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n VARIABLE LOGARITHMIC MEAN AND n VARIABLE IDENTRIC MEAN

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Abstract. It is well known that the Hermite-Hadamard inequality refines the definition of convexity of function $f(x)$ defined on $[a,b]$ by using the integral of $f(x)$ from a to b. There are many generalizations or refinements of the Hermite-Hadamard inequality. In this article, we give an n variable Hermite-Hadamard inequality and apply to give the definition of n variable logarithmic mean and n variable identric mean.

1. Introduction

A function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on $[a,b]$ if the inequality

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \tag{1.1}$$

holds for all $x, y \in [a,b]$. If inequality (1.1) reverses, then f is said to be concave on $[a,b]$. Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval $[a.b]$. Then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b f(t)dt \\ &= \int_0^1 f((1-t)a+tb)dt \\ &\le \frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.2}$$

This double inequality is known in the literature as the Hermite-Hadamard integral inequality for convex functions. It has many applications in more different areas of pure and applied mathematics. In Section 2, we try to obtain an n variable Hermite-Hadamard inequality. As applications, we give the definitions of n variable logarithmic mean and n variable operator logarithmic mean. In Section 3, we state other definitions of n variable logarithmic mean and n variable identric mean which have been given by [10, 11]. Finally in Section 4, we compare our 3 variable logarithmic mean and 3 variable identric mean with other 3 variable logarithmic mean and 3 variable identric mean.

2. Hermite-Hadamard inequality

We need the following result.

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Lemma 2.1 ([15]). *Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$ or $x_1, x_2, \ldots, x_n \in X$, where X is a linear space. Then*

$$\sum_{i=1}^{n} x_i = \frac{1}{n-1} \sum_{i<j} (x_i + x_j).$$

Proof.

$$\begin{aligned}
\sum_{i=1}^{n} x_i &= \frac{1}{2}\left\{\sum_{i=1}^{n} x_i + \sum_{j=1}^{n} x_j\right\} = \frac{1}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i + x_j) \\
&= \frac{1}{2n}\left\{2\sum_{i=1}^{n} x_i + \sum_{i\neq j}(x_i + x_j)\right\} \\
&= \frac{1}{n}\sum_{i=1}^{n} x_i + \frac{1}{2n}\left\{\sum_{i<j}(x_i + x_j) + \sum_{i>j}(x_i + x_j)\right\} \\
&= \frac{1}{n}\sum_{i=1}^{n} x_i + \frac{1}{n}\sum_{i<j}(x_i + x_j).
\end{aligned}$$

Then

$$\left(1 - \frac{1}{n}\right)\sum_{i=1}^{n} x_i = \frac{1}{n}\sum_{i<j}(x_i + x_j).$$

That is

$$\sum_{i=1}^{n} x_i = \frac{1}{n-1}\sum_{i<j}(x_i + x_j).$$

□

We have the following n variable Hermite-Hadamard inequality.

Theorem 2.2 ([15]). *Let $f(x)$ be a convex function on $\mathbb{R}$ and let $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Then*

$$\begin{aligned}
f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) &= f\left(\frac{1}{n(n-1)}\sum_{i<j}(x_i + x_j)\right) \\
&= f\left(\frac{2}{n(n-1)}\sum_{i<j}\frac{x_i + x_j}{2}\right) \\
&\leq \frac{2}{n(n-1)}\sum_{i<j} f\left(\frac{x_i + x_j}{2}\right) \\
&\leq \frac{2}{n(n-1)}\sum_{i<j}\int_0^1 f((1-t)x_i + t x_j)dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2}{n(n-1)}\sum_{i<j}\frac{f(x_i)+f(x_j)}{2}\\
&= \frac{1}{n(n-1)}\sum_{i<j}(f(x_i)+f(x_j))\\
&= \frac{1}{n}\sum_{i=1}^{n} f(x_i).
\end{aligned}
$$

Proof. The first equality is given by Lemma 2.1. The first inequality is given by the convexity of $f(x)$. From the second inequality to the third inequality are given by (1.2). And the last equality is given by Lemma 2.1. □

When $f(x) = e^x$, we have the following corollary.

Corollary 2.3. *Let $f(x) = e^x$. We suppose that $x_i \neq x_j$ for $i \neq j$. Then*

$$
\begin{aligned}
exp\left\{\frac{1}{n}\sum_{i=1}^{n} x_i\right\} &\leq \frac{2}{n(n-1)}\sum_{i<j}\frac{e^{x_i}-e^{x_j}}{x_i-x_j}\\
&\leq \frac{1}{n}\sum_{i=1}^{n} e^{x_i}.
\end{aligned}
$$

By putting $e^{x_i} = y_i, e^{x_j} = y_j$, we obtain

$$
\begin{aligned}
\left(\prod_{i=1}^{n} y_i\right)^{1/n} &\leq \frac{2}{n(n-1)}\sum_{i<j}\frac{y_i-y_j}{\log y_i-\log y_j}\\
&\leq \frac{1}{n}\sum_{i=1}^{n} y_i.
\end{aligned}
$$

Then we define n variable logarithmic mean as follows:

Definition 2.4. *Let $x_1, x_2 \ldots, x_n \in \mathbb{R}$ and let $x_i \neq x_j$ for $i \neq j$. Then n variable logarithmic mean is defined by*

$$
L_n^{(1)} = \frac{2}{n(n-1)}\sum_{i<j}\frac{x_i-x_j}{\log x_i-\log x_j}.
$$

We also define n variable operator logarithmic mean as follows:

Definition 2.5. *Let $A_1, A_2, \ldots, A_n$ be positive bounded linear operators on Hilbert space. Then n variable operator logarithmic mean is defined by*

$$
\frac{2}{n(n-1)}\sum_{i<j} A_i \ell A_j,
$$

where $A_i \ell A_j = \int_0^1 A_i \sharp_x A_j dx$ and $A_i \sharp_x A_j = A_i^{1/2}(A_i^{-1/2} A_j A_i^{-1/2})^x A_i^{1/2}$.

When $f(x) = -\log x$, we have the following corollary.

Corollary 2.6. *Let $f(x) = -\log x$. We suppose that $x_i \neq x_j$ for $i \neq j$. Then*

$$-\log \frac{1}{n}\sum_{i=1}^{n} x_i \leq \frac{2}{n(n-1)}\sum_{i<j}\left\{\frac{x_i \log x_i}{x_j - x_i} - \frac{x_j \log x_j}{x_j - x_i} + 1\right\} \leq -\frac{1}{n}\sum_{i=1}^{n}\log x_i.$$

That is

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} x_i &\geq \exp\left\{\frac{2}{n(n-1)}\sum_{i<j}\left\{\frac{x_i \log x_i}{x_i - x_j} + \frac{x_j \log x_j}{x_j - x_i} - 1\right\}\right\} \\ &= \exp\left\{\frac{2}{n(n-1)}\sum_{i<j}\log\left(\frac{1}{e}x_i^{\frac{x_i}{x_i - x_j}} x_j^{\frac{x_j}{x_j - x_i}}\right)\right\} \\ &\geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}. \end{aligned}$$

Then we define n variable identric mean as follows:

Definition 2.7. *Let $x_1, x_2 \ldots, x_n \in \mathbb{R}$ and let $x_i \neq x_j$ for $i \neq j$. Then n variable identric mean is defined by*

$$I_n^{(1)} = \exp\left\{\frac{2}{n(n-1)}\sum_{i<j}\log\left(\frac{1}{e}x_i^{\frac{x_i}{x_i - x_j}} x_j^{\frac{x_j}{x_j - x_i}}\right)\right\}$$

3. n Variable logarithmic mean and n variable identric mean

As another extension to n variable Hermite-Hadamard inequality, the following theorem has been given by [10, 11].

Theorem 3.1. *Let $f(x)$ be a convex function on $\mathbb{R}$ and let $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Then*

$$\begin{aligned} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) &\leq (n-1)!\int_{\Delta_{n-1}} f\left(\sum_{i=1}^{n} t_i x_i\right) dt_1 \cdots dt_{n-1} \\ &\leq \frac{1}{n}\sum_{i=1}^{n} f(x_i), \end{aligned}$$

where $\Delta_{n-1} = \{(t_1, t_2, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} : t_1 + \cdots + t_{n-1} \leq 1, t_i \geq 0\}$ and $t_n = 1 - \sum_{i=1}^{n-1} t_i$.

When $f(x) = e^x$, we have the following corollary.

Corollary 3.2. *Let $f(x) = e^x$. Then*

$$\exp\left\{\frac{1}{n}\sum_{i=1}^{n} x_i\right\} \le (n-1)! \int_{\Delta_{n-1}} \exp\left\{\sum_{i=1}^{n} t_i x_i\right\} dt_1 \cdots dt_{n-1}$$
$$\le \frac{1}{n}\sum_{i=1}^{n} \exp\{x_i\}.$$

By putting $e^{x_i} = y_i, e^{x_j} = y_j$, we obtain

$$\exp\left\{\frac{1}{n}\sum_{i=1}^{n} \log y_i\right\} \le (n-1)! \int_{\Delta_{n-1}} \exp\left\{\sum_{i=1}^{n} t_i \log y_i\right\} dt_1 \cdots dt_{n-1}$$
$$\le \frac{1}{n}\sum_{i=1}^{n} \exp\{\log y_i\}.$$

Then we have

$$\left(\prod_{i=1}^{n} y_i\right)^{1/n} \le (n-1)! \int_{\Delta_{n-1}} \exp\left\{\sum_{i=1}^{n} t_i \log y_i\right\} dt_1 \cdots dt_{n-1}$$
$$\le \frac{1}{n}\sum_{i=1}^{n} y_i.$$

Then we define n variable logarithmic mean as follows:

Definition 3.3. *Let $x_i, x_2, \ldots, x_n \in \mathbb{R}$. Then n variable logarithmic mean is defined by*

$$L_n^{(2)} = (n-1)! \int_{\Delta_{n-1}} \left(\prod_{i=1}^{n} x_i^{t_i}\right) dt_1 \cdots dt_{n-1}$$
$$= (n-1)! \sum_{i=1}^{n} \frac{x_i}{\prod_{j=1, j\neq i}^{n} (\log x_i - \log x_j)}.$$

When $f(x) = -\log x$, we have the following corollary.

Corollary 3.4. *Let $f(x) = -\log x$. Then*

$$-\log\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le -(n-1)! \int_{\Delta_{n-1}} \log\left(\sum_{i=1}^{n} t_i x_i\right) dt_1 \cdots dt_{n-1} \le -\frac{1}{n}\sum_{i=1}^{n} \log x_i.$$

Then we have

$$\frac{1}{n}\sum_{i=1}^{n} x_i \ge \exp\left\{(n-1)! \int_{\Delta_{n-1}} \log\left(\sum_{i=1}^{n} t_i x_i\right) dt_1 \cdots dt_{n-1}\right\} \ge \left(\prod_{i=1}^{n} x_i\right)^{1/n}.$$

Then n variable identric mean is defined as follows:

Definition 3.5. *Let $x_i, x_2, \ldots, x_n \in \mathbb{R}$. Then n variable identric mean is defined by*

$$I_n^{(2)} = \exp\left\{(n-1)! \int_{\Delta_{n-1}} \log\left(\sum_{i=1}^{n} t_i x_i\right) dt_1 \cdots dt_{n-1}\right\}.$$

4. The comparison between two types of 3 variable logarithmic mean and 3 variable identric mean

When $n = 3$, $L_3^{(1)}$ and $L_3^{(2)}$ are represented in the followings.

$$L_3^{(1)} = \frac{1}{3}\left\{\frac{x_1 - x_2}{\log \frac{x_1}{x_2}} + \frac{x_1 - x_3}{\log \frac{x_1}{x_3}} + \frac{x_2 - x_3}{\log \frac{x_2}{x_3}}\right\}$$

and

$$L_3^{(2)} = 2\left\{\frac{x_1}{\log \frac{x_1}{x_2} \log \frac{x_1}{x_3}} + \frac{x_2}{\log \frac{x_2}{x_1} \log \frac{x_2}{x_3}} + \frac{x_3}{\log \frac{x_3}{x_1} \log \frac{x_3}{x_2}}\right\}.$$

We compare $L_3^{(1)}$ with $L_3^{(2)}$. When $x_1 = 1000, x_2 = 1001, x_3 = 1002$, we have

$$L_3^{(1)} = 1000.999\cdots, \ L_3^{(2)} = 1002.$$

Then $L_3^{(1)} < L_3^{(2)}$. When $x_1 = 1000, x_2 = 1010, x_3 = 2000$, we have

$$L_3^{(1)} = 1298.918\cdots, \ L_3^{(2)} = 1281.339\cdots.$$

Then $L_3^{(1)} > L_3^{(2)}$. We can't compare between $L_3^{(1)}$ and $L_3^{(2)}$.
When $n = 3$, $I_3^{(1)}$ and $I_3^{(2)}$ are represented in the followings.

$$I_3^{(1)} = \frac{1}{e}(x_1^{x_1})^{\frac{2x_1 - x_2 - x_3}{3(x_1 - x_2)(x_1 - x_3)}} (x_2^{x_2})^{\frac{2x_2 - x_1 - x_3}{3(x_2 - x_1)(x_2 - x_3)}} (x_3^{x_3})^{\frac{2x_3 - x_1 - x_2}{3(x_3 - x_1)(x_3 - x_2)}} \tag{4.1}$$

and

$$I_3^{(2)} = e^{-\frac{3}{2}} (x_1^{x_1})^{-\frac{x_1}{(x_1 - x_2)(x_3 - x_1)}} (x_2^{x_2})^{-\frac{x_2}{(x_1 - x_2)(x_2 - x_3)}} (x_3^{x_3})^{-\frac{x_3}{(x_2 - x_3)(x_3 - x_1)}}. \tag{4.2}$$

We compare between $I_3^{(1)}$ and $I_3^{(2)}$.

Theorem 4.1. *If $0 < x_1 < x_2 < x_3$, then $I_3^{(1)} < I_3^{(2)}$.*

Proof. By taking the logarithm for (4.1) and (4.2).

$$\begin{aligned} &\log I_3^{(1)} \\ = \ &\frac{1}{3}\left\{\left(\frac{1}{x_1 - x_2} + \frac{1}{x_1 - x_3}\right) x_1 \log x_1 + \left(\frac{1}{x_2 - x_1} + \frac{1}{x_2 - x_3}\right) x_2 \log x_2\right\}. \\ &+\frac{1}{3}\left\{\left(\frac{1}{x_3 - x_1} + \frac{1}{x_3 - x_2}\right) x_3 \log x_3\right\} - 1 \end{aligned}$$

and

$$\begin{aligned} &\log I_3^{(2)} \\ = \ &\frac{x_1}{(x_1 - x_2)(x_1 - x_3)} x_1 \log x_1 + \frac{x_2}{(x_2 - x_1)(x_2 - x_3)} x_2 \log x_2 \end{aligned}$$

$$+\frac{x_3}{(x_3-x_1)(x_3-x_2)}x_3\log x_3-\frac{3}{2}.$$

Then

$$\log I_3^{(2)}-\log I_3^{(1)}=\frac{x_1+x_2+x_3}{3}\sum_{i=1}^{3}\frac{x_i\log x_i}{\prod_{j=1,j\neq i}(x_i-x_j)}-\frac{1}{2}.$$

We put $x_1=x, x_2=ax, x_3=bx$, where $x>0, 1<a<b$. Then we have

$$\begin{aligned}&\log I_3^{(2)}-\log I_3^{(1)}\\ =\ &\frac{1+a+b}{3}\left\{\frac{b\log b}{(b-1)(b-a)}-\frac{a\log a}{(a-1)(b-a)}\right\}-\frac{1}{2}\\ =\ &\frac{1+a+b}{3(b-a)}\left\{\frac{b\log b}{b-1}-\frac{a\log a}{a-1}-\frac{3(b-a)}{2(1+a+b)}\right\}.\end{aligned}$$

Now we put

$$F(a,b)=\frac{b\log b}{b-1}-\frac{a\log a}{a-1}-\frac{3(b-a)}{2(1+a+b)}.$$

Then

$$\frac{\partial F}{\partial b}(a,b)=\frac{(\log b+1)(b-1)-b\log b}{(b-1)^2}-\frac{3}{2}\frac{1+2a}{(1+a+b)^2}$$

and

$$\frac{\partial F}{\partial a\partial b}(a,b)=-\frac{3(b-a)}{(1+a+b)^3}<0.$$

Since $\frac{\partial F}{\partial b}(a,b)$ is a decreasing function of a,

$$\frac{\partial F}{\partial b}(a,b)>\frac{\partial F}{\partial b}(b,b)=\frac{b^2+4b-5-(4b+2)\log b}{2(b-1)^2(2b+1)}.$$

We put $k(b)=b^2+4b-5-(4b+2)\log b$. Then

$$\begin{aligned}k'(b)&=2b+4-4\log b-\frac{4b+2}{b}\\ &=2(b-b^{-1}-2\log b)\\ &=2b^{-1}(b^2-1-2b\log b).\end{aligned}$$

Furthermore we put $\ell(b)=b^2-1-2b\log b$. Since $\ell'(b)=2(b-1-\log b)>0$ and $\ell(1)=0$, we get $\ell(b)>0$. Then $k'(b)>0$. By $k(1)=0$, $k(b)>0$. Since

$$\frac{\partial F}{\partial b}(a,b)>\frac{\partial F}{\partial b}(b,b)>0,$$

$F(a,b)$ is a increasing function of b. That is $F(a,b)>F(a,a)=0$. We prove the result. □

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