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AN APPLICATION OF BORSUK-ULAM'S THEOREM TO PARAMETRIC OPTIMIZATION

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ABSTRACT. Borsuk-Ulam's theorem is a useful tool of algebraic topology. It states that for any continuous mapping f from the n-sphere S^n to $\mathbb{R}^n$, there exists a pair of antipodal points such that $f(x) = f(-x)$. As for its applications, ham-sandwich theorem, necklace theorem and coloring of Kneser graph by Lovász [3] are well-known. Recently [2] applied Borsuk-Ulam's theorem to an n-tuple of parametric optimization problems with parameter $u \in S^n$. This paper sharpens the results of [2].

1. Introduction

Borsuk-Ulam's theorem [1] is an important theorem of algebraic topology. It states that for any continuous mapping f from the n-sphere S^n to the Euclidean space $\mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(x) = f(-x)$. It has several equivalent statements: Tucker's lemma is a combinatorial version and LSB theorem is a set-cover version, see e.g. Matoušek [4]. This is reminiscent of Brouwer's fixed point theorem, which also has many equivalent statements: Sperner's lemma is a combinatorial version and KKM lemma is a set-cover version. Borsuk-Ulam's theorem implies Brouwer's fixed point theorem. However, the converse is unknown for 100 years. In this sense, Borsuk-Ulam's theorem seems stronger than Brouwer's fixed point theorem.

Ham-sandwich theorem is one of the most famous applications of Borsuk-Ulam's theorem. Let $A_1, \ldots, A_n \subset \mathbb{R}^n$ be compact sets with positive Lebesgue measure μ. Then ham-sandwich theorem states that there is a hyperplane H which divides each A_i in half, that is, $\mu(A_i \cap H^+) = \mu(A_i \cap H^-)$ for any $i = 1, \ldots, n$, where H^+ and H^- denote closed half spaces determined by H.

Recently in [2] we applied Borsuk-Ulam's theorem to an n-tuple of parametric optimization problems with parameter $u \in S^n$ by using a technique of ham-sandwich theorem, and presented Theorems 1.1 and 1.2 below.

Before quoting them, we explain our notations. For any $\boldsymbol{u} = (u_1, \ldots, u_{n+1}) \in S^n$, we write $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $\boldsymbol{u} = (u, u_{n+1})$. We assign to $\boldsymbol{u} \in S^n$ a

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hyperplane $H_{\boldsymbol{u}} = \{x \in \mathbb{R}^n \mid \langle u, x\rangle = u_{n+1}\}$ and two closed half-spaces:

$$H_{\boldsymbol{u}}^+ = \{x \in \mathbb{R}^n \mid \langle u, x\rangle \geq u_{n+1}\},\ H_{\boldsymbol{u}}^- = \{x \in \mathbb{R}^n \mid \langle u, x\rangle \leq u_{n+1}\},$$

where $\langle u, x\rangle$ denotes the inner product $u_1x_1+\cdots+u_nx_n$. It is clear that $H_{-\boldsymbol{u}}^+ = H_{\boldsymbol{u}}^-$. In the case of $u \neq \mathbf{0}$, both $H_{\boldsymbol{u}}^+$ and $H_{\boldsymbol{u}}^-$ are non-empty. In the case of $u = \mathbf{0}$, one of $H_{\boldsymbol{u}}^+$ and $H_{\boldsymbol{u}}^-$ is $\mathbb{R}^n$, and the other is empty.

We assume the following strict convexity of A_i in Theorem 1.1 and Sections 2.

(SC) $A_i \cap H = \{x\}$ for any boundary point x of A_i and for any supporting hyperplane H of A_i at x.

Theorem 1.1 ([2]). *Let $A_i \subset \mathbb{R}^n$ be a compact convex set whose interior is non-empty, and Q_i be a non-singular matrix of order n for any $i = 1, \ldots, n$. Assume (SC) for any $i = 1, \ldots, n$. Then there exists some $\boldsymbol{u} \in S^n$ such that both $A_i \cap H_{\boldsymbol{u}}^+$ and $A_i \cap H_{\boldsymbol{u}}^-$ are non-empty, and*

$$\max_{x\in A_i\cap H_{\boldsymbol{u}}^+}\langle u, Q_ix\rangle - \min_{x\in A_i\cap H_{\boldsymbol{u}}^+}\langle u, Q_ix\rangle = \max_{x\in A_i\cap H_{\boldsymbol{u}}^-}\langle u, Q_ix\rangle - \min_{x\in A_i\cap H_{\boldsymbol{u}}^-}\langle u, Q_ix\rangle. \tag{1.1}$$

Theorem 1.2 ([2]). *Let $A_i \subset \mathbb{R}^n$ be a compact set whose convex hull has a non-empty interior for any $i = 1, \ldots, n$. Then there exists some $\boldsymbol{u} \in S^n$ such that both $A_i \cap H_{\boldsymbol{u}}^+$ and $A_i \cap H_{\boldsymbol{u}}^-$ are non-empty and*

$$\delta^*(u \mid \mathrm{co}A_i\cap H_{\boldsymbol{u}}^+)-\delta_*(u \mid \mathrm{co}A_i\cap H_{\boldsymbol{u}}^+) = \delta^*(u \mid \mathrm{co}A_i\cap H_{\boldsymbol{u}}^-)-\delta_*(u \mid \mathrm{co}A_i\cap H_{\boldsymbol{u}}^-) \tag{1.2}$$

for all $i = 1, \ldots, n$, where $\mathrm{co}A_i$ denotes the convex hull of A_i, and

$$\delta^*(u \mid X) := \max_{x\in X}\langle u, x\rangle,\ \delta_*(u \mid X) := \min_{x\in X}\langle u, x\rangle.$$

In this paper, we extend Theorem 1.1 to any antipodal function $f_i(x, \boldsymbol{u})$ w.r.t. $\boldsymbol{u}$, that is, $f_i(x, -\boldsymbol{u}) = -f_i(x, \boldsymbol{u})$ for any $(x, \boldsymbol{u}) \in \mathbb{R}^n \times S^n$. Further we remove some assumptions on the interior of A_i from Theorems 1.1 and 1.2.

In Section 2, we introduce n-tuple of parametric optimization problems with parameter $\boldsymbol{u} \in S^n$, and show the continuity of its optimal-value function w.r.t. $\boldsymbol{u}$. By applying Borsuk-Ulam's theorem to the optimal-value functions, we obtain the main theorem (Theorem 2.3).

2. Parametric optimal-value functions

In this section, we consider a family of parametric optimization problems, and show the continuity of optimal-value functions.

Let A_i be a non-empty compact convex subset of $\mathbb{R}^n$ and $f_i : \mathbb{R}^n \times S^n \to \mathbb{R}$ be a continuous function for any $i = 1, \ldots, n$. We consider n-tuple of optimal-value functions $\varphi = (\varphi_1, \ldots, \varphi_n) : S^n \to \mathbb{R}^n$:

$$\varphi_i(\boldsymbol{u}) := \begin{cases} \max\limits_{x\in A_i\cap H_{\boldsymbol{u}}^+} f_i(x, \boldsymbol{u}) - \min\limits_{x\in A_i\cap H_{\boldsymbol{u}}^+} f_i(x, \boldsymbol{u}) & (A_i \cap H_{\boldsymbol{u}}^+ \neq \emptyset), \\ \delta^*(u \mid A_i) - u_{n+1} & (A_i \cap H_{\boldsymbol{u}}^+ = \emptyset). \end{cases} \tag{2.1}$$

The essential part of (2.1) is the first case. The second case comes from the fact that $A_i \cap H_{\boldsymbol{u}}^+ = \emptyset$ if and only if $\delta^*(u \mid A_i) - u_{n+1} < 0$, see Lemma 2.1 below. We note that we defined $\varphi_i(\boldsymbol{u}) = 0$ for $A_i \cap H_{\boldsymbol{u}}^+ = \emptyset$ in [2]. The present φ_i works better than the old one.

Set U_i^-, U_i^+ and U_i^0 as follows.

$$U_i^- := \{\boldsymbol{u} \in \mathbb{R}^{n+1} \mid \delta^*(u \mid A_i) < u_{n+1}\}, \tag{2.2}$$

$$U_i^+ := \{\boldsymbol{u} \in \mathbb{R}^{n+1} \mid \delta^*(u \mid A_i) > u_{n+1}\}, \tag{2.3}$$

$$U_i^0 := \{\boldsymbol{u} \in \mathbb{R}^{n+1} \mid \delta^*(u \mid A_i) = u_{n+1}\}. \tag{2.4}$$

Since A_i is a compact convex set, $\delta^*(u \mid A_i) - u_{n+1}$ is a continuous convex function of $\boldsymbol{u} = (u, u_{n+1})$. Hence U_i^- is open and convex, U_i^+ is open, and U_i^0 is closed.

Lemma 2.1.

(1) *$\varphi_i(\boldsymbol{u}^*) = -1$ at the north pole $\boldsymbol{u}^* := (0, \dots, 0, 1) \in S^n$.*

(2) *$A_i \cap H_{\boldsymbol{u}}^+$ is empty if and only if $\boldsymbol{u} \in U_i^-$.*

Further, under the assumption (SC), it holds that

(3) *If $\delta^*(u \mid A_i) = u_{n+1}$, then $A_i \cap H_{\boldsymbol{u}}^+$ is a singleton and $\varphi_i(\boldsymbol{u}) = 0$. For any converging sequence $\boldsymbol{u}^k$ to $\boldsymbol{u}$, the diameter of $A_i \cap H_{\boldsymbol{u}^k}^+$ converges to 0.*

(4) *φ_i is continuous on S^n.*

Proof. Since $A_i \cap H_{\boldsymbol{u}^*}^+ = \{x \in A_i \mid \langle 0, x \rangle \geq 1\} = \emptyset$ at the north pole $\boldsymbol{u}^* = (\mathbf{0}, 1)$, and since $\delta^*(\mathbf{0} \mid A_i) - 1 = -1$, (1) is apparent from the second case of (2.1).

(2) Since A_i is a compact convex set, it holds that

$$A_i \cap H_{\boldsymbol{u}}^+ = \emptyset \;\Leftrightarrow\; \max\{\langle u, x \rangle \mid x \in A_i\} < u_{n+1} \;\Leftrightarrow\; \boldsymbol{u} \in U_i^-.$$

(3) If $\delta^*(u \mid A_i) = u_{n+1}$, then it follows from (2) that $A_i \cap H_{\boldsymbol{u}}^+$ is non-empty. Since A_i is compact, there exists $x^* \in A_i$ such that $\langle u, x^* \rangle = \delta^*(u \mid A_i) = u_{n+1}$. Hence $H := \{x \in \mathbb{R}^n \mid \langle u, x \rangle = u_{n+1}\}$ is a supporting hyperplane of A_i at x^*. By (SC), $A_i \cap H$ is a singleton. Since $A_i \cap H_{\boldsymbol{u}}^+ = A_i \cap H = \{x^*\}$, we have

$$\varphi_i(\boldsymbol{u}) = \max_{x \in A_i \cap H_{\boldsymbol{u}}^+} f_i(x, \boldsymbol{u}) - \min_{x \in A_i \cap H_{\boldsymbol{u}}^+} f_i(x, \boldsymbol{u}) = 0.$$

Deny the latter half assertion, then there exist a sequence $\boldsymbol{u}^k$ converging to $\boldsymbol{u}$ and $\delta > 0$ such that $\operatorname{diam}(A_i \cap H_{\boldsymbol{u}^k}^+) \geq \delta$. That is, there exist y^k, $z^k \in A_i$ such that

$$\| y^k - z^k \| \geq \delta, \quad \langle u^k, y^k \rangle \geq u_{n+1}^k, \quad \langle u^k, z^k \rangle \geq u_{n+1}^k. \tag{2.5}$$

By compactness of A_i, we may assume that y^k and z^k converge to some y, $z \in A_i$, respectively. Taking $k \to \infty$ in (2.5), we have

$$\| y - z \| \geq \delta, \quad \langle u, y \rangle \geq u_{n+1}, \quad \langle u, z \rangle \geq u_{n+1}.$$

Therefore, $A_i \cap H_{\boldsymbol{u}}^+$ includes distinct points y and z, which contradicts the first assertion.

(4) Since $\max\{f_i(x, \boldsymbol{u}) \mid x \in A_i \cap H_{\boldsymbol{u}}^+\} - \min\{f_i(x, \boldsymbol{u}) \mid x \in A_i \cap H_{\boldsymbol{u}}^+\}$ is continuous on the open set U_i^+, and since $\delta^*(u \mid A_i) - u_{n+1}$ is continuous on the open set U_i^-, it suffices to prove that φ_i is continuous at any $\boldsymbol{u} \in U_i^0$. Assume that $\boldsymbol{u}^k = (u^k, u_{n+1}^k)$ converges to $\boldsymbol{u}$. Since $\varphi_i(\boldsymbol{u}) = 0$, we may assume that $\varphi_i(\boldsymbol{u}^k) \neq 0$ for all k.

(i) If $\varphi_i(\boldsymbol{u}^k) < 0$, then

$$\varphi_i(\boldsymbol{u}^k) = \delta^*(u^k \mid A_i) - u^k_{n+1} \to \delta^*(u \mid A_i) - u_{n+1} = 0 = \varphi_i(\boldsymbol{u}).$$

(ii) If $\varphi_i(\boldsymbol{u}^k) > 0$, then

$$\varphi_i(\boldsymbol{u}^k) = \max_{x \in A_i \cap H^+_{\boldsymbol{u}^k}} f_i(x, \boldsymbol{u}^k) - \min_{x \in A_i \cap H^+_{\boldsymbol{u}^k}} f_i(x, \boldsymbol{u}^k).$$

Since f_i is uniformly continuous on the compact set $A_i \times S^n$, and since the diameter of $A_i \cap H^+_{\boldsymbol{u}^k}$ tends to 0, we see for any $\varepsilon > 0$

$$|f_i(y, \boldsymbol{u}^k) - f_i(z, \boldsymbol{u}^k)| < \varepsilon \quad (y,\, z \in A_i \cap H^+_{\boldsymbol{u}^k})$$

for all sufficiently large k. Therefore

$$0 \leq \max_{x \in A_i \cap H^+_{\boldsymbol{u}^k}} f_i(x, \boldsymbol{u}^k) - \min_{x \in A_i \cap H^+_{\boldsymbol{u}^k}} f_i(x, \boldsymbol{u}^k) < \varepsilon,$$

that is, $|\varphi_i(\boldsymbol{u}^k) - \varphi_i(\boldsymbol{u})| = |\varphi_i(\boldsymbol{u}^k)| < \varepsilon$. □

Lemma 2.2. *Assume that $f_i(x, \boldsymbol{u})$ is antipodal w.r.t. $\boldsymbol{u}$ for any $x \in \mathbb{R}^n$. Then*

$$\varphi_i(-\boldsymbol{u}) = \begin{cases} \max\limits_{x \in A_i \cap H^-_{\boldsymbol{u}}} f_i(x, \boldsymbol{u}) - \min\limits_{x \in A_i \cap H^-_{\boldsymbol{u}}} f_i(x, \boldsymbol{u}) & (A_i \cap H^-_{\boldsymbol{u}} \neq \emptyset), \\ u_{n+1} - \delta_*(u \mid A_i) & (A_i \cap H^-_{\boldsymbol{u}} = \emptyset). \end{cases} \tag{2.6}$$

Proof. Since $H^+_{-\boldsymbol{u}} = H^-_{\boldsymbol{u}}$, $A_i \cap H^-_{\boldsymbol{u}}$ is non-empty if and only if $A_i \cap H^+_{-\boldsymbol{u}}$ is non-empty. Then it follows from definition of φ_i that

$$\begin{aligned} \varphi_i(-\boldsymbol{u}) &= \max_{x \in A_i \cap H^+_{-\boldsymbol{u}}} f_i(x, -\boldsymbol{u}) - \min_{x \in A_i \cap H^+_{-\boldsymbol{u}}} f_i(x, -\boldsymbol{u}) \\ &= \max_{x \in A_i \cap H^-_{\boldsymbol{u}}} (-f_i(x, \boldsymbol{u})) - \min_{x \in A_i \cap H^-_{\boldsymbol{u}}} (-f_i(x, \boldsymbol{u})) \\ &= \max_{x \in A_i \cap H^-_{\boldsymbol{u}}} f_i(x, \boldsymbol{u}) - \min_{x \in A_i \cap H^-_{\boldsymbol{u}}} f_i(x, \boldsymbol{u}). \end{aligned}$$

If $A_i \cap H^-_{\boldsymbol{u}}$ is empty, then $A_i \cap H^+_{-\boldsymbol{u}}$ is empty. Hence, by definition of φ_i, we have

$$\varphi_i(-\boldsymbol{u}) = \delta^*(-u \mid A_i) + u_{n+1} = u_{n+1} - \delta_*(u \mid A_i).$$

□

By applying Borsuk-Ulam's theorem to $\varphi : S^n \to \mathbb{R}^n$, we obtain the following, which is an extension of Theorem 1.1.

Theorem 2.3. *Assume that $f_i(x, \boldsymbol{u})$ is antipodal w.r.t. $\boldsymbol{u}$ and (SC) is satisfied for any $i = 1, \ldots, n$. Then there exists some $\boldsymbol{u} \in S^n$ such that both $A_i \cap H^+_{\boldsymbol{u}}$ and $A_i \cap H^-_{\boldsymbol{u}}$ are non-empty, and*

$$\max_{x \in A_i \cap H^+_{\boldsymbol{u}}} f_i(x, \boldsymbol{u}) - \min_{x \in A_i \cap H^+_{\boldsymbol{u}}} f_i(x, \boldsymbol{u}) = \max_{x \in A_i \cap H^-_{\boldsymbol{u}}} f_i(x, \boldsymbol{u}) - \min_{x \in A_i \cap H^-_{\boldsymbol{u}}} f_i(x, \boldsymbol{u}) \tag{2.7}$$

for any $i = 1, \ldots, n$.

Proof. By Borsuk-Ulam's theorem, there exists $\boldsymbol{u} \in S^n$ such that $\varphi(\boldsymbol{u}) = \varphi(-\boldsymbol{u})$. Suppose that $A_i \cap H_{\boldsymbol{u}}^+$ is empty for some i. Then Lemma 2.1 (2) implies $\varphi_i(\boldsymbol{u}) < 0$. On the other hand, Since $A_i \cap H_{\boldsymbol{u}}^- = A_i \neq \emptyset$, we see from (2.6) that $\varphi_i(-\boldsymbol{u}) \geq 0$, which contradicts $\varphi_i(\boldsymbol{u}) = \varphi_i(-\boldsymbol{u})$. Hence $A_i \cap H_{\boldsymbol{u}}^+$ is non-empty. Similarly, $A_i \cap H_{\boldsymbol{u}}^-$ is non-empty. Therefore (2.7) is a direct consequence of (2.1) and (2.6). □

3. Special case of $f_i(x, \boldsymbol{u})$

In Section 2, we required the strict convexity (SC) for A_i to guarantee the continuity of φ_i. When we take $f_i(x, \boldsymbol{u}) = \langle u, x \rangle$, we do not need (SC).

Lemma 3.1. *When we take $f_i(x, \boldsymbol{u}) = \langle u, x \rangle$ for any $i = 1, \ldots, n$, φ_i is continuous on the whole S^n without assuming (SC).*

Proof. It suffices to prove that φ_i is continuous at any $\boldsymbol{u} = (u, u_{n+1})$ such that $\varphi_i(\boldsymbol{u}) = 0$. Then, it follows from Lemma 2.1 (2) that $A_i \cap H_{\boldsymbol{u}}^+$ is non-empty. So, by definition of φ_i, we have

$$\max_{x \in A_i \cap H_{\boldsymbol{u}}^+} \langle u, x \rangle - \min_{x \in A_i \cap H_{\boldsymbol{u}}^+} \langle u, x \rangle = \varphi_i(\boldsymbol{u}) = 0,$$

which implies that $\langle u, x \rangle$ is constant on $A_i \cap H_{\boldsymbol{u}}^+ = \{x \in A_i \mid \langle u, x \rangle \geq u_{n+1}\}$.

Now, assume that $\boldsymbol{u}^k = (u^k, u_{n+1}^k)$ converges to $\boldsymbol{u}$. Since $\varphi_i(\boldsymbol{u}) = 0$, it's enough to consider k such that $\varphi_i(\boldsymbol{u}^k) \neq 0$.

(i) If there are infinitely many k such that $\varphi_i(\boldsymbol{u}^k) > 0$, then it holds that

$$\varphi_i(\boldsymbol{u}^k) = \max_{x \in A_i \cap H_{\boldsymbol{u}^k}^+} \langle u^k, x \rangle - \min_{x \in A_i \cap H_{\boldsymbol{u}^k}^+} \langle u^k, x \rangle = \langle u^k, x^k \rangle - \langle u^k, y^k \rangle \tag{3.1}$$

for some x^k, $y^k \in A_i \cap H_{\boldsymbol{u}^k}$. By taking subsequences, we may assume that x^k and y^k converge to some $x^* \in A_i$ and $y^* \in A_i$, respectively. Taking $k \to \infty$ in (3.1), we have

$$\lim_{k \to \infty} \varphi_i(\boldsymbol{u}^k) = \langle u, x^* \rangle - \langle u, y^* \rangle. \tag{3.2}$$

Also, since $\langle u^k, x^k \rangle \geq u_{n+1}^k$, we have $\langle u, x^* \rangle \geq u_{n+1}$, that is, $x^* \in A_i \cap H_{\boldsymbol{u}}^+$. Similarly we have $y^* \in A_i \cap H_{\boldsymbol{u}}^+$. Since $\langle u, x \rangle$ is constant on $A_i \cap H_{\boldsymbol{u}}^+$, it follows from (3.2) that $\varphi_i(\boldsymbol{u}^k) \to 0 = \varphi_i(\boldsymbol{u})$.

(ii) If there are infinitely many k such that $\varphi_i(\boldsymbol{u}^k) < 0$, then by definition of φ_i, it holds that

$$\varphi_i(\boldsymbol{u}^k) = \delta^*(\boldsymbol{u}^k \mid A_i) - u_{n+1}^k \to \delta^*(\boldsymbol{u} \mid A_i) - u_{n+1}. \tag{3.3}$$

Hence $\delta^*(\boldsymbol{u} \mid A_i) - u_{n+1} \leq 0$. On the other hand, taking $x^0 \in A_i \cap H_{\boldsymbol{u}}^+ \subset A_i$, we have $\delta^*(\boldsymbol{u} \mid A_i) - u_{n+1} \geq \langle u, x^0 \rangle - u_{n+1} \geq 0$. Therefore

$$\varphi_i(\boldsymbol{u}^k) \to \delta^*(\boldsymbol{u} \mid A_i) - u_{n+1} = 0 = \varphi_i(\boldsymbol{u}).$$

□

Theorem 3.2. *Let $A_i \subset \mathbb{R}^n$ $(i = 1, \ldots, n)$ be non-empty compact convex sets. Then there exists some $\boldsymbol{u} \in S^n$ such that both $A_i \cap H_{\boldsymbol{u}}^+$ and $A_i \cap H_{\boldsymbol{u}}^-$ are non-empty and*

$$\delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^+) - \delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^+) = \delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^-) - \delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^-) \tag{3.4}$$

for all $i = 1, \ldots, n$.

Proof. Take $f_i(x, \boldsymbol{u}) = \langle u, x\rangle$ for any $i = 1, \ldots, n$. Since φ_i is continuous by Lemma 3.1, we obtain (2.7) as well as Theorem 2.3. Then LHS of (2.7) turns into

$$\varphi_i(\boldsymbol{u}) = \max_{x \in A_i \cap H_{\boldsymbol{u}}^+} \langle u, x\rangle - \min_{x \in A_i \cap H_{\boldsymbol{u}}^+} \langle u, x\rangle = \delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^+) - \delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^+),$$

and RHS of (2.7) turns into

$$\varphi_i(-\boldsymbol{u}) = \max_{x \in A_i \cap H_{\boldsymbol{u}}^-} \langle u, x\rangle - \min_{x \in A_i \cap H_{\boldsymbol{u}}^-} \langle u, x\rangle = \delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^-) - \delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^-).$$

Therefore, we obtain (3.4). □

Example 3.3. $n = 1$. Take $f_1(x_1, \boldsymbol{u}) = u_1 x_1$ and $A_1 = [-1, 1]$. Representing $\boldsymbol{u} \in S^1$ in polar coordinates as $\boldsymbol{u} = (\cos\theta, \sin\theta)$ $(-\pi/4 \le \theta \le 7\pi/4)$, we have

$$A_1 \cap H_{\boldsymbol{u}}^+ = \{x_1 \in [-1, 1] \mid x_1 \cos\theta \ge \sin\theta\} = \begin{cases} [\tan\theta, 1] & (-\pi/4 \le \theta \le \pi/4) \\ \emptyset & (\pi/4 < \theta < 3\pi/4) \\ [-1, \tan\theta] & (3\pi/4 \le \theta \le 5\pi/4) \\ [-1, 1] & (5\pi/4 \le \theta \le 7\pi/4). \end{cases}$$

$$\delta^*(u_1 \mid A_1) - u_2 = \begin{cases} |u_1| - u_2 & (u_1 \ne 0) \\ -u_2 & (u_1 = 0) \end{cases} = |u_1| - u_2 = |\cos\theta| - \sin\theta.$$

$$\delta^*(u_1 \mid A_1) - \delta_*(u_1 \mid A_1) = \begin{cases} \cos(\theta) - \sin(\theta) & (-\pi/4 \le \theta \le \pi/4) \\ \sin(\theta) - \cos(\theta) & (3\pi/4 \le \theta \le 5\pi/4) \\ 2|\cos(\theta)| & (5\pi/4, 7\pi/4). \end{cases}$$

Therefore, the optimal-value function (2.1) turns into

$$\varphi_1(\boldsymbol{u}) = \begin{cases} |\cos(\theta) - \sin(\theta)| & \text{on } [-\pi/4, \pi/4] \cup [3\pi/4, 5\pi/4] \\ |\cos\theta| - \sin\theta & \text{on } [\pi/4, 3\pi/4] \\ 2|\cos(\theta)| & \text{on } [5\pi/4, 7\pi/4]. \end{cases}$$

There is a pair of antipodal points where φ_1 has the same value. For $\boldsymbol{u} = (1, 0)$ corresponding to $\theta = 0$, π, it holds that $\varphi_1(\boldsymbol{u}) = \varphi_1(-\boldsymbol{u}) = 1$.

We have just removed the assumption of the strict convexity of A_i. By taking its convex hull, we do not need to require convexity either. So A_i can be finite.

Corollary 3.4. *Let $A_i \subset \mathbb{R}^n$ $(i = 1, \ldots, n)$ be non-empty compact sets. Then there exists some $\boldsymbol{u} \in S^n$ such that both $A_i \cap H_{\boldsymbol{u}}^+$ and $A_i \cap H_{\boldsymbol{u}}^-$ are non-empty and*

$$\delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^+) - \delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^+) = \delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^-) - \delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^-) \tag{3.5}$$

for all $i = 1, \ldots, n$.

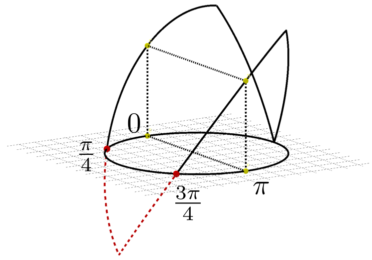

FIGURE 1. The graph of φ_1 on S^1. The doted curve corresponds to the case $A_1 \cap H_{\boldsymbol{u}}^+ = \emptyset$.

Proof. This is a combination of Theorem 3.2,

$$\delta^*(u \mid \mathrm{co}A_i \cap H_{\boldsymbol{u}}^{\pm}) = \delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^{\pm})$$

and

$$\delta_*(u \mid \mathrm{co}A_i \cap H_{\boldsymbol{u}}^{\pm}) = \delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^{\pm}).$$

□

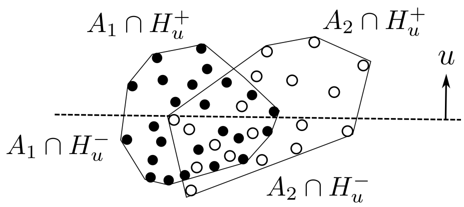

FIGURE 2. This figure represents the equal division of the widths of data A_1 and A_2. The black points represent data from A_1, and the white points represent data from A_2.

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