Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167

Volume 9, Number 2, 2023, 127–138

A SINGLE FACILITY MINISUM LOCATION PROBLEM WITH UNCERTAINTY

MASAMICHI KON

Abstract. In this paper, we consider a single facility location problem. It is assumed that demand points and associated weights are given. The demand points represent locations of users of the facility. For each demand point, the weight associated with the demand point represents the demand quantity of the facility. The single facility minisum location problem is a problem to minimize the total sum of weighted distances from the variable location of the facility to the demand points. We consider the demand points and weights as parameters. The demand points and weights with uncertainty are represented as bounded and interval uncertainty sets, respectively. First, we consider minimax and maximin problems to minimize and maximize with respect to the variable location of the facility and the parameters, respectively. Next, we propose a new interval-valued approach in order to treat such uncertainty. Furthermore, we also present a procedure to find all (weak) efficient solutions of the one-dimensional interval-valued minisum location problem derived by the interval-valued approach.

1. Introduction

Given demand points which are fixed points in $\mathbb{R}^n$, a *single facility minisum location problem* or simply a *minisum location problem* is a problem to find the variable location $\boldsymbol{x} \in \mathbb{R}^n$ of the facility, which minimizes the total weighted distances from $\boldsymbol{x}$ to the demand points. Let $\boldsymbol{a}_i^0 \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$ be m demand points, and let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Then, the minisum location problem is formulated as follows:

$$\text{(P)} \qquad \min_{\boldsymbol{x}\in\mathbb{R}^n} \sum_{i=1}^{m} w_i^0 \|\boldsymbol{x} - \boldsymbol{a}_i^0\|$$

where $w_i^0 > 0$ is a weight associated with the demand point $\boldsymbol{a}_i^0$ for each $i \in \{1, 2, \ldots, m\}$. We set $\boldsymbol{w}^0 = (w_1^0, w_2^0, \ldots, w_m^0)$.

The minisum location problem without uncertainty is dealt in [6, 8, 13, 14, 15, 16] and many other literature. A suitable iterative method or linear programming can be applied to the minisum location problems without uncertainty according to the problems in them. However, in real-world problems, the locations of demand points and/or the weights are often not known precisely. Such uncertainty is dealt as uncertainty sets in [4, 5, 9, 10, 11, 18], and as random variables in [1, 3, 12, 17]. A suitable iterative method, (conic) linear programming, heuristic method,

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2020 *Mathematics Subject Classification.* 90B85.
Key words and phrases. Minisum location problem, minimax problem, maximin problem, interval-valued approach, efficient solution, weak efficient solution.

second-order cone programming (SOCP), or semidefinite programming (SDP) can be applied to the minisum location problems with uncertainty sets according to the problems in them. A suitable iterative method can be applied to the minisum location problems with random variables according to the problems in them.

In this paper, we deal with the uncertainty as uncertainty sets. It is suitable when probability distributions can not be determined, for example, because of lack of data. We consider the demand points and weights as parameters, and represent uncertainty of the demand points and weights as bounded and interval uncertainty sets, respectively.

First, we consider *minimax* and *maximin problems* to minimize and maximize with respect to the variable location of the facility and the parameters, respectively. Then, a relationship between them is derived. The minimax problem is a robust optimization problem [2], and the maximin problem is a problem to find the worst case of the optimal value in the variable location of the facility with respect to the parameters.

Jamalian and Salahi [9] considered the minisum location problem in $\mathbb{R}^2$ with block norm in which uncertainty of the weights was represented as the interval uncertainty sets, and the minisum location problem in $\mathbb{R}^2$ with Euclidean norm in which uncertainty of the demand points and weights were represented as the bounded and interval uncertainty sets, respectively (see also [18]). Juel [11] considered the maximin problem in which uncertainty of the demand points was represented as the bounded uncertainty sets (see also [4, 5, 10]). In other words, [9, 18] take a robust optimization approach, and [11, 4, 5, 10] take another approach which analyzes the range of optimal value and solution of the minisum location problem with respect to the parameters.

Next, we propose an *interval-valued approach* as a new approach, in which the uncertainty sets themselves are considered as the demand points and weights, and the derived minimization problem with an interval-valued objective function is considered. Furthermore, we also present a procedure to find all (weak) efficient solutions of the one-dimensional interval-valued minisum location problem derived by the interval-valued approach.

The remainder of the present paper is organized as follows. In Section 2, some notations and terminologies are presented. In Section 3, we present a relationship between the minimax problem and the maximin problem. In Section 4, we present an example such that optimal values of the minimax and maximin problems do not coincide when some condition is not satisfied. In Section 5, we propose an interval-valued approach as a new approach handling uncertainty. In Section 6, we present a procedure to find all (weak) efficient solutions of the one-dimensional interval-valued minisum location problem derived by the interval-valued approach. In Section 7, we present a numerical example to illustrate the procedure of Section 6. Finally, conclusions are presented in Section 8.

2. Preliminaries

We consider the minisum location problem (P) with the demand points and weights involving uncertainty. We set $\underline{\boldsymbol{w}} = (\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_m) \in \mathbb{R}^m$ and $\overline{\boldsymbol{w}} = (\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_m) \in \mathbb{R}^m$, and assume that $0 < \underline{w}_i \leq w_i^0 \leq \overline{w}_i$, $i = 1, 2, \ldots, m$. Then

$$[\underline{\boldsymbol{w}}, \overline{\boldsymbol{w}}] = \{\boldsymbol{w} \in \mathbb{R}^m : \underline{\boldsymbol{w}} \leq \boldsymbol{w} \leq \overline{\boldsymbol{w}}\} \tag{2.1}$$

is called an *interval uncertainty set* for the weights. For each $i \in \{1, 2, \ldots, m\}$ and $r_i \geq 0$,

$$\mathcal{U}_i = \{\boldsymbol{a}_i^0 + \boldsymbol{a}_i : \boldsymbol{a}_i \in \mathbb{R}^n, \|\boldsymbol{a}_i\| \leq r_i\} = \{\boldsymbol{b}_i \in \mathbb{R}^n : \|\boldsymbol{b}_i - \boldsymbol{a}_i^0\| \leq r_i\} \tag{2.2}$$

is called a *bounded uncertainty set* for the demand point $\boldsymbol{a}_i^0$.

We define $f : \mathbb{R}^n \times \mathbb{R}^m \times \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{m \text{ times}} \to \mathbb{R}$ as

$$f(\boldsymbol{x}; \boldsymbol{w}, \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_m) = \sum_{i=1}^{m} w_i \|\boldsymbol{x} - \boldsymbol{b}_i\| \tag{2.3}$$

for each $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{w} = (w_1, w_2, \ldots, w_m) \in \mathbb{R}^m$, and $\boldsymbol{b}_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$. Then, the problem

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \max_{\boldsymbol{w} \in [\underline{\boldsymbol{w}}, \overline{\boldsymbol{w}}], \boldsymbol{b}_i \in \mathcal{U}_i, i=1,\ldots,m} f(\boldsymbol{x}; \boldsymbol{w}, \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_m) \tag{2.4}$$

is called the *minimax problem*. Let $\boldsymbol{x}^* \in \mathbb{R}^n$ be its optimal solution, and let $f_{\min\max}$ be its optimal value. The problem

$$\max_{\boldsymbol{w} \in [\underline{\boldsymbol{w}}, \overline{\boldsymbol{w}}], \boldsymbol{b}_i \in \mathcal{U}_i, i=1,\ldots,m} \min_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}; \boldsymbol{w}, \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_m) \tag{2.5}$$

is called the *maximin problem*. Let $\boldsymbol{w}^* \in [\underline{\boldsymbol{w}}, \overline{\boldsymbol{w}}]$ and $\boldsymbol{b}_i^* \in \mathcal{U}_i$, $i = 1, 2, \ldots, m$ be its optimal solution, and let $f_{\max\min}$ be its optimal value.

The following two lemmas are very useful for our analysis hereafter.

Lemma 2.1. (Juel [11, Lemma 1]) *For fixed $\boldsymbol{x}, \boldsymbol{a} \in \mathbb{R}^n$ and $r \geq 0$, consider the problem*

$$\left| \begin{array}{ll} \max & \|\boldsymbol{x} - \boldsymbol{y}\|, \\ s.t. & \|\boldsymbol{y} - \boldsymbol{a}\| \leq r. \end{array} \right. \tag{2.6}$$

Then, its optimal value is $\|\boldsymbol{x} - \boldsymbol{a}\| + r$. In addition, its optimal solution is $\boldsymbol{y} = \boldsymbol{a} + \frac{r}{\|\boldsymbol{x} - \boldsymbol{a}\|}(\boldsymbol{a} - \boldsymbol{x})$ if $\boldsymbol{x} \neq \boldsymbol{a}$, and is any $\boldsymbol{y} \in \mathbb{R}^n$ with $\|\boldsymbol{y} - \boldsymbol{a}\| = r$ if $\boldsymbol{x} = \boldsymbol{a}$.

Lemma 2.2. (Juel [11, Lemma 2]) *For fixed $\boldsymbol{x}, \boldsymbol{a} \in \mathbb{R}^n$ and $r \geq 0$, consider the problem*

$$\left| \begin{array}{ll} \min & \|\boldsymbol{x} - \boldsymbol{y}\|, \\ s.t. & \|\boldsymbol{y} - \boldsymbol{a}\| \leq r. \end{array} \right. \tag{2.7}$$

Then, its optimal value is $\max\{\|\boldsymbol{x} - \boldsymbol{a}\| - r, 0\}$. *In addition, its optimal solution is $\boldsymbol{y} = \boldsymbol{a} - \frac{r}{\|\boldsymbol{x} - \boldsymbol{a}\|}(\boldsymbol{a} - \boldsymbol{x})$ if $\|\boldsymbol{x} - \boldsymbol{a}\| > r$, and is $\boldsymbol{y} = \boldsymbol{x}$ if $\|\boldsymbol{x} - \boldsymbol{a}\| \leq r$.*

3. Minimax and maximin problems

This section derives a relationship between the minimax problem (2.4) and the maximin problem (2.5).

The following theorem provides an optimal solution and value of the minimax problem (2.4).

Theorem 3.1. *Consider the minisum location problem*

$$(\overline{\mathrm{P}}) \qquad \min_{\boldsymbol{x}\in\mathbb{R}^n} f(\boldsymbol{x};\overline{\boldsymbol{w}},\boldsymbol{a}_1^0,\boldsymbol{a}_2^0,\dots,\boldsymbol{a}_m^0).$$

Let $\overline{\boldsymbol{x}}\in\mathbb{R}^n$ be its optimal solution, and let $\overline{f}$ be its optimal value. Then, $\boldsymbol{x}^*=\overline{\boldsymbol{x}}$ and $f_{\min\max}=\overline{f}+\sum_{i=1}^m \overline{w}_i r_i$.

Proof. From Lemma 2.1, it follows that, for each $\boldsymbol{x}\in\mathbb{R}^n$,

$$\begin{aligned}
&\max_{\boldsymbol{w}\in[\underline{\boldsymbol{w}},\overline{\boldsymbol{w}}],\boldsymbol{b}_i\in\mathcal{U}_i,i=1,\dots,m} f(\boldsymbol{x};\boldsymbol{w},\boldsymbol{b}_1,\boldsymbol{b}_2,\dots,\boldsymbol{b}_m)\\
&=\max_{\boldsymbol{w}\in[\underline{\boldsymbol{w}},\overline{\boldsymbol{w}}],\boldsymbol{b}_i\in\mathcal{U}_i,i=1,\dots,m} \sum_{i=1}^m w_i\|\boldsymbol{x}-\boldsymbol{b}_i\|\\
&=\sum_{i=1}^m \overline{w}_i(\|\boldsymbol{x}-\boldsymbol{a}_i^0\|+r_i)\\
&=\sum_{i=1}^m \overline{w}_i\|\boldsymbol{x}-\boldsymbol{a}_i^0\|+\sum_{i=1}^m \overline{w}_i r_i\\
&=f(\boldsymbol{x};\overline{\boldsymbol{w}},\boldsymbol{a}_1^0,\boldsymbol{a}_2^0,\dots,\boldsymbol{a}_m^0)+\sum_{i=1}^m \overline{w}_i r_i.
\end{aligned}$$

Therefore, we have the conclusions. □

Theorem 3.2. (Juel [11, Theorem 2]) *Fix any $\boldsymbol{w}=(w_1,w_2,\dots,w_m)\in[\underline{\boldsymbol{w}},\ \overline{\boldsymbol{w}}]$, and consider the minisum location problem*

$$(\mathrm{P}^{\boldsymbol{w}}) \qquad \min_{\boldsymbol{x}\in\mathbb{R}^n} f(\boldsymbol{x};\boldsymbol{w},\boldsymbol{a}_1^0,\boldsymbol{a}_2^0,\dots,\boldsymbol{a}_m^0).$$

Let $\boldsymbol{x}^{\boldsymbol{w}}\in\mathbb{R}^n$ be its optimal solution, and let $f^{\boldsymbol{w}}$ be its optimal value. In addition, consider the problem

$$\max_{\boldsymbol{b}_i\in\mathcal{U}_i,i=1,\dots,m}\ \min_{\boldsymbol{x}\in\mathbb{R}^n} f(\boldsymbol{x};\boldsymbol{w},\boldsymbol{b}_1,\boldsymbol{b}_2,\dots,\boldsymbol{b}_m).$$

Let $\boldsymbol{b}_i^{\boldsymbol{w}}\in\mathcal{U}_i$, $i=1,2,\dots,m$ be its optimal solution, and let $f_{\max\min}^{\boldsymbol{w}}$ be its optimal value. If $\boldsymbol{x}^{\boldsymbol{w}}\neq\boldsymbol{a}_i^0$, $i=1,2,\dots,m$, then $\boldsymbol{b}_i^{\boldsymbol{w}}=\boldsymbol{a}_i^0+\frac{r_i}{\|\boldsymbol{x}^{\boldsymbol{w}}-\boldsymbol{a}_i^0\|}(\boldsymbol{a}_i^0-\boldsymbol{x}^{\boldsymbol{w}})$, $i=1,2,\dots,m$ and $f_{\max\min}^{\boldsymbol{w}}=f^{\boldsymbol{w}}+\sum_{i=1}^m w_i r_i$.

The following theorem provides a relationship between optimal values of the minimax problem (2.4) and the maximin problem (2.5), and presents an optimal solution of the maximin problem (2.5).

Theorem 3.3. *Let $\boldsymbol{x}^{\overline{\boldsymbol{w}}} \in \mathbb{R}^n$ be an optimal solution of* $(\mathrm{P}^{\overline{\boldsymbol{w}}})$ *(or* $(\overline{\mathrm{P}})$*). If $\boldsymbol{x}^{\overline{\boldsymbol{w}}} \neq \boldsymbol{a}_i^0$, $i = 1, 2, \ldots, m$, then $f_{\max\min} = f_{\min\max}$, $\boldsymbol{w}^* = \overline{\boldsymbol{w}}$, and $\boldsymbol{b}_i^* = \boldsymbol{a}_i^0 + \frac{r_i}{\|\boldsymbol{x}^{\overline{\boldsymbol{w}}} - \boldsymbol{a}_i^0\|}(\boldsymbol{a}_i^0 - \boldsymbol{x}^{\overline{\boldsymbol{w}}})$, $i = 1, 2, \ldots, m$.*

Proof. Setting $\boldsymbol{w} = \overline{\boldsymbol{w}}$ in Theorem 3.2, we use the same notations as in Theorems 3.1 and 3.2. Since $\boldsymbol{x}^{\overline{\boldsymbol{w}}} \neq \boldsymbol{a}_i^0$, $i = 1, 2, \ldots, m$, it follows from Theorems 3.1 and 3.2 that

$$\begin{aligned}
f_{\max\min} &= \max_{\boldsymbol{w} \in [\underline{\boldsymbol{w}}, \overline{\boldsymbol{w}}], \boldsymbol{b}_i \in \mathcal{U}_i, i=1,\ldots,m} \min_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}; \boldsymbol{w}, \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_m) \\
&\geq \max_{\boldsymbol{b}_i \in \mathcal{U}_i, i=1,\ldots,m} \min_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}; \overline{\boldsymbol{w}}, \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_m) \\
&= f^{\overline{\boldsymbol{w}}}_{\max\min} \\
&= f^{\overline{\boldsymbol{w}}} + \sum_{i=1}^m \overline{w}_i r_i \\
&= \min_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}; \overline{\boldsymbol{w}}, \boldsymbol{a}_1^0, \boldsymbol{a}_2^0, \ldots, \boldsymbol{a}_m^0) + \sum_{i=1}^m \overline{w}_i r_i \\
&= \overline{f} + \sum_{i=1}^m \overline{w}_i r_i \\
&= f_{\min\max}.
\end{aligned}$$

Since $f_{\max\min} \leq f_{\min\max}$ by elementary calculus, we have $f_{\max\min} = f_{\min\max}$. The last part of the theorem is derived from Theorem 3.2. □

4. Examples

We consider examples given by Juel [11] in our settings. Set $n = 1$ and $m = 2$, and let $\|\cdot\| = |\cdot|$. In addition, let $x^{\overline{\boldsymbol{w}}} \in \mathbb{R}$ and $f^{\overline{\boldsymbol{w}}}$ be an optimal solution and value of $(\mathrm{P}^{\overline{\boldsymbol{w}}})$ (or $(\overline{\mathrm{P}})$), respectively.

Example 4.1. Let $a_1^0 = 0$, $a_2^0 = 4$, $r_1 = 1$, $r_2 = 2$, and let $w_1^0 = w_2^0 = 1$, $\underline{w}_1 = \overline{w}_1 = \underline{w}_2 = \overline{w}_2 = 1$. Then, an optimal solution of $(\mathrm{P}^{\overline{\boldsymbol{w}}})$ (or $(\overline{\mathrm{P}})$) is any $x^{\overline{\boldsymbol{w}}} \in [0, 4]$, and the optimal value is $f^{\overline{\boldsymbol{w}}} = 4$. From Theorem 3.1, an optimal solution $x^* \in \mathbb{R}$ of the minimax problem (2.4) is any $x^* = x^{\overline{\boldsymbol{w}}} \in [0, 4]$, and the optimal value $f_{\min\max}$ is $f_{\min\max} = f^{\overline{\boldsymbol{w}}} + \overline{w}_1 r_1 + \overline{w}_2 r_2 = 7$. From Theorem 3.3, an optimal solution $\boldsymbol{w}^* \in [\underline{\boldsymbol{w}}, \overline{\boldsymbol{w}}]$ and $b_1^*, b_2^* \in \mathbb{R}$ of the maximin problem (2.5) is $\boldsymbol{w}^* = \overline{\boldsymbol{w}} = (1, 1)$ and $b_1^* = -1$, $b_2^* = 6$ (also $b_1^* = 1$ when $x^{\overline{\boldsymbol{w}}} = a_1^0 = 0$, and also $b_2^* = 2$ when $x^{\overline{\boldsymbol{w}}} = a_2^0 = 4$), and the optimal value $f_{\max\min}$ is $f_{\max\min} = f_{\min\max} = 7$.

Example 4.2. Let $a_1^0 = 0$, $a_2^0 = 4$, $r_1 = r_2 = 1$, and let $w_1^0 = 1$, $w_2^0 = 2$, $\underline{w}_1 = \overline{w}_1 = 1$, $\underline{w}_2 = \overline{w}_2 = 2$. Then, an optimal solution of $(\mathrm{P}^{\overline{\boldsymbol{w}}})$ (or $(\overline{\mathrm{P}})$) is $x^{\overline{\boldsymbol{w}}} = a_2^0 = 4$, and the optimal value is $f^{\overline{\boldsymbol{w}}} = 4$. From Theorem 3.1, an optimal solution $x^* \in \mathbb{R}$ of the minimax problem (2.4) is $x^* = x^{\overline{\boldsymbol{w}}} = 4$, and the optimal value $f_{\min\max}$ is $f_{\min\max} = f^{\overline{\boldsymbol{w}}} + \overline{w}_1 r_1 + \overline{w}_2 r_2 = 7$. On the other hand, an optimal solution $\boldsymbol{w}^* \in [\underline{\boldsymbol{w}}, \overline{\boldsymbol{w}}]$ and $b_1^*, b_2^* \in \mathbb{R}$ of the maximin problem (2.5) is $\boldsymbol{w}^* = \overline{\boldsymbol{w}} = (1, 2)$ and $b_1^* = -1$, $b_2^* = 5$, and the optimal value $f_{\max\min}$ is $f_{\max\min} = 6$. Therefore, we have $f_{\max\min} = 6 < 7 = f_{\min\max}$.

5. Interval-valued approach

This section proposes an *interval-valued approach* as a new approach handling uncertainty.

We define $f^R : \mathbb{R}^n \to \mathbb{R}$ as

$$f^R(\boldsymbol{x}) = \max_{\boldsymbol{w} \in [\underline{\boldsymbol{w}}, \overline{\boldsymbol{w}}], \boldsymbol{b}_i \in \mathcal{U}_i, i=1,\dots,m} f(\boldsymbol{x}; \boldsymbol{w}, \boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_m) \tag{5.1}$$

for each $\boldsymbol{x} \in \mathbb{R}^n$. From Lemma 2.1, it follows that, for each $\boldsymbol{x} \in \mathbb{R}^n$,

$$\begin{aligned} f^R(\boldsymbol{x}) &= \max_{\boldsymbol{w} \in [\underline{\boldsymbol{w}}, \overline{\boldsymbol{w}}], \boldsymbol{b}_i \in \mathcal{U}_i, i=1,\dots,m} f(\boldsymbol{x}; \boldsymbol{w}, \boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_m) \\ &= \max_{\boldsymbol{w} \in [\underline{\boldsymbol{w}}, \overline{\boldsymbol{w}}], \boldsymbol{b}_i \in \mathcal{U}_i, i=1,\dots,m} \sum_{i=1}^m w_i \|\boldsymbol{x} - \boldsymbol{b}_i\| \\ &= \sum_{i=1}^m \overline{w}_i (\|\boldsymbol{x} - \boldsymbol{a}_i^0\| + r_i) \\ &= \sum_{i=1}^m \overline{w}_i \|\boldsymbol{x} - \boldsymbol{a}_i^0\| + \sum_{i=1}^m \overline{w}_i r_i \\ &= f(\boldsymbol{x}; \overline{\boldsymbol{w}}, \boldsymbol{a}_1^0, \boldsymbol{a}_2^0, \dots, \boldsymbol{a}_m^0) + \sum_{i=1}^m \overline{w}_i r_i. \end{aligned}$$

It also can be seen that f^R is a convex function. In addition, we define $f^L : \mathbb{R}^n \to \mathbb{R}$ as

$$f^L(\boldsymbol{x}) = \min_{\boldsymbol{w} \in [\underline{\boldsymbol{w}}, \overline{\boldsymbol{w}}], \boldsymbol{b}_i \in \mathcal{U}_i, i=1,\dots,m} f(\boldsymbol{x}; \boldsymbol{w}, \boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_m) \tag{5.2}$$

for each $\boldsymbol{x} \in \mathbb{R}^n$. From Lemma 2.2, it follows that, for each $\boldsymbol{x} \in \mathbb{R}^n$,

$$\begin{aligned} f^L(\boldsymbol{x}) &= \min_{\boldsymbol{w} \in [\underline{\boldsymbol{w}}, \overline{\boldsymbol{w}}], \boldsymbol{b}_i \in \mathcal{U}_i, i=1,\dots,m} f(\boldsymbol{x}; \boldsymbol{w}, \boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_m) \\ &= \min_{\boldsymbol{w} \in [\underline{\boldsymbol{w}}, \overline{\boldsymbol{w}}], \boldsymbol{b}_i \in \mathcal{U}_i, i=1,\dots,m} \sum_{i=1}^m w_i \|\boldsymbol{x} - \boldsymbol{b}_i\| \\ &= \sum_{i=1}^m \underline{w}_i \max\{\|\boldsymbol{x} - \boldsymbol{a}_i^0\| - r_i, 0\}. \end{aligned}$$

It also can be seen that f^L is a convex function. Since f is continuous, we have

$$f(\boldsymbol{x}; [\underline{\boldsymbol{w}}, \overline{\boldsymbol{w}}], \mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_m) = \left[f^L(\boldsymbol{x}), f^R(\boldsymbol{x})\right] \tag{5.3}$$

for each $\boldsymbol{x} \in \mathbb{R}^n$. Then, we consider the folloing *interval-valued minisum location problem*:

$$\text{(IP)} \qquad \min_{\boldsymbol{x} \in \mathbb{R}^n} \left[f^L(\boldsymbol{x}), f^R(\boldsymbol{x}) \right].$$

Now, for $a, b, c, d \in \mathbb{R}$ with $a \leq b$ and $c \leq d$, we define orderings of $[a, b]$ and $[c, d]$ as follows:

$$\begin{aligned} [a,b] \leq [c,d] &\overset{\text{def}}{\Leftrightarrow} a \leq c, b \leq d, \\ [a,b] < [c,d] &\overset{\text{def}}{\Leftrightarrow} a < c, b < d. \end{aligned}$$

Then, it follows that, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$,

$$\begin{aligned} \left[f^L(\boldsymbol{x}), f^R(\boldsymbol{x})\right] \leq \left[f^L(\boldsymbol{y}), f^R(\boldsymbol{y})\right] &\overset{\text{iff}}{\Leftrightarrow} \left(f^L(\boldsymbol{x}), f^R(\boldsymbol{x})\right) \leq \left(f^L(\boldsymbol{y}), f^R(\boldsymbol{y})\right), \\ \left[f^L(\boldsymbol{x}), f^R(\boldsymbol{x})\right] \neq \left[f^L(\boldsymbol{y}), f^R(\boldsymbol{y})\right] &\overset{\text{iff}}{\Leftrightarrow} \left(f^L(\boldsymbol{x}), f^R(\boldsymbol{x})\right) \neq \left(f^L(\boldsymbol{y}), f^R(\boldsymbol{y})\right), \\ \left[f^L(\boldsymbol{x}), f^R(\boldsymbol{x})\right] < \left[f^L(\boldsymbol{y}), f^R(\boldsymbol{y})\right] &\overset{\text{iff}}{\Leftrightarrow} \left(f^L(\boldsymbol{x}), f^R(\boldsymbol{x})\right) < \left(f^L(\boldsymbol{y}), f^R(\boldsymbol{y})\right). \end{aligned}$$

Thus, the interval-valued minisum location problem (IP) is equivalent to the following bicriteria programming problem:

$$\text{(MP)} \qquad \min_{\boldsymbol{x} \in \mathbb{R}^n} \left(f^L(\boldsymbol{x}), f^R(\boldsymbol{x}) \right).$$

Definition 5.1. Let $\overline{\boldsymbol{x}} \in \mathbb{R}^n$.

(i) $\overline{\boldsymbol{x}}$ is said to be an *efficient solution* or *globally efficient solution* of (IP) and (MP) if there is no $\boldsymbol{x} \in \mathbb{R}^n$ such that $(f^L(\boldsymbol{x}), f^R(\boldsymbol{x})) \leq (f^L(\overline{\boldsymbol{x}}), f^R(\overline{\boldsymbol{x}}))$ and $(f^L(\boldsymbol{x}), f^R(\boldsymbol{x})) \neq (f^L(\overline{\boldsymbol{x}}), f^R(\overline{\boldsymbol{x}}))$.

(ii) $\overline{\boldsymbol{x}}$ is said to be a *locally efficient solution* of (IP) and (MP) if for some neighborhood W of $\overline{\boldsymbol{x}}$, there is no $\boldsymbol{x} \in W$ such that $(f^L(\boldsymbol{x}), f^R(\boldsymbol{x})) \leq (f^L(\overline{\boldsymbol{x}}), f^R(\overline{\boldsymbol{x}}))$ and $(f^L(\boldsymbol{x}), f^R(\boldsymbol{x})) \neq (f^L(\overline{\boldsymbol{x}}), f^R(\overline{\boldsymbol{x}}))$.

(iii) $\overline{\boldsymbol{x}}$ is said to be a *weak efficient solution* or *globally weak efficient solution* of (IP) and (MP) if there is no $\boldsymbol{x} \in \mathbb{R}^n$ such that $(f^L(\boldsymbol{x}), f^R(\boldsymbol{x})) < (f^L(\overline{\boldsymbol{x}}), f^R(\overline{\boldsymbol{x}}))$.

(iv) $\overline{\boldsymbol{x}}$ is said to be a *locally weak efficient solution* of (IP) and (MP) if for some neighborhood W of $\overline{\boldsymbol{x}}$, there is no $\boldsymbol{x} \in W$ such that $(f^L(\boldsymbol{x}), f^R(\boldsymbol{x})) < (f^L(\overline{\boldsymbol{x}}), f^R(\overline{\boldsymbol{x}}))$.

Let $\overline{\boldsymbol{x}} \in \mathbb{R}^n$. By the convexity of f^R and f^L, if $\overline{\boldsymbol{x}}$ is a locally (weak) efficient solution of (IP) and (MP), then $\overline{\boldsymbol{x}}$ is a globally (weak) efficient solution of (IP) and (MP). Let E and WE be the sets of all efficient solutions and all weak efficient solutions of (IP) and (MP), respectively.

The scalarization method [7] can be applied to the bicriteria programming problem (MP). However, it is not trivial whether all (weak) efficient solutions can be determined even if the scalarized problems can be solved.

6. One-dimensional interval-valued minisum location problem

This section presents a procedure to find all (weak) efficient solutions of the one-dimensional interval-valued minisum location problem (IP).

Throughout this and next sections, we set $n = 1$, and let $\|\cdot\| = |\cdot|$. In addition, we represent $\boldsymbol{a}_i^0$ and $\boldsymbol{x}$ as a_i^0 and x, respectively. In this case,

$$\text{(P)} \qquad \min_{x\in\mathbb{R}} \sum_{i=1}^{m} w_i^0 |x - a_i^0|,$$

$$\text{(IP)} \qquad \min_{x\in\mathbb{R}} \left[f^L(x), f^R(x) \right],$$

$$\text{(MP)} \qquad \min_{x\in\mathbb{R}} \left(f^L(x), f^R(x) \right),$$

and

$$f^R(x) = \sum_{i=1}^{m} \overline{w}_i |x - a_i^0| + \sum_{i=1}^{m} \overline{w}_i r_i,$$

$$f^L(x) = \sum_{i=1}^{m} \underline{w}_i \max\{|x - a_i^0| - r_i, 0\}.$$

Let $c_{(1)}^R, c_{(2)}^R, \dots, c_{(p)}^R$ $(p \le m)$ be all distinct values among $a_1^0, a_2^0, \dots, a_m^0$ sorted in ascending order, namely, $c_{(1)}^R < c_{(2)}^R < \cdots < c_{(p)}^R (p \le m)$, and let $c_{(1)}^L, c_{(2)}^L, \dots, c_{(q)}^L$ $(q \le 3m)$ be all distinct values among $a_1^0, a_2^0, \dots, a_m^0, a_1^0 - r_1, a_2^0 - r_2, \dots, a_m^0 - r_m, a_1^0 + r_1, a_2^0 + r_2, \dots, a_m^0 + r_m$ sorted in ascending order, namely, $c_{(1)}^L < c_{(2)}^L < \cdots < c_{(q)}^L$. For convenience, we set $c_{(0)}^R = c_{(0)}^L = -\infty$ and $c_{(p+1)}^R = c_{(q+1)}^L = \infty$. For each $k \in \{1, 2, \dots, p+1\}$, we set $K_k^R = \left[c_{(k-1)}^R, c_{(k)}^R \right] = \{x \in \mathbb{R} : c_{(k-1)}^R \le x \le c_{(k)}^R\}$, and fix any $x_k^R \in \operatorname{int}(K_k^R)$, where $\operatorname{int}(K_k^R)$ is the interior of K_k^R. For each $k \in \{1, 2, \dots, q+1\}$, we set $K_k^L = \left[c_{(k-1)}^L, c_{(k)}^L \right] = \{x \in \mathbb{R} : c_{(k-1)}^L \le x \le c_{(k)}^L\}$, and fix any $x_k^L \in \operatorname{int}(K_k^L)$.

It can be seen that f^R is a piecewise linear convex function, and is linear on each K_k^R, $k \in \{1, 2, \dots, p+1\}$. In addition, it follows that

$$(6.1) \qquad (f^R)'(x_k^R) = \sum_{i \in \{j \in \{1,2,\dots,m\} : a_j^0 < x_k^R\}} \overline{w}_i - \sum_{i \in \{j \in \{1,2,\dots,m\} : a_j^0 > x_k^R\}} \overline{w}_i$$

for each $k \in \{1, 2, \dots, p+1\}$, especially,

$$(6.2) \qquad (f^R)'(x_1^R) = -\sum_{i=1}^{m} \overline{w}_i < 0, \quad (f^R)'(x_{p+1}^R) = \sum_{i=1}^{m} \overline{w}_i > 0.$$

Moreover, it follows that

$$(6.3) \qquad (f^R)'(x_1^R) < (f^R)'(x_2^R) < \cdots < (f^R)'(x_{p+1}^R).$$

Similarly, it can be seen that f^L is a piecewise linear convex function, and is linear on each K_k^L, $k \in \{1, 2, \dots, q+1\}$. In addition, it follows that

$$(f^L)'(x_k^L) = \sum_{i \in \{j \in \{1,2,\dots,m\}: a_j^0 + r_j < x_k^L\}} \underline{w}_i - \sum_{i \in \{j \in \{1,2,\dots,m\}: a_j^0 - r_j > x_k^L\}} \underline{w}_i \tag{6.4}$$

for each $k \in \{1, 2, \dots, q+1\}$, especially,

$$(f^L)'(x_1^L) = -\sum_{i=1}^{m} \underline{w}_i < 0, \quad (f^L)'(x_{q+1}^L) = \sum_{i=1}^{m} \underline{w}_i > 0. \tag{6.5}$$

Moreover, it follows that

$$(f^L)'(x_1^L) < (f^L)'(x_2^L) < \cdots < (f^L)'(x_{q+1}^L). \tag{6.6}$$

Let S^R be the set of all optimal solutions of the problem

$$\min_{x \in \mathbb{R}} f^R(x), \tag{6.7}$$

and let S^L be the set of all optimal solutions of the problem

$$\min_{x \in \mathbb{R}} f^L(x). \tag{6.8}$$

We set

$$k^R = \min\left\{k \in \{2, \dots, p+1\} : (f^R)'(x_k^R) \geq 0\right\}, \tag{6.9}$$

$$k^L = \min\left\{k \in \{2, \dots, q+1\} : (f^L)'(x_k^L) \geq 0\right\}. \tag{6.10}$$

Then, we have

$$S^R = \begin{cases} \left\{c_{(k^R-1)}^R\right\} & \text{if } (f^R)'(x_{k^R}^R) > 0, \\ K_{k^R}^R = \left[c_{(k^R-1)}^R, c_{(k^R)}^R\right] & \text{if } (f^R)'(x_{k^R}^R) = 0, \end{cases} \tag{6.11}$$

$$S^L = \begin{cases} \left\{c_{(k^L-1)}^L\right\} & \text{if } (f^L)'(x_{k^L}^L) > 0, \\ K_{k^L}^L = \left[c_{(k^L-1)}^L, c_{(k^L)}^L\right] & \text{if } (f^L)'(x_{k^L}^L) = 0, \end{cases} \tag{6.12}$$

and E and WE can be obtained as follows:

(i) Assume that $S^R = \left\{c_{(k^R-1)}^R\right\}$ and $S^L = \left\{c_{(k^L-1)}^L\right\}$. Then

$$E = WE = \left[\min\left\{c_{(k^R-1)}^R, c_{(k^L-1)}^L\right\}, \max\left\{c_{(k^R-1)}^R, c_{(k^L-1)}^L\right\}\right].$$

(ii) Assume that $S^R = \left\{c_{(k^R-1)}^R\right\}$ and $S^L = K_{k^L}^L = \left[c_{(k^L-1)}^L, c_{(k^L)}^L\right]$.

(ii-1) If $S^R \subset S^L$, then

$$E = S^R = \left\{c_{(k^R-1)}^R\right\}, \quad WE = S^L = K_{k^L}^L = \left[c_{(k^L-1)}^L, c_{(k^L)}^L\right].$$

(ii-2) If $c_{(k^R-1)}^R < c_{(k^L-1)}^L$, then

$$E = \left[c_{(k^R-1)}^R, c_{(k^L-1)}^L\right], \quad WE = \left[c_{(k^R-1)}^R, c_{(k^L)}^L\right].$$

(ii-3) If $c^L_{(k^L)} < c^R_{(k^R-1)}$, then

$$E = \left[c^L_{(k^L)}, c^R_{(k^R-1)}\right], \quad WE = \left[c^L_{(k^L-1)}, c^R_{(k^R-1)}\right].$$

(iii) Assume that $S^R = K^R_{k^R} = \left[c^R_{(k^R-1)}, c^R_{(k^R)}\right]$ and $S^L = \left\{c^L_{(k^L-1)}\right\}$.

(iii-1) If $S^L \subset S^R$, then

$$E = S^L = \left\{c^L_{(k^L-1)}\right\}, \quad WE = S^R = K^R_{k^R} = \left[c^R_{(k^R-1)}, c^R_{(k^R)}\right].$$

(iii-2) If $c^L_{(k^L-1)} < c^R_{(k^R-1)}$, then

$$E = \left[c^L_{(k^L-1)}, c^R_{(k^R-1)}\right], \quad WE = \left[c^L_{(k^L-1)}, c^R_{(k^R)}\right].$$

(iii-3) If $c^R_{(k^R)} < c^L_{(k^L-1)}$, then

$$E = \left[c^R_{(k^R)}, c^L_{(k^L-1)}\right], \quad WE = \left[c^R_{(k^R-1)}, c^L_{(k^L-1)}\right].$$

(iv) Assume that $S^R = K^R_{k^R} = \left[c^R_{(k^R-1)}, c^R_{(k^R)}\right]$ and $S^L = K^L_{k^L} = \left[c^L_{(k^L-1)}, c^L_{(k^L)}\right]$.

(iv-1) If $S^R \cap S^L \neq \emptyset$, then

$$E = S^R \cap S^L = \left[\max\left\{c^R_{(k^R-1)}, c^L_{(k^L-1)}\right\}, \min\left\{c^R_{(k^R)}, c^L_{(k^L)}\right\}\right],$$

$$WE = S^R \cup S^L = \left[\min\left\{c^R_{(k^R-1)}, c^L_{(k^L-1)}\right\}, \max\left\{c^R_{(k^R)}, c^L_{(k^L)}\right\}\right].$$

(iv-2) If $c^R_{(k^R)} < c^L_{(k^L-1)}$, then

$$E = \left[c^R_{(k^R)}, c^L_{(k^L-1)}\right], \quad WE = \left[c^R_{(k^R-1)}, c^L_{(k^L)}\right].$$

(iv-3) If $c^L_{(k^L)} < c^R_{(k^R-1)}$, then

$$E = \left[c^L_{(k^L)}, c^R_{(k^R-1)}\right], \quad WE = \left[c^L_{(k^L-1)}, c^R_{(k^R)}\right].$$

7. Numerical example

Using the similar settings as in the previous section, we set $m = 3$, and set $a^0_1 = 3$, $a^0_2 = 5$, $a^0_3 = 8$, $r_1 = r_2 = r_3 = 1$, $w^0_1 = w^0_2 = 3$, $w^0_3 = 4$, $\underline{w}_1 = 2$, $\underline{w}_2 = \underline{w}_3 = 1$, $\overline{w}_1 = \overline{w}_2 = 4$, $\overline{w}_3 = 8$. Then, it follows that (Figures 1 and 2)

$$\begin{aligned} f^R(x) &= \sum_{i=1}^m \overline{w}_i |x - a^0_i| + \sum_{i=1}^m \overline{w}_i r_i \\ &= 4|x-3| + 4|x-5| + 8|x-8| + 16, \end{aligned}$$

$$\begin{aligned} f^L(x) &= \sum_{i=1}^m \underline{w}_i \max\{|x - a^0_i| - r_i, 0\} \\ &= 2\max\{|x-3|-1, 0\} + \max\{|x-5|-1, 0\} \\ &\quad + \max\{|x-8|-1, 0\}. \end{aligned}$$

From Figures 1 and 2, it follows that $S^R = [5, 8]$ and $S^L = \{4\}$. Therefore, we have $E = [4, 5]$ and $WE = [4, 8]$ from (iii-2) in the previous section.

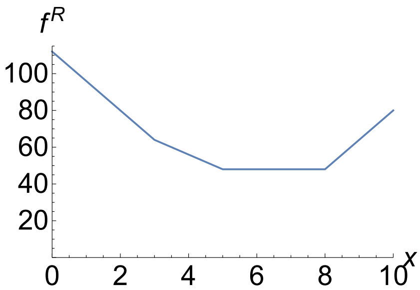

Figure 1 f^R.

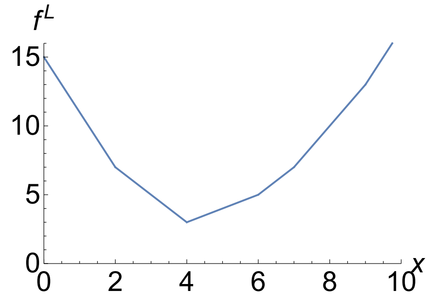

Figure 2 f^L.

8. Conclusion

In this paper, we considered a single facility location problem, where demand points and associated weights were given. The single facility location problem was a problem to minimize the total sum of weighted distances from the variable location of the facility to the demand points. Then, the demand points and weights were considered as parameters, and the demand points and weights with uncertainty were represented as bounded and interval uncertainty sets, respectively. First, we considered minimax and maximin problems to minimize and maximize with respect to the variable location of the facility and the parameters, respectively. Theorem 3.1 provided an optimal solution and value of the minimax problem. Theorem 3.3 provided the condition of coincidence between optimal values of the maximin and minimax problems, and presented an optimal solution of the maximin problem. Furthermore, we gave an example such that optimal values of the minimax and maximin problems did not coincide when the condition of Theorem 3.3 was not satisfied. Next, we proposed an interval-valued approach as a new approach handling uncertainty. Moreover, we also presented a procedure to find all (weak) efficient solutions of the one-dimensional interval-valued minisum location problem derived by the interval-valued approach.

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Manuscript received 20 October 2023
revised 29 November 2023

Masamichi Kon
Graduate School of Science and Technology, Hirosaki University, 3 Bunkyo, Hirosaki, Aomori 036-8561, Japan
E-mail address: masakon@hirosaki-u.ac.jp