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N VARIABLE LOGARITHMIC MEAN

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Abstract. It is well known that the Hermite-Hadamard inequality refines the definition of convexity of function $f(x)$ defined on $[a,b]$ by using the integral of $f(x)$ from a to b. There are many generalizations or refinements of the Hermite-Hadamard inequality. In this article, we give an N variable Hermite-Hadamard inequality and apply to give the definition of N variable logarithmic mean.

1. Introduction

A function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on $[a,b]$ if the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{1.1}$$

holds for all $x, y \in [a,b]$. If inequality (1.1) reverses, then f is said to be concave on $[a,b]$. Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval $[a.b]$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt = \int_0^1 f((1-t)a+tb)dt \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

This double inequality is known in the literature as the Hermite-Hadamard integral inequality for convex functions. It has many applications in more different areas of pure and applied mathematics. In Section 2, we try to obtain an N variable Hermite-Hadamard inequality. As applications, we give the definition of N variable logarithmic mean and N variable operator logarithmic mean. In Section 3, we compare our N variable logarithmic mean and another N variable logarithmic mean which has been defined in [10, 11]. We show that we can't compare those means by taking examples.

2. Hermite-Hadamard inequality

We need the following result.

Lemma 2.1 ([15]). *Let $x_1, x_2, \ldots, x_N \in \mathbb{R}$ or $x_1, x_2, \ldots, x_N \in X$, where X is a linear space. Then*

$$\sum_{i=1}^N x_i = \frac{1}{N-1}\sum_{i<j}(x_i + x_j).$$

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Proof.

$$\begin{aligned}
\sum_{i=1}^{N} x_i &= \frac{1}{2}\left\{\sum_{i=1}^{N} x_i + \sum_{j=1}^{N} x_j\right\} = \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}(x_i + x_j) \\
&= \frac{1}{2N}\left\{2\sum_{i=1}^{N} x_i + \sum_{i\neq j}(x_i + x_j)\right\} \\
&= \frac{1}{N}\sum_{i=1}^{N} x_i + \frac{1}{2N}\left\{\sum_{i<j}(x_i + x_j) + \sum_{i>j}(x_i + x_j)\right\} \\
&= \frac{1}{N}\sum_{i=1}^{N} x_i + \frac{1}{N}\sum_{i<j}(x_i + x_j).
\end{aligned}$$

Then

$$\left(1 - \frac{1}{N}\right)\sum_{i=1}^{N} x_i = \frac{1}{N}\sum_{i<j}(x_i + x_j).$$

That is

$$\sum_{i=1}^{N} x_i = \frac{1}{N-1}\sum_{i<j}(x_i + x_j).$$

□

We have the following N variable Hermite-Hadamard inequality.

Theorem 2.2 ([15])**.** *Let $f(x)$ be a convex function on $\mathbb{R}$ and let $x_1, x_2, \ldots, x_N \in \mathbb{R}$. Then*

$$\begin{aligned}
f\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right) &= f\left(\frac{1}{N(N-1)}\sum_{i<j}(x_i + x_j)\right) = f\left(\frac{2}{N(N-1)}\sum_{i<j}\frac{x_i + x_j}{2}\right) \\
&\leq \frac{2}{N(N-1)}\sum_{i<j} f\left(\frac{x_i + x_j}{2}\right) \\
&\leq \frac{2}{N(N-1)}\sum_{i<j}\int_0^1 f((1-t)x_i + tx_j)dt \\
&\leq \frac{2}{N(N-1)}\sum_{i<j}\frac{f(x_i) + f(x_j)}{2} \\
&= \frac{1}{N(N-1)}\sum_{i<j}(f(x_i) + f(x_j)) = \frac{1}{N}\sum_{i=1}^{N} f(x_i).
\end{aligned}$$

Proof. The first equality is given by Lemma 2.1. The first inequality is given by the convexity of $f(x)$. From the second inequality to the third inequality are given by (1.2). And the last equality is given by Lemma 2.1. □

When $f(x) = e^x$, we have the following corollary.

Corollary 2.3. *Let $f(x) = e^x$. We suppose that $x_i \neq x_j$ for $i \neq j$. Then*

$$exp\left\{\frac{1}{N}\sum_{i=1}^{N} x_i\right\} \leq \frac{2}{N(N-1)}\sum_{i<j}\frac{e^{x_i}-e^{x_j}}{x_i - x_j} \leq \frac{1}{N}\sum_{i=1}^{N} e^{x_i}.$$

By putting $e^{x_i} = y_i, e^{x_j} = y_j$ we obtain

$$\left(\prod_{i=1}^{N} y_i\right)^{1/N} \leq \frac{2}{N(N-1)}\sum_{i<j}\frac{y_i - y_j}{\log y_i - \log y_j} \leq \frac{1}{N}\sum_{i=1}^{N} y_i.$$

Then we define N variable logarithmic mean as follows:

Definition 2.4. *Let $x_1, x_2 \ldots, x_N \in \mathbb{R}$ and let $x_i \neq x_j$ for $i \neq j$. Then N variable logarithmic mean is defined by*

$$I_1 = \frac{2}{N(N-1)}\sum_{i<j}\frac{x_i - x_j}{\log x_i - \log x_j}.$$

We also define N variable operator logarithmic mean as follows:

Definition 2.5. *Let $A_1, A_2, \ldots, A_N$ be positive bounded linear operators on Hilbert space. Then N variable operator logarithmic mean is defined by*

$$\frac{2}{N(N-1)}\sum_{i<j} A_i \ell A_j,$$

where $A_i \ell A_j = \int_0^1 A_i \sharp_x A_j dx$ and $A_i \sharp_x A_j = A_i^{1/2}(A_i^{-1/2} A_j A_i^{-1/2})^x A_i^{1/2}$.

3. N Variable logarithmic mean

The another definition of N variable Hermite-Hadamard inequality has been given by [10, 11].

Definition 3.1. *Let $f(x)$ be a convex function on $\mathbb{R}$ and let $x_1, x_2, \ldots, x_N \in \mathbb{R}$. Then*

$$\begin{aligned} f\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right) &\leq (N-1)!\int_{\Delta_{N-1}} f\left(\sum_{i=1}^{N} t_i x_i\right) dt_1 dt_2 \cdots dt_{N-1} \\ &\leq \frac{1}{N}\sum_{i=1}^{N} f(x_i), \end{aligned}$$

where $\Delta_{N-1} = \{(t_1, t_2, \cdots, t_{N-1}) \in \mathbb{R}^{N-1} : t_1 + \cdots + t_{N-1} \leq 1, t_i \geq 0\}$ and $t_N = 1 - \sum_{i=1}^{N-1} t_i$.

When $f(x) = x^{-1}$, we have the following corollary.

Corollary 3.2. *Let $f(x) = x^{-1}$. Then*

$$\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)^{-1} \leq (N-1)!\int_{\Delta_{N-1}} \left(\sum_{i=1}^{N} t_i x_i\right)^{-1} dt_1 \cdots dt_{N-1} \leq \frac{1}{N}\sum_{i=1}^{N} x_i^{-1}.$$

That is

$$\left(\frac{1}{N}\sum_{i=1}^{N} x_i^{-1}\right)^{-1} \leq \left((N-1)!\int_{\Delta_{N-1}} \left(\sum_{i=1}^{N} t_i x_i\right)^{-1} dt_1 \cdots dt_{N-1}\right)^{-1} \leq \frac{1}{N}\sum_{i=1}^{N} x_i.$$

Then N variable logarithmic mean is defined as follows:

Definition 3.3. *Let $x_i, x_2, \cdots, x_N \in \mathbb{R}$. Then N variable logarithmic mean is defined by*

$$I_2 = \left((N-1)!\int_{\Delta_{N-1}} \left(\sum_{i=1}^{N} t_i x_i\right)^{-1} dt_1 \cdots dt_{N-1}\right)^{-1}.$$

4. The comparison between two definitions

,
When $N = 3$, I_1 and I_2 are represented in the followings.

$$I_1 = \frac{1}{3}\left\{\frac{x_1 - x_2}{\log x_1 - \log x_2} + \frac{x_2 - x_3}{\log x_2 - \log x_3} + \frac{x_3 - x_1}{\log x_3 - \log x_1}\right\},$$

and

$$I_2 = \frac{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}{2\{x_1(x_3 - x_2)\log x_1 + x_2(x_1 - x_3)\log x_2 + x_3(x_2 - x_1)\log x_3\}}.$$

In order to compare I_1 with I_2, we put $x_1 = x, x_2 = (1+s)x, x_3 = (1+t)x$. Then

$$I_1 = \frac{x}{3}\left\{\frac{s}{\log(1+s)} + \frac{t-s}{\log(1+t) - \log(1+s)} + \frac{t}{\log(1+t)}\right\},$$

and

$$I_2^{-1} = \frac{2}{(t-s)x}\left\{\frac{(1+t)\log(1+t)}{t} - \frac{(1+s)\log(1+s)}{s}\right\}.$$

When $s = 1, t = 2$, we have $I_1 I_2^{-1} = 0.999312\cdots$. Then $I_1 < I_2$. On the other hand when $s = 1, t = 100$, we have $I_1 I_2^{-1} = 1.0663634\cdots$. Then $I_1 > I_2$. Therefore we can't compare I_1 with I_2.

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