{P_r}\rangle$ for $r = 1,\ldots,t$ are quasiconvex at ξ, then by (5.5) and (5.6), we get

$$Re[\Phi'_{MD}(\xi)(\zeta-\xi)] < 0 \qquad \text{and} \qquad Re\langle h'_{P_r}(\xi)(\xi-\zeta), \mu_{P_r}\rangle \leq 0,\ r = 1,\ldots,t.$$

Thus,

$$\begin{aligned}
&\Big\{ \sum_{i=1}^{k} \lambda_i \big[\overline{\nabla_z f(\xi,\eta_i)} + \nabla_{\overline{z}} f(\xi,\eta_i)\big] + Aw_1 + \mu_{P_0}^T\ \overline{\nabla_z h_{P_0}(\xi)} + \mu_{P_0}^H \nabla_{\overline{z}} h_{P_0}(\xi)\Big\} \times\\
&\Big(\sum_{i=1}^{k} \lambda_i\ \big[g(\xi,\eta_i) - (\alpha^H B\alpha)^{1/2}\big]\Big) - \Big(\sum_{i=1}^{k} \lambda_i\ \big[f(\xi,\eta_i) + (\alpha^H A\alpha)^{1/2} + \langle h_{P_0}(\xi), \mu_{P_0}\rangle\big]\Big) \times\\
&\quad \Big\{ \sum_{i=1}^{k} \lambda_i \big[\overline{\nabla_z g(\xi,\eta_i)} + \nabla_{\overline{z}} g(\xi,\eta_i)\big] - Bw_2\Big\} + \sum_{r=1}^{t} \Big(\mu_{P_r}^T\ \overline{\nabla_z h_{P_r}(\xi)} + \mu_{P_r}^H \nabla_{\overline{z}} h_{P_r}(\xi)\Big)\\
&< 0. \qquad (5.7)
\end{aligned}$$

This contradicts the equality of (4.13).

If hypothesis (ii) holds, from quasiconvexity of $\Phi_{MD}(\bullet)$ at ξ and (5.5), we have

$$Re[\Phi'_{MD}(\xi)(\zeta-\xi)] \leq 0.$$

From $\langle h_{P_1}(\bullet), \mu_{P_1}\rangle$ for $r=1,\dots,t$ are srtictly pseudoconvex at ξ and (5.6), we get

$$Re\langle h'_{P_r}(\xi)(\xi-\zeta), \mu_{P_r}\rangle < 0,\ r=1,\dots,t.$$

This contradicts the equality of (4.13), since we can get the inequality (5.7) again.

Hence the proof is complete. □

Suppose that ζ_0 is an optimal solution of the primary problem (P). Using ζ_0 and the optimality conditions of (P), we can find a feasible solution for its dual problem (D). Furthermore, if we assume that some suitable conditions are fulfilled, then problems (P) and (D) have the same optimal vale ($\min(P) = \max(D)$), and we have the following strong duality theorem.

Theorem 5.2 (Strong Duality Theorem). *Let $\zeta_0 = (z_0, \overline{z_0})$ be an optimal solution of problem (P) satisfying the hypothesis of Theorem 3.1 (Necessary Optimality Conditions). Then there exist $(k, \widetilde{\lambda}, \widetilde{\eta}) \in K(\zeta_0)$ and $(\zeta_0, \mu, w_1, w_2) \in X(k, \widetilde{\lambda}, \widetilde{\eta})$ such that $(k, \widetilde{\lambda}, \widetilde{\eta}, \zeta_0, \mu, w_1, w_2)$ is a feasible solution of the dual problem (MD). If the hypotheses of Theorem 5.1 are fulfilled, then $(k, \widetilde{\lambda}, \widetilde{\eta}, \zeta_0, \mu, w_1, w_2)$ is an optimal solution of (MD), and the two problems (P) and (MD) have the same optimal values.*

Proof. If $\zeta_0 = (z_0, \overline{z_0}) \in Q$ is an optimal solution of problem (P) with optimal value

$$v^* = \varphi(\zeta_0) = \frac{\sum_{i=1}^k \lambda_i Re[f(\zeta_0,\eta_i) + (z_0^H A z_0)^{1/2}]}{\sum_{i=1}^k \lambda_i Re[g(\zeta_0,\eta_i) - (z_0^H B z_0)^{1/2}]},$$

then by Theorem 3.1, there exist $0 \neq \mu \in S^* \subset \mathbb{C}^p$, $w_1, w_2 \in \mathbb{C}^n$ and positive integer k to satisfy the following equality:

$$\Big\{\sum_{i=1}^k \lambda_i \big[\overline{\nabla_z f(\zeta_0,\eta_i)} + \nabla_{\overline{z}} f(\zeta_0,\eta_i)\big] + Aw_1 + \mu^T\ \overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\overline{z}} h(\zeta_0)\Big\}\times$$
$$\Big(\sum_{i=1}^k \lambda_i\ [g(\zeta_0,\eta_i) - (z_0^H B z_0)^{1/2}]\Big) - \Big(\sum_{i=1}^k \lambda_i\ [f(\zeta_0,\eta_i) + (z_0^H A z_0)^{1/2} + \langle h(\zeta_0), \mu\rangle]\Big)\times$$
$$\Big\{\sum_{i=1}^k \lambda_i \big[\overline{\nabla_z g(\zeta_0,\eta_i)} + \nabla_{\overline{z}} g(\zeta_0,\eta_i)\big] - Bw_2\Big\} = 0.$$

Now, let constant $C_g = \Big(\sum_{i=1}^k \lambda_i\ [g(\zeta_0,\eta_i) - (z_0^H B z_0)^{1/2}]\Big)$, and replace μ_{P_r} with $\mu_{P_r} \times C_g$ for $r = 1,\dots,t$. Thus

$$\Big\{\sum_{i=1}^k \lambda_i \big[\overline{\nabla_z f(\zeta_0,\eta_i)} + \nabla_{\overline{z}} f(\zeta_0,\eta_i)\big] + Aw_1 + \mu_{P_0}^T\ \overline{\nabla_z h_{P_0}(\zeta_0)} + \mu_{P_0}^H \nabla_{\overline{z}} h_{P_0}(\zeta_0)\Big\}\times$$
$$\Big(\sum_{i=1}^k \lambda_i\ [g(\zeta_0,\eta_i) - (z_0^H B z_0)^{1/2}]\Big) - \Big(\sum_{i=1}^k \lambda_i\ [f(\zeta_0,\eta_i) + (z_0^H A z_0)^{1/2} + \langle h_{P_0}(\zeta_0), \mu_{P_0}\rangle]\Big)\times$$
$$\Big\{\sum_{i=1}^k \lambda_i \big[\overline{\nabla_z g(\zeta_0,\eta_i)} + \nabla_{\overline{z}} g(\zeta_0,\eta_i)\big] - Bw_2\Big\} + \sum_{r=1}^t \Big(\mu_{P_r}^T\ \overline{\nabla_z h_{P_r}(\zeta_0)} + \mu_{P_r}^H \nabla_{\overline{z}} h_{P_r}(\zeta_0)\Big) = 0.$$

It follows that $(k, \widetilde{\lambda}, \widetilde{\eta}) \in K(\zeta_0)$ and $(\zeta_0, \mu, w_1, w_2) \in X(k, \widetilde{\lambda}, \widetilde{\eta})$ such that $(k, \widetilde{\lambda}, \widetilde{\eta}, \zeta_0, \mu, w_1, w_2)$ is a feasible solution of the dual problem (MD).

If the hypotheses of Theorem 5.1 are also fulfilled, then $(k, \widetilde{\lambda}, \widetilde{\eta}, \zeta_0, \mu, w_1, w_2)$ is an optimal solution of the dual problem (MD). □

Next, we state the strict converse duality theorem.

Theorem 5.3 (Strict Converse Duality Theorem). *Let $\widehat{\zeta}$ and $(\widehat{k}, \widehat{\lambda}, \widehat{\eta}, \widehat{\xi}, \widehat{\mu}, \widehat{w_1}, , \widehat{w_1})$ be optimal solutions of (P) and (MD), respectively, and assume that the assumptions of Theorem 5.2 are fulfilled. If $\Phi_{MD}(\bullet)$ is strictly pseudoconvex at $\xi \in Q$ and $\langle h_{P_r}(\bullet), \mu_{P_r}\rangle$ for $r = 1, \ldots, t$ is quasiconvex at $\xi \in Q$, then $\widehat{\zeta} = \widehat{\xi}$; and the optimal values of (P) and (MD) are equal.*

Proof. Assume that $(\widehat{z}, \overline{\widehat{z}}) = \widehat{\zeta} \neq \widehat{\xi} = (\widehat{\alpha}, \overline{\widehat{\alpha}})$, and reach a contradiction.

By Theorem 5.2, we know that

$$\max_{\eta\in Y} \frac{Re[f(\widehat{\zeta}, \eta) + (\widehat{z}^H A\widehat{z})^{1/2}]}{Re[g(\widehat{\zeta}, \eta) - (\widehat{z}^H B\widehat{z})^{1/2}]} = \frac{\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re\ \left[f(\widehat{\xi}, \widehat{\eta}_i) + (\widehat{\alpha}^H A\widehat{\alpha})^{1/2} + \langle h_{P_0}(\widehat{\xi}), \widehat{\mu_{P_0}}\rangle\right]}{\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re\ \left[g(\widehat{\xi}, \widehat{\eta}_i) - (\widehat{\alpha}^H B\widehat{\alpha})^{1/2}\right]}.$$

Then for each $\eta \in Y$,

$$\frac{Re[f(\widehat{\zeta}, \eta) + (\widehat{z}^H A\widehat{z})^{1/2}]}{Re[g(\widehat{\zeta}, \eta) - (\widehat{z}^H B\widehat{z})^{1/2}]} \leq \frac{\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re\ \left[f(\widehat{\xi}, \widehat{\eta}_i) + (\widehat{\alpha}^H A\widehat{\alpha})^{1/2} + \langle h_{P_0}(\widehat{\xi}), \widehat{\mu_{P_0}}\rangle\right]}{\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re\ \left[g(\widehat{\xi}, \widehat{\eta}_i) - (\widehat{\alpha}^H B\widehat{\alpha})^{1/2}\right]}.$$

That is, for each $\eta \in Y$,

$$\begin{aligned}
&\Big(Re[f(\widehat{\zeta}, \eta) + (\widehat{z}^H A\widehat{z})^{1/2}]\Big) \times \Big\{\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re\ \left[g(\widehat{\xi}, \widehat{\eta}_i) - (\widehat{\alpha}^H B\widehat{\alpha})^{1/2}\right]\Big\}\\
&\Big(Re[g(\widehat{\zeta}, \eta) - (\widehat{z}^H B\widehat{z})^{1/2}]\Big) \times \Big\{\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re\ \left[f(\widehat{\xi}, \widehat{\eta}_i) + (\widehat{\alpha}^H A\widehat{\alpha})^{1/2} + \langle h_{P_0}(\widehat{\xi}), \widehat{\mu_{P_0}}\rangle\right]\Big\}\ \leq 0.
\end{aligned}$$

It implies that

$$\begin{aligned}
&\Big(\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re[f(\widehat{\zeta}, \widehat{\eta}_i) + (\widehat{z}^H A\widehat{z})^{1/2}]\Big) \times \Big\{\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re\ \left[g(\widehat{\xi}, \widehat{\eta}_i) - (\widehat{\alpha}^H B\widehat{\alpha})^{1/2}\right]\Big\}\\
&- \Big(\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re[g(\widehat{\zeta}, \widehat{\eta}_i) - (\widehat{z}^H B\widehat{z})^{1/2}]\Big) \times \Big\{\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re\ \left[f(\widehat{\xi}, \widehat{\eta}_i) + (\widehat{\alpha}^H A\widehat{\alpha})^{1/2} + \langle h_{P_0}(\widehat{\xi}), \widehat{\mu_{P_0}}\rangle\right]\Big\}\\
&\leq 0. \quad (5.8)
\end{aligned}$$

From inequality (5.8), one can easily obtain

$$\Phi_{MD}(\widehat{\zeta}) \leq \Phi_{MD}(\widehat{\xi})$$

since we could used the same lines as the proof of Theorem 5.1.

Now, $\zeta = (z, \overline{z})$ and $\xi = (\alpha, \overline{\alpha})$ are feasible solutions of (P) and (MD), respectively. We then have

$$Re\ \langle h_{P_r}(\zeta), \mu_{P_r}\rangle \le 0 \le Re\ \langle h_{P_r}(\xi), \mu_{P_r}\rangle \quad , \ r = 1, \ldots, t.$$

By hypothesis, $\Phi_{MD}(\bullet)$ is strictly pseudoconvex at ξ and $\langle h_{P_r}(\bullet), \mu_{P_r}\rangle$, $r = 1, \ldots, t$ is quasiconvexat at ξ, it implies that

$$\Big\{\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i \big[\overline{\nabla_z f(\widehat{\xi}, \widehat{\eta}_i)} + \nabla_{\overline{z}} f(\widehat{\xi}, \widehat{\eta}_i)\big] + A\widehat{w}_1 + \widehat{\mu_{P_0}}^T\ \overline{\nabla_z h_{P_0}(\widehat{\xi})} + \widehat{\mu_{P_0}}^H \nabla_{\overline{z}} h_{P_0}(\widehat{\xi})\Big\}\times$$
$$\Big(\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i\ [g(\widehat{\xi}, \widehat{\eta}_i) - (\widehat{\alpha}^H B\widehat{\alpha})^{1/2}]\Big) - \ \Big(\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i\ [f(\widehat{\xi}, \widehat{\eta}_i) + (\widehat{\alpha}^H A\widehat{\alpha})^{1/2} + \langle h_{P_0}(\widehat{\xi}), \widehat{\mu_{P_0}}\rangle]\Big)\times$$
$$\Big\{\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i \big[\overline{\nabla_z g(\widehat{\xi}, \widehat{\eta}_i)} + \nabla_{\overline{z}} g(\widehat{\xi}, \widehat{\eta}_i)\big] - B\widehat{w}_2\Big\} + \sum_{r=1}^{t}\Big(\widehat{\mu_{P_r}}^T\ \overline{\nabla_z h_{P_r}(\widehat{\xi})} + \widehat{\mu_{P_r}}^H \nabla_{\overline{z}} h_{P_r}(\widehat{\xi})\Big)$$
$$< 0$$

which contradicts the equality of (4.13). Hence the proof is complete. □

6 Conclusions

In this paper, we construct a mixed type dual problem (MD) for a nondifferentiable minimax fractional complex problem (P). The merit of problem (MD) is that it can include the wolfe type dual problem (WD) and Mond-Weir type dual problem (MWD) of problem (P). Furthermore, we have proved the weak, strong and strict converse duality theorems of (MD).

References

[1] I. Ahmand, Optimality conditions and mixed duality in differentiable programming, *J. Nonlinear Convex Anal.* 5 (2004) 71–83.

[2] C.R. Bector, On mixed duality in mathematical programming, *J. Math. Anal. Appl.* 259 (2001) 346–356.

[3] B. Mond and T. Weir, Generalized concavity and duality, in *Generalized Concavity in Optimization and Economics*, S. Schaible, W.T. Ziemba (eds.), Academic Press, New York, 1981, pp. 263–279.

[4] H.C. Lai and J.C. Lee, On duality theorems for a nondifferentiable minimax fractional programming, *J. Computational Appl. Math.* 146 (2002) 115–126.

[5] H.C. Lai, J.C. Lee and S.C. Ho, Parametric duality on minimax programming involving generalized convexity in complex space, *J. Math. Anal. Appl.* 323 (2006) 1104–1115.

[6] H. C. Lai and J. C. Liu, Duality for nondifferentiable minimax programming in complex spaces, *Nonlinear Anal.* 71 (2009) e224–233.

[7] H.C. Lai and J.C. Liu, Complex fractional programming involving generalized quasi/pseudo convex functions, *ZAMM Z. Angew. Math. Mech.* 8 (2002), 159–166.

[8] H.C. Lai, J.C. Liu and S. Schaible, Complex minimax fractional programming of analytic functions, *J. Optim. Theory Appl.* 137 (2008) 171–184.

[9] H.C. Lai and T.Y. Huang, Optimality conditions for a nondifferentiable minimax programming in complex spaces, *Nonlinear Anal.* 71 (2009) 1205–1212.

[10] H.C. Lai and T.Y. Huang, Optimality conditions for nondifferentiable minimax fractional programming with complex variables, *J. Math. Anal. Appl.* 359 (2009) 229–239.

[11] H.C. Lai and T.Y. Huang, Complex Analysis Methods Related an Optimization Problem with Complex Variables, *Euro. J. Pure Appl. Math.* 3 (2010) 989–1005.

[12] H.C. Lai and T.Y. Huang, Nondifferentiable minimax fractional programming in complex spaces with parametric duality, *J. Global Optim.* 53 (2012) 243–254.

[13] H.C. Lai and T.Y. Huang, Mixed type Duality for a Nondifferentiable Minimax Fractional Complex Programming, *Proceedings of NAO-Asia2012, Matsue, Japan*, Yokohama Publishes, 2012, preprint.

[14] P. Wolfe, A duality theorem for nonlinear programming, *Q. Appl. Math.* 19 (1961) 239–244.

Manuscript received 22 November 2012
revised 10 April 2013
accepted for publication 8 May 2013

Hang-Chin Lai
Department of Mathematics, National Tsing Hua University
Hsing-Chu, Taiwan
E-mail address: laihc@mx.nthu.edu.tw

Tone-Yau Huang
Department of Applied Mathematics, Feng-Chia University
Tai-Chung, Taiwan
E-mail address: huangty@fcu.edu.tw