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SKEWNESS OF BANAŚ-FRĄCZEK SPACE

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Abstract. Let $s(X)$ be the skewness of a Banach space X. In this paper, we compute the $s(X)$-constant for X being Banaś-Frączek space $\mathbb{R}^2_\lambda$, where $\lambda > 1$. Moreover, it is shown that the inequality $s(X) \leq 2\rho_X(1)$ is strict for such a space X, where ρ_X is the modulus of smoothness of X.

1. Introduction and preliminaries

Throughout this paper, let X be a real Banach space with $\dim X \geq 2$, $S_X = \{x \in X : \|x\| = 1\}$ and $B_X = \{x \in X : \|x\| \leq 1\}$. The *skewness* of X was introduced by Fitzpatrick-Reznick [3], as follows:

$$s(X) = \sup\left\{\langle x, y\rangle - \langle y, x\rangle : \ x, y \in S_X\right\},$$

where

$$\langle x, y\rangle = \|x\| \lim_{t\to+0} \frac{\|x + ty\| - \|x\|}{t}$$

for $x, y \in X$ (cf.[9]). Clearly, $0 \leq s(X) \leq 2$ for all spaces X. It is known that $s(X) = 0$ if and only if X is a Hilbert space. The skewness for X being L_p spaces were computed. Some geometrical properties of X can be characterized by means of $s(X)$. In fact, it was shown that $s(X) < 2$ if and only if X is uniformly non-square, that is, there exists a $\delta > 0$ such that for any $x, y \in S_X$, either $\|x + y\| \leq 2(1 - \delta)$ or $\|x - y\| \leq 2(1 - \delta)$ ([3]). Moreover, it is known that

$$s(X) \leq 2\rho_X(1) \tag{1.1}$$

for any Banach space X, where ρ_X is the modulus of smoothness of X ([2]). In [11], Yang introduced the Banaś-Frączek space $\mathbb{R}^2_\lambda$, i.e., the space $\mathbb{R}^2$ with the norm $\|\cdot\|_{\lambda,2}$ defined by

$$\|(x, y)\|_{\lambda,2} = \max\left\{\lambda|x|, \|(x, y)\|_2\right\},$$

where $\lambda > 1$ (see also [1]). As stated in [8], this space may be considered as a generalization of Day-James space ℓ_2-ℓ_1. Indeed, $\mathbb{R}^2_{\sqrt{2}}$ is isometric to ℓ_2-ℓ_1. Recently, Mitani-Saito-Komuro [7] computed $s(X)$-constant for X being ℓ_p-ℓ_1, where $1 \leq p \leq \infty$ (cf. [2, 8]).

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In this paper, we compute $s(X)$-constant for X being $\mathbb{R}^2_\lambda$. This calculation method is similar to that of $s(\ell_p\text{-}\ell_1)$ in [7]. Indeed, as in [7], we will use the following result in [3]; for any Banach space X,

$$s(X) = \sup\{x^*(y) - y^*(x) : x, y \in S_X, x^* \in D(X, x), y^* \in D(X, y)\}, \tag{1.2}$$

where $D(X, x)$ is the set of all norming functionals of x. We first determine the dual norm of $\|\cdot\|_{\lambda,2}$ and the set $D(\mathbb{R}^2_\lambda, x)$. By using this result and (1.2), we can obtain the value of $s(\mathbb{R}^2_\lambda)$ for all $\lambda > 1$. As an application, we discuss the inequality (1.1) for a Banach space X and prove that the inequality (1.1) is strict under the case $X = \mathbb{R}^2_\lambda$.

We recall some definitions and notations. Throughout this paper, we denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of real numbers and non-negative real integers, respectively. An element $x^* \in S_{X^*}$ is said to be a *norming functional* of $x \in X$ with $x \neq 0$ if $x^*(x) = \|x\|$. The *modulus of smoothness* of X is defined by

$$\rho_X(\tau) = \sup\left\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in S_X\right\}, \quad \tau \geq 0.$$

It is well-known that X is uniformly non-square if and only if $\rho_X(1) < 1$ (see [10]). For $1 \leq p, q \leq \infty$, the norm $|\cdot|_{p,q}$ on $\mathbb{R}^2$ is defined by

$$|(x, y)|_{p,q} = \begin{cases} \|(x, y)\|_p & xy \geq 0 \\ \|(x, y)\|_q & xy \leq 0, \end{cases}$$

where $\|\cdot\|_p$ is ℓ_p-norm on $\mathbb{R}^2$. The space $(\mathbb{R}^2, |\cdot|_{p,q})$ is called the *Day-James space* and is denoted by $\ell_p\text{-}\ell_q$ (cf. [4]).

2. Results

The following lemma is an improvement of (1.2), which is useful for computing the constant $s(\mathbb{R}^2_\lambda)$.

Lemma 2.1 ([6], Lemma 3.3)**.** *Let $X = (\mathbb{R}^2, \|\cdot\|)$. Then*

$$s(X) = \sup\{s(X, x, y) : x, y \in \mathrm{ext}(B_X) \cap \mathbb{R} \times \mathbb{R}^+\},$$

where

$$s(X, x, y) = \sup\{|x^*(y) - y^*(x)| : x^* \in D(X, x),\ y^* \in D(X, y)\}$$

and $\mathrm{ext}(B_X)$ *is the set of all extreme points of* B_X.

Let $\|\cdot\|^*_{\lambda,2}$ be the dual norm of $\|\cdot\|_{\lambda,2}$, that is,

$$\|(x, y)\|^*_{\lambda,2} = \sup\{|xz + yw| : (z, w) \in \mathbb{R}^2, \|(z, w)\|_{\lambda,2} = 1\} \tag{2.1}$$

for $(x, y) \in \mathbb{R}^2$.

Lemma 2.2. *Fix $t > 0$. We define a function*

$$f(s) = ts + \sqrt{1 - s^2}, \quad 0 \leq s \leq 1. \tag{2.2}$$

Then f is strictly increasing on $(0, \alpha)$ and is strictly decreasing on (α, ∞), where $\alpha = \frac{t}{\sqrt{t^2+1}}$.

Proof. The derivative of f is

$$f'(s) = t - \frac{s}{\sqrt{1-s^2}}.$$

Hence, if $0 < s < \alpha$, then $f'(s) > 0$ and if $\alpha < s$, then $f'(s) < 0$. This completes the proof. $\square$

Using this lemma we can determine the norm $\|\cdot\|^*_{\lambda,2}$.

Proposition 2.3. *Let* $\lambda > 1$. *Then*

$$\|(x,y)\|^*_{\lambda,2} = \begin{cases} \frac{1}{\lambda}|x| + \sqrt{1-\frac{1}{\lambda^2}}|y|, & \text{if } \sqrt{\lambda^2-1}|x| \geq |y| \\ \sqrt{x^2+y^2}, & \text{if } \sqrt{\lambda^2-1}|x| \leq |y| \end{cases}$$

for any $(x,y) \in \mathbb{R}^2$.

Proof. Put $\alpha = \frac{t}{\sqrt{t^2+1}}$. Let f be the function as in (2.2). By (2.1) we have $\|(|x|,|y|)\|^*_{\lambda,2} = \|(x,y)\|^*_{\lambda,2}$ for any $(x,y) \in \mathbb{R}^2$. Hence it is enough to show that for all $t > 0$,

$$\begin{aligned} \|(t,1)\|^*_{\lambda,2} &= \sup\{|tz+w| : (z,w) \in \mathbb{R}^2, \|(z,w)\|_{\lambda,2} = 1\} \\ &= \begin{cases} \frac{t}{\lambda} + \sqrt{1-\frac{1}{\lambda^2}}, & \text{if } t \geq \frac{1}{\sqrt{\lambda^2-1}} \\ \sqrt{t^2+1}, & \text{if } t \leq \frac{1}{\sqrt{\lambda^2-1}}. \end{cases} \end{aligned}$$

Take $(z,w) \in \mathbb{R}^2$ with $\|(z,w)\|_{\lambda,2} = 1$. It follows from the definition of $\|\cdot\|_{\lambda,2}$ that $\lambda|z| \leq 1$ and $\sqrt{z^2+w^2} \leq 1$. Namely, $|z| \leq \frac{1}{\lambda}$ and $|w| \leq \sqrt{1-z^2}$. Hence

$$|tz+w| \leq t|z| + |w| \leq t|z| + \sqrt{1-z^2} = f(|z|).$$

Let $t \geq \frac{1}{\sqrt{\lambda^2-1}}$. Then $|z| \leq \frac{1}{\lambda} \leq \alpha$. By Lemma 2.2,

$$f(|z|) \leq f\Big(\frac{1}{\lambda}\Big) = \frac{t}{\lambda} + \sqrt{1-\frac{1}{\lambda^2}}.$$

Thus

$$|tz+w| \leq \frac{t}{\lambda} + \sqrt{1-\frac{1}{\lambda^2}}.$$

We have equality in above when $z = \frac{1}{\lambda}$ and $w = \sqrt{1-\frac{1}{\lambda^2}}$.

Let $t \leq \frac{1}{\sqrt{\lambda^2-1}}$. Then $|z| \leq \alpha \leq \frac{1}{\lambda}$. By Lemma 2.2,

$$f(|z|) \leq f(\alpha) = t\alpha + \sqrt{1-\alpha^2} = \sqrt{t^2+1}.$$

Thus

$$|tz+w| \leq \sqrt{t^2+1}.$$

We have equality in above when $z = \frac{t}{\sqrt{t^2+1}}$ and $w = \frac{1}{\sqrt{t^2+1}}$. This completes the proof. $\square$

In the following, we shall determine norming functionals of $\mathbb{R}^2_\lambda$. Let $\lambda > 1$. For $0 \le \theta < 2\pi$, we put $z(\theta)$ as

$$z(\theta) = \frac{(\cos\theta, \sin\theta)}{\|(\cos\theta, \sin\theta)\|_{\lambda,2}}.$$

It is clear that

$$\text{ext}(B_{\mathbb{R}^2_\lambda}) \cap \mathbb{R} \times \mathbb{R}^+ = \{z(\theta) : \theta_\lambda \le \theta \le \pi - \theta_\lambda\}, \tag{2.3}$$

where $\theta_\lambda = \tan^{-1}\sqrt{\lambda^2 - 1}$. Note that

$$z(\theta_\lambda) = (\cos\theta_\lambda, \sin\theta_\lambda) = \left(\frac{1}{\lambda}, \sqrt{1 - \frac{1}{\lambda^2}}\right).$$

Lemma 2.4. *Let* $\lambda > 1$.
(i) *If* $\theta_\lambda < \theta < \pi - \theta_\lambda$*, then*

$$D(\mathbb{R}^2_\lambda, z(\theta)) = \big\{z(\theta)\big\}.$$

(ii) *If* $\theta = \theta_\lambda$*, then*

$$D(\mathbb{R}^2_\lambda, z(\theta_\lambda)) = \left\{\left(\frac{\mu}{\lambda} + (1-\mu)\lambda, \mu\sqrt{1 - \frac{1}{\lambda^2}}\right) : 0 \le \mu \le 1\right\}. \tag{2.4}$$

(iii) *If* $\theta = \pi - \theta_\lambda$*, then*

$$D(\mathbb{R}^2_\lambda, z(\pi - \theta_\lambda)) = \left\{\left(-\frac{\mu}{\lambda} - (1-\mu)\lambda, \mu\sqrt{1 - \frac{1}{\lambda^2}}\right) : 0 \le \mu \le 1\right\}. \tag{2.5}$$

Proof. (i) Let $\theta_\lambda < \theta < \pi - \theta_\lambda$. Then $z(\theta) = (\cos\theta, \sin\theta)$. By $\|z(\theta)\|^*_{\lambda,2} = \|z(\theta)\|^*_2 = 1$ and $\langle z(\theta), z(\theta)\rangle = 1$, it follows that $z(\theta)$ is a norming functional of $z(\theta)$. Since $z(\theta)$ is a point of smoothness of $B_{\mathbb{R}^2_\lambda}$, we obtain (i).
(ii) Let $\theta = \theta_\lambda$. Take any $\alpha = (a, b) \in D(\mathbb{R}^2_\lambda, z(\theta_\lambda))$. Then we have

$$\begin{aligned} 1 = \langle (a,b), z(\theta_\lambda)\rangle &\le \langle (|a|, |b|), z(\theta_\lambda)\rangle \\ &\le \|(|a|,|b|)\|^*_{\lambda,2}\|z(\theta)\|_{\lambda,2} = \|(a,b)\|^*_{\lambda,2}\|z(\theta)\|_{\lambda,2} = 1, \end{aligned}$$

which gives $|a| = a$ and $|b| = b$. Hence $a \ge 0$ and $b \ge 0$. We also have $\|(a,b)\|^*_{\lambda,2} = 1$ and

$$1 = \langle \alpha, z(\theta_\lambda)\rangle = \frac{a}{\lambda} + \sqrt{1 - \frac{1}{\lambda^2}}\, b. \tag{2.6}$$

Hence, it follows from Proposition 2.3 that $\sqrt{\lambda^2 - 1}\, a \ge b$. Namely, $a \ge \frac{b}{\sqrt{\lambda^2 - 1}}$. This inequality and (2.6) imply that

$$1 = \frac{a}{\lambda} + \sqrt{1 - \frac{1}{\lambda^2}}\, b \ge \frac{b}{\lambda\sqrt{\lambda^2 - 1}} + \sqrt{1 - \frac{1}{\lambda^2}}\, b$$

and so $b \le \sqrt{1-\frac{1}{\lambda^2}}$. Here, we put $b = \mu\sqrt{1-\frac{1}{\lambda^2}}$, where $0 \le \mu \le 1$. Then we have by (2.6),

$$a = \lambda - b\sqrt{\lambda^2-1} = \lambda - \mu\sqrt{1-\frac{1}{\lambda^2}}\sqrt{\lambda^2-1} = \frac{\mu}{\lambda} + (1-\mu)\lambda.$$

Thus we can write

$$(a,b) = \left(\frac{\mu}{\lambda} + (1-\mu)\lambda, \mu\sqrt{1-\frac{1}{\lambda^2}}\right).$$

Conversely, take any μ with $0 \le \mu \le 1$. It is easy to see that

$$\left(\frac{\mu}{\lambda} + (1-\mu)\lambda, \mu\sqrt{1-\frac{1}{\lambda^2}}\right) \in D(\mathbb{R}^2_\lambda, z(\theta_\lambda)).$$

Thus we get (2.4).
In the same way, we can obtain (iii). □

We now show the main theorem.

Theorem 2.5. *Let* $\lambda > 1$.
(i) *If* $\lambda \ge \sqrt{2}$, *then* $s(\mathbb{R}^2_\lambda) = 2\left(1-\frac{1}{\lambda^2}\right)$.

(ii) *If* $\lambda \le \sqrt{2}$, *then* $s(\mathbb{R}^2_\lambda) = \sqrt{\lambda^2-1}$.

Proof. Let $X = \mathbb{R}^2_\lambda$. Lemma 2.1 and (2.3) yield that

$$s(X) = \sup\left\{s(X,x,y) : x = z(\theta),\ y = z(\theta'),\ \theta,\theta' \in [\theta_\lambda, \pi-\theta_\lambda]\right\}, \tag{2.7}$$

where

$$s(X,x,y) = \sup\{|x^*(y) - y^*(x)| : x^* \in D(X,x),\ y^* \in D(X,y)\}.$$

Let $x = z(\theta)$ and $y = z(\theta')$, where $\theta,\theta' \in [\theta_\lambda, \pi-\theta_\lambda]$. Take any $x^* \in D(X,x)$ and $y^* \in D(X,y)$. It is enough to consider the following cases:

Case 1: $\theta_\lambda < \theta, \theta' < \pi - \theta_\lambda$. By Lemma 2.4 it follows that $x^* = z(\theta)$ and $y^* = z(\theta')$. Hence we have

$$x^*(y) - y^*(x) = \langle z(\theta), z(\theta')\rangle - \langle z(\theta'), z(\theta)\rangle = 0,$$

which gives $s(X,x,y) = 0$.

Case 2: $\theta = \theta_\lambda$, $\theta_\lambda < \theta' < \pi - \theta_\lambda$. Then $x = z(\theta_\lambda) = \left(\frac{1}{\lambda}, \sqrt{1-\frac{1}{\lambda^2}}\right)$ and $y = z(\theta') = (\cos\theta', \sin\theta')$. By Lemma 2.4, we have $x^* = \left(\frac{\mu}{\lambda} + (1-\mu)\lambda, \mu\sqrt{1-\frac{1}{\lambda^2}}\right)$ and $y^* = z(\theta') = (\cos\theta', \sin\theta')$, where $0 \le \mu \le 1$. Hence,

$$\begin{aligned} &x^*(y) - y^*(x) \\ &= \left\langle \left(\frac{\mu}{\lambda} + (1-\mu)\lambda, \mu\sqrt{1-\frac{1}{\lambda^2}}\right), (\cos\theta', \sin\theta')\right\rangle \end{aligned}$$

$$- \left\langle (\cos\theta', \sin\theta'), \left(\frac{1}{\lambda}, \sqrt{1-\frac{1}{\lambda^2}}\right)\right\rangle$$
$$= \left(\frac{\mu}{\lambda} + (1-\mu)\lambda\right)\cos\theta' + \mu\sqrt{1-\frac{1}{\lambda^2}}\sin\theta' - \frac{1}{\lambda}\cos\theta' - \sqrt{1-\frac{1}{\lambda^2}}\sin\theta'$$
$$= -(1-\mu)\sqrt{1-\frac{1}{\lambda^2}}\left(\sin\theta' - \sqrt{\lambda^2-1}\cos\theta'\right)$$

and so

$$|x^*(y) - y^*(x)| \leq \sqrt{1-\frac{1}{\lambda^2}}\ |\sin\theta' - \sqrt{\lambda^2-1}\cos\theta'|.$$

We define a function

$$g(t) = \sin t - \sqrt{\lambda^2-1}\cos t.$$

The derivative of g is $g'(t) = \cos t + \sqrt{\lambda^2-1}\sin t$ and so

$$g'(\pi - \theta_\lambda) = -\cos\theta_\lambda + \sqrt{\lambda^2-1}\sin\theta_\lambda$$
$$= -\frac{1}{\lambda} + \sqrt{\lambda^2-1}\sqrt{1-\frac{1}{\lambda^2}} = \frac{\lambda^2-2}{\lambda}.$$

Let $\lambda \geq \sqrt{2}$. Then $g'(t) > 0$ for all $\theta_\lambda < t < \pi - \theta_\lambda$. This implies that g is strictly increasing on $(\theta_\lambda, \pi - \theta_\lambda)$ and so

$$g(t) \leq g(\pi - \theta_\lambda) = 2\sqrt{1-\frac{1}{\lambda^2}}$$

for all $\theta_\lambda \leq t \leq \pi - \theta_\lambda$. Moreover, $g(\theta_\lambda) = 0$. Hence

$$|x^*(y) - y^*(x)| \leq \sqrt{1-\frac{1}{\lambda^2}} \cdot 2\sqrt{1-\frac{1}{\lambda^2}} = 2\left(1-\frac{1}{\lambda^2}\right).$$

Thus

$$s(X, x, y) \leq 2\left(1-\frac{1}{\lambda^2}\right).$$

Let $\lambda < \sqrt{2}$. Then g is strictly increasing on (θ_λ, t_0) and is strictly decreasing on $(t_0, \pi - \theta_\lambda)$, where t_0 is the unique solution of $g'(t) = 0$ $(\theta_\lambda < t < \pi - \theta_\lambda)$, that is, $t_0 = \cos^{-1}\left(-\sqrt{1-\frac{1}{\lambda^2}}\right)$. From $\cos t_0 = -\sqrt{1-\frac{1}{\lambda^2}}$ and $\sin t_0 = \frac{1}{\lambda}$, we have

$$g(t) \leq g(t_0) = \frac{1}{\lambda} - \sqrt{\lambda^2-1} \cdot \left(-\sqrt{1-\frac{1}{\lambda^2}}\right) = \lambda$$

for all $\theta_\lambda \leq t \leq \pi - \theta_\lambda$. Hence,

$$|x^*(y) - y^*(x)| \leq \sqrt{1-\frac{1}{\lambda^2}} \cdot \lambda = \sqrt{\lambda^2-1}.$$

Moreover, we have equality in above when $t = t_0$ and $\mu = 0$. Thus we obtain

$$s(X, x, y) \leq \sqrt{\lambda^2-1}.$$

In particular, $s(X, x, y) = \sqrt{\lambda^2-1}$, whenever $x = z(\theta_\lambda)$ and $y = z(t_0)$.

Case 3: $\theta = \theta_\lambda$, $\theta' = \pi - \theta_\lambda$. Note that $x = \left(\frac{1}{\lambda}, \sqrt{1 - \frac{1}{\lambda^2}}\right)$ and $y = \left(-\frac{1}{\lambda}, \sqrt{1 - \frac{1}{\lambda^2}}\right)$. By Lemma 2.4, we have $x^* = \left(\frac{\mu}{\lambda} + (1-\mu)\lambda, \mu\sqrt{1 - \frac{1}{\lambda^2}}\right)$ and $y^* = \left(-\frac{\mu'}{\lambda} - (1-\mu')\lambda, \mu'\sqrt{1 - \frac{1}{\lambda^2}}\right)$, where $0 \le \mu, \mu' \le 1$. Then

$$\begin{aligned}
&x^*(y) - y^*(x) \\
&= \left\langle \left(\frac{\mu}{\lambda} + (1-\mu)\lambda, \mu\sqrt{1 - \frac{1}{\lambda^2}}\right), \left(-\frac{1}{\lambda}, \sqrt{1 - \frac{1}{\lambda^2}}\right) \right\rangle \\
&\quad - \left\langle \left(-\frac{\mu'}{\lambda} - (1-\mu')\lambda, \mu'\sqrt{1 - \frac{1}{\lambda^2}}\right), \left(\frac{1}{\lambda}, \sqrt{1 - \frac{1}{\lambda^2}}\right) \right\rangle \\
&= \left(\frac{\mu}{\lambda} + (1-\mu)\lambda\right)\left(-\frac{1}{\lambda}\right) + \mu\sqrt{1 - \frac{1}{\lambda^2}}\sqrt{1 - \frac{1}{\lambda^2}} \\
&\quad - \left(-\frac{\mu'}{\lambda} - (1-\mu')\lambda\right)\frac{1}{\lambda} - \mu'\sqrt{1 - \frac{1}{\lambda^2}}\sqrt{1 - \frac{1}{\lambda^2}} \\
&= 2(\mu - \mu')\left(1 - \frac{1}{\lambda^2}\right)
\end{aligned}$$

and so

$$|x^*(y) - y^*(x)| \le 2\left(1 - \frac{1}{\lambda^2}\right).$$

We have equality in above when $\mu = 1$ and $\mu' = 0$. Therefore

$$s(X, x, y) = 2\left(1 - \frac{1}{\lambda^2}\right).$$

Case 4: $\theta_\lambda < \theta < \pi - \theta_\lambda, \theta' = \pi - \theta_\lambda$. By an argument similar to that in Case 2, it follows that $s(X, x, y) \le 2\left(1 - \frac{1}{\lambda^2}\right)$ if $\lambda \ge \sqrt{2}$, and $s(X, x, y) \le \sqrt{\lambda^2 - 1}$ if $\lambda < \sqrt{2}$.

By $2(1 - \frac{1}{\lambda^2}) \le \sqrt{\lambda^2 - 1}$ for all $\lambda > 1$, this completes the proof. □

3. Applications

The following is due to Baronti-Papini [2].

Proposition 3.1 ([2])**.** *For any Banach space X,*

$$s(X) \le 2\rho_X(1). \tag{3.1}$$

If X is uniformly convex, then the inequality (3.1) is strict ([8], cf.[5]). If X is not uniformly non-square, then $s(X) = 2\rho_X(1) = 2$. There exists a uniformly non-square (not uniformly convex) Banach space X such that $s(X) = 2\rho_X(1)$. In fact, if X is Day-James space ℓ_∞-ℓ_1, then $s(X) = 2\rho_X(1) = 1$. As in [7], it was shown that if X is Day-James space ℓ_p-ℓ_1, where $1 < p < \infty$, then the inequality

(3.1) is strict. We discuss the inequality (3.1) for the case $X = \mathbb{R}^2_\lambda$. The *James type constant* $J_{X,t}(\tau)$ for a Banach space X is defined by

$$J_{X,t}(\tau) = \sup\left\{\left(\frac{\|x+\tau y\|^t + \|x-\tau y\|^t}{2}\right)^{1/t} : x, y \in S_X\right\},$$

where $\tau \geq 0$ and $-\infty \leq t < \infty$ ([10]). Note that $\rho_X(1) = J_{X,1}(1) - 1$. C. Yang-X. Yang [12] introduced the *Banaś-Frączek type space* $X_{\lambda,p}$, i.e., $\mathbb{R}^2$ with the norm $\|\cdot\|_{\lambda,p}$ defined by

$$\|(x,y)\|_{\lambda,p} = \max\{\lambda|x|, \|(x,y)\|_p\},$$

where $\lambda > 1$ and $p \geq 1$. Note that $X_{\lambda,2} = \mathbb{R}^2_\lambda$.

Theorem 3.2 ([12], Theorem 2.3)**.** *Let $p \geq 2$, $\lambda > 1$.*
(i) *If $t \geq p$, then*

$$J_{X_{\lambda,p},t}(1) = 2^{1-\frac{1}{t}}\left(1 + \left(1 - \frac{1}{\lambda^p}\right)^{\frac{t}{p}}\right)^{\frac{1}{t}} \tag{3.2}$$

(ii) *If $t < p$ and $\lambda^p \leq 1 + \lambda^{\frac{tp}{t-p}}$, then*

$$J_{X_{\lambda,p},t}(1) = 2^{1-\frac{1}{t}}\lambda\left(1 + \lambda^{\frac{tp}{t-p}}\right)^{\frac{1}{t}-\frac{1}{p}}$$

(iii) *If $t < p$ and $\lambda^p \geq 1 + \lambda^{\frac{tp}{t-p}}$, then* (3.2) *is also valid.*

In particular,

Corollary 3.3. *Let $X = \mathbb{R}^2_\lambda$, where $\lambda > 1$.*

(i) *If $\lambda \leq \sqrt{\frac{1+\sqrt{5}}{2}}$, then $\rho_X(1) = \sqrt{\lambda^2+1} - 1$.*

(ii) *If $\lambda \geq \sqrt{\frac{1+\sqrt{5}}{2}}$, then $\rho_X(1) = \sqrt{1 - \frac{1}{\lambda^2}}$.*

By this result and Theorem 2.5 we obtain the following.

Corollary 3.4. *Let $X = \mathbb{R}^2_\lambda$, where $\lambda > 1$. Then $s(X) < 2\rho_X(1)$.*

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