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EXISTENCE AND CONVERGENCE OF BEST PROXIMITY POINTS FOR CYCLIC ENRICHED CONTRACTIONS

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Abstract. In this manuscript, we give some convergence and existence results of best proximity points for the class of cyclic enriched type contraction mappings. We also obtain the existence of a fixed point under weaker conditions. Also, some illustrative examples are provided to show the validity of the results obtained.

1. Introduction and preliminaries

Let Y be a nonempty subset of a metric space (X, d). A mapping $T : Y \to X$ is said to have a fixed point in Y, if the fixed point equation $Tx = x$ has at least one solution. That is, $x \in Y$ is a fixed point of T if $d(x, Tx) = 0$. The case when fixed point equation $Tx = x$ does not have a solution, then $d(x, Tx) > 0$ for all $x \in Y$. In such circumstances, we are in searching for an element $x \in Y$ such that $d(x, Tx)$ is minimum in some sense. The best approximation theory and best proximity pair theorems are studied in this direction. Consider a pair of nonempty subsets (A, B) of a metric space (X, d). A mapping $T : A \to B$ is said to have a best proximity point if $d(x, Tx) = d(A, B)$. If $d(A, B) = 0$, best proximity point is nothing but a fixed point of T.

In this paper, we deal with a problem of optimization which is at par with the approximate solution of a fixed point equation $d(x, Tx) = 0$. The problem is of global minima which has nothing to do with the establishment of such theory of best approximation while we are inclined to investigate best proximity theorems. Motivated with enriched type contraction mappings introduced by Berinde et al. [5, 4], we introduce cyclic enriched type contraction mappings in the setting of convex metric spaces. We give some convergence and existence results of best proximity points for the class of cyclic enriched type contraction mappings. We also illustrate some examples to show the validity of the results obtained.

Definition 1.1. A Banach space X is said to be uniformly convex if there exists a strictly increasing function $\delta : (0, 2] \to [0, 1]$ such that the following implication holds for all $x, y, p \in X$, $R > 0$ and $r \in [0, 2R]$ such that

$$\left.\begin{array}{r} ||x-p|| \leq R \\ ||y-p|| \leq R \\ ||x-y|| \geq r \end{array}\right\} \quad \Longrightarrow \quad ||(x+y)/2 - p|| \leq (1 - \delta(r/R))R.$$

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Definition 1.2. Let (X,d) be a metric space. A continuous mapping $W : X \times X \times [0,1] \to X$ is said to be a convex structure on X if for all $x, y \in X$ and $\lambda \in [0,1]$,

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y),$$

holds for all $u \in X$. The metric space (X,d) together with a convex structure W, that is (X,d,W), is called a convex metric space (see [9]).

Definition 1.3. A nonempty subset A of a convex metric space (X,d,W) is said to be a convex set [9] if $W(x,y,\lambda) \in A$ for all $x, y \in A$ and $\lambda \in [0,1]$.

A normed linear space and each of its convex subset are simple examples of convex metric spaces with W given by $W(x,y,\lambda) = \lambda x + (1-\lambda)y$ for $x, y \in X$ and $0 \leq \lambda \leq 1$. There are many convex metric spaces which are not normed linear spaces (see [9]).

The following result present some fundamental properties of a convex metric space in the sense of Definition 1.2 (see [2, 9] for more details).

Lemma 1.4. *Let (X,d,W) be a convex metric space. For each $x, y \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0,1]$, we have the following:*

(i) $W(x,x,\lambda) = x$, $W(x,y,0) = y$ *and* $W(x,y,1) = x$.
(ii) $d(x, W(x,y,\lambda)) = (1-\lambda)d(x,y)$ *and* $d(y, W(x,y,\lambda)) = \lambda d(x,y)$.
(iii) $d(x,y) = d(x, W(x,y,\lambda)) + d(W(x,y,\lambda), y)$.
(iv) $|\lambda_1 - \lambda_2| d(x,y) \leq d(W(x,y,\lambda_1), W(x,y,\lambda_2))$.

Definition 1.5. Let (X,d,W) be a convex metric space and $T : X \to X$ be a self mapping. Define the mapping $T_\lambda : X \to X$ as $T_\lambda x = W(x,Tx;\lambda)$, for all $x \in X$. T is said to be an enriched contraction (see [5]) if there exist $c \in [0,1)$ and $\lambda \in [0,1)$ such that

$$d(T_\lambda x, T_\lambda y) = d(W(x,Tx;\lambda), W(y,Ty;\lambda)) \leq c\ d(x,y),$$

for all $x, y \in X$.

Definition 1.6. Let A and B be nonempty subsets of a metric space X. A map $T : A \cup B \to A \cup B$ is a cyclic contraction map if it satisfies:

(i) $T(A) \subseteq B$ and $T(B) \subseteq A$;
(ii) for some $k \in (0,1)$ we have

$$d(Tx,Ty) \leq k\ d(x,y) + (1-k)dist(A,B),$$

for all $x \in A$, $y \in B$.

2. Main results

2.1. **Cyclic enriched contraction map.** To start with, we give the following definition.

Definition 2.1. Let A and B be nonempty subsets of a convex metric space (X,d,W). A map $T : A \cup B \to A \cup B$ is a cyclic enriched contraction map if it satisfies:

(i) $T(A) \subseteq B$ and $T(B) \subseteq A$.

(ii) there exist $c \in [0,1)$ and $\lambda \in [0,1)$ such that

$$d(T_\lambda x, T_\lambda y) = d(W(x,Tx;\lambda), W(y,Ty;\lambda)) \leq c\, d(x,y) + (1-c)\, dist(A,B),$$

for all $x \in A$, $y \in B$.

Here, it is to note that (ii) implies that T satisfies

$$d(W(x,Tx;\lambda), W(y,Ty;\lambda)) \leq d(x,y),$$

for all $x \in A$, $y \in B$.

Also (ii) can be rewritten as

$$(d(W(x,Tx;\lambda), W(y,Ty;\lambda)) - dist(A,B)) \leq k\ (d(x,y) - dist(A,B)),$$

for all $x \in A$, $y \in B$.

Notice that if $F(T)$ is the set of fixed points of a cyclic enriched contraction map $T : A \cup B \to A \cup B$, then $F(T) \subseteq A \cap B$. Also $F(T)$ is convex.

The following approximation result will be needed in what follows.

Theorem 2.2. *Let A and B be nonempty subsets of a convex metric space (X,d,W). Suppose that $T : A \cup B \to A \cup B$ is a cyclic enriched contraction map. Then starting with any x_0 in $A \cup B$ we have $d(x_n, T_\lambda x_n) \to dist(A,B)$, where $x_{n+1} = W(x_n, Tx_n;\lambda) = T_\lambda x_n$, $n \geq 0$.*

Proof. Now, using definition of cyclic enriched contraction map, we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T_\lambda x_n, T_\lambda x_{n-1}) \\ &\leq c\, d(x_n, x_{n-1}) + (1-c)\, dist(A,B) \\ &\leq c\ (c\, d(x_{n-1}, x_{n-2}) + (1-c)\, dist(A,B)) + (1-c) dist(A,B) \\ &= c^2\, d(x_{n-1}, x_{n-2}) + (1-c^2)\, dist(A,B). \end{aligned}$$

Inductively, we have $d(x_{n+1}, x_n) \leq c^n\, d(x_{n-1}, x_{n-2}) + (1-c^n)\, dist(A,B)$. Therefore, $d(x_n, T_\lambda x_n) \to dist(A,B)$. □

Proposition 3.1 of [3] and Theorem 3 of [1] are special cases of the above theorem.

Next, we give an existence result for a best proximity point.

Theorem 2.3. *Let A and B be nonempty subsets of a convex metric space (X,d,W). Suppose that $T : A \cup B \to A \cup B$ is a cyclic enriched contraction map, x_0 in A and $x_{n+1} = W(x_n, Tx_n;\lambda) = T_\lambda x_n$, $n \geq 0$. If $\{x_{2n}\}$ has a convergent subsequence in A, then there exists $x \in A$ such that $d(x, T_\lambda x) = dist(A,B)$.*

Proof. Let $\{x_{2n(k)}\}$ be a subsequence of $\{x_{2n}\}$ converging to x in A. Since T is a cyclic enriched contraction, it follows that $dist(A,B) \leq d(x_{2n(k)}, T_\lambda x) \leq d(x_{2n(k)-1}, x) \leq d(x_{2n(k)-1}, x_{2n(k)}) + d(x_{2n(k)}, x)$. Thus $d(x_{2n(k)-1}, x) \to d(A,B)$ and $d(x, T_\lambda x) = d(A,B)$. □

Proposition 3.2 of [3] and Theorem 4 of [1] are special cases of the above result.

The following result ascertains the boundedness of the Krasnoselskij ' s iterates for a cyclic enriched contraction.

Theorem 2.4. *Let A and B be nonempty subsets of a convex metric space (X, d, W). Suppose that $T : A \cup B \to A \cup B$ is a cyclic enriched contraction map, x_0 in $A \cup B$ and $x_{n+1} = W(x_n, Tx_n; \lambda) = T_\lambda x_n$, $n \geq 0$. Then the sequence $\{x_n\}$ is bounded.*

Proof. Since T is a cyclic enriched contraction, $\{d(x_{2n}, x_{2n+1})\}$ is a decreasing sequence of non-negative terms. Therefore, $\{d(x_{2n}, x_{2n+1})\}$ is bounded. Further, because T is a cyclic enriched contraction, we have

$$\begin{aligned} d(x_{2n+1}, x_2) &= d(T_\lambda x_{2n}, T_\lambda x_1) \\ &\leq c\ d(x_{2n}, x_1) + (1-c)\ dist(A, B) \\ &\leq c\ [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_2) + d(x_2, x_1)] + (1-c)\ dist(A, B). \end{aligned}$$

Therefore, it can be concluded that

$$d(x_{2n+1}, x_2) \leq \frac{c}{1-c}[d(x_{2n}, x_{2n+1}) + d(x_2, x_1)] + dist(A, B).$$

Thus, it follows that the sequence $\{x_{2n+1}\}$ is bounded. Similarly, it can be shown that the sequence $\{x_{2n}\}$ is also bounded. Therefore, the sequence $\{x_n\}$ is bounded. This completes the proof of the proposition. □

The following result furnishes some sufficient conditions under which a cyclic enriched contraction has a best approximation.

Theorem 2.5. *Let A and B be nonempty subsets of a convex metric space (X, d, W). Suppose that $T : A \cup B \to A \cup B$ is a cyclic enriched contraction map. If either A or B is boundedly compact, then there exists x in $A \cup B$ with $d(x, T_\lambda x) = dist(A, B)$.*

Proof. The result follows directly from Theorems 2.3 and 2.4. □

Lemma 2.6. *Let A be a nonempty convex subset and B be a nonempty subset of a uniformly convex Banach space X. Suppose that $T : A \cup B \to A \cup B$ is a map such that $T(A) \subseteq B$ and $T(B) \subseteq A$. For $x_0 \in A$, define $x_{n+1} = T_\lambda x_n = (1-\lambda)x_n + \lambda T x_n$ for each $n \geq 0$. Then $||x_{2n+2} - x_{2n}|| \to 0$ and $||x_{2n+3} - x_{2n+1}|| \to 0$ as $n \to \infty$.*

Proof. To show that $||x_{2n+2} - x_{2n}|| \to 0$ as $n \to \infty$, assume the contrary. Then there exists $\varepsilon_0 > 0$ such that for each $n \geq 1$, there exists $n(k) > n$ so that

$$||x_{2n(k)+2} - x_{2n(k)}|| \geq \varepsilon_0. \tag{2.1}$$

Choose $0 < \gamma < 1$ such that $\dfrac{\varepsilon_0}{\gamma} > dist(A, B)$.

Take ε such that $0 < \varepsilon < \min\{\dfrac{\varepsilon_0}{\gamma} - dist(A, B), \dfrac{dist(A, B)\ \delta(\gamma)}{1 - \delta(\gamma)}\}$.

By Theorem 2.2, there exists N_1 such that

$$||x_{2n(k)+2} - x_{2n(k)+1}|| \leq dist(A, B) + \varepsilon, \tag{2.2}$$

for all $n(k) \geq N_1$. Also, there exists N_2 such that

$$||x_{2n(k)} - x_{2n(k)+1}|| \leq dist(A, B) + \varepsilon, \tag{2.3}$$

for all $n(k) \geq N_2$. Let $N = \max\{N_1, N_2\}$. It follows from the uniform convexity of X and (2.1)-(2.3) that

$$||(x_{2n(k)+2} + x_{2n(k)})/2 - x_{2n(k)+1}|| \leq \left(1 - \delta\left(\frac{\varepsilon_0}{dist(A,B)+\varepsilon}\right)\right)(dist(A,B)+\varepsilon),$$

for all $n(k) \geq N$. As $(x_{2n(k)+2} + x_{2n(k)})/2 \in A$, the choice of ε and the fact that δ is strictly increasing implies that

$$||(x_{2n(k)+2} + x_{2n(k)})/2 - x_{2n(k)+1}|| < dist(A,B),$$

for all $n(k) \geq N$, a contradiction.

A similar argument will show that $||x_{2n+3} - x_{2n+1}|| \to 0$. □

2.2. **Cyclic (b,θ)-enriched contraction.** Now, we introduce the following cyclic (b,θ)-enriched contraction map for our next results.

Definition 2.7. Let A and B be nonempty subsets of a uniformly convex Banach space X. A map $T : A \cup B \to A \cup B$ is a cyclic (b,θ)-enriched contraction map if it satisfies:

(1) $T(A) \subseteq B$ and $T(B) \subseteq A$.
(2) If there exist $b \in [0,+\infty)$ and $\theta \in [0, b+1)$ such that $||b(x-y)+Tx-Ty|| \leq \theta\ ||x-y|| + (1-\theta)\ dist(A,B)$, for all $x \in A$, $y \in B$.

Here, it is to note that (2) implies that T is cyclic enriched contraction for $\lambda = \frac{1}{b+1}$, and choosing $c = \frac{\theta}{b+1} = \lambda\theta$.

For $b = 0$, if we take $\lambda = 1$, and $c = \theta$, then for all $x \in A$, $y \in B$, we have $||Tx - Ty|| \leq c\ ||x-y|| + (1-c)\ dist(A,B)$.

For $b > 0$, if we take $\lambda = \frac{1}{b+1}$, and $c = \lambda\theta$, then for all $x \in A$, $y \in B$, we have $||(\frac{1}{\lambda} - 1)(x-y) + Tx - Ty|| \leq \theta\ ||x-y|| + (1-\theta)\ dist(A,B)$. Hence $||T_\lambda x - T_\lambda y|| \leq c\ ||x-y|| + (\lambda - c)\ dist(A,B) < c\ ||x-y|| + (1-c)\ dist(A,B)$.

Theorem 2.8. *Let A and B be nonempty subsets of a uniformly convex Banach space X. Suppose that A is convex and $T : A \cup B \to A \cup B$ is a cyclic (b,θ)-enriched contraction map. Further, if $x_0 \in A$ and $x_{n+1} = (1-\lambda)x_n + \lambda Tx_n = T_\lambda x_n$ with $\lambda = \frac{1}{b+1}$, for each $n \geq 0$. Then for each $\varepsilon > 0$, there exists a positive integer N_0 such that for all $m > n \geq N_0$,*

$$||x_{2m} - x_{2n+1}|| < dist(A,B) + \varepsilon.$$

Proof. Suppose the contrary. Then there exists $\varepsilon_0 > 0$, such that for each $k \geq 1$, there is $m(k) > n(k) \geq k$, satisfying

$$||x_{2m(k)} - x_{2n(k)+1}|| \geq dist(A,B) + \varepsilon_0 \tag{2.4}$$

$$||x_{2m(k)-2} - x_{2n(k)+1}|| < dist(A,B) + \varepsilon_0. \tag{2.5}$$

It follows that

$$dist(A,B) + \varepsilon_0 \leq ||x_{2m(k)} - x_{2n(k)+1}||$$

$$\leq ||x_{2m(k)} - x_{2m(k)-2}|| + ||x_{2m(k)-2} - x_{2n(k)+1}||$$
$$< ||x_{2m(k)} - x_{2m(k)-2}|| + dist(A,B) + \varepsilon_0.$$

Taking $k \to \infty$, using Lemma 2.6, we have

$$\lim_{k\to\infty} ||x_{2m(k)} - x_{2n(k)+1}|| = dist(A,B) + \varepsilon_0.$$

As T is cyclic (b,θ)-enriched contraction map, we obtain

$$\begin{aligned} ||x_{2m(k)} - x_{2n(k)+1}|| &\leq ||x_{2m(k)} - x_{2m(k)+2}|| + ||x_{2m(k)+2} - x_{2n(k)+3}|| \\ &\quad + ||x_{2n(k)+3} - x_{2n(k)+1}|| \\ &\leq ||x_{2m(k)} - x_{2m(k)+2}|| + c^2||x_{2m(k)} - x_{2n(k)+1}|| \\ &\quad + (1-c^2)dist(A,B) + ||x_{2n(k)+3} - x_{2n(k)+1}||. \end{aligned}$$

Letting $k \to \infty$, we have $dist(A,B)+\varepsilon_0 \leq c^2(dist(A,B)+\varepsilon_0)+(1-c^2)dist(A,B) = dist(A,B) + c^2\varepsilon_0$, a contradiction. This completes the proof. □

Theorem 2.9. *Let A and B be nonempty subsets of a uniformly convex Banach space X. Suppose that A is closed and $T : A\cup B \to A\cup B$ is a cyclic (b,θ)-enriched contraction map. Further, if $x_0 \in A$ and $x_{n+1} = (1-\lambda)x_n + \lambda Tx_n = T_\lambda x_n$ with $\lambda = \dfrac{1}{b+1}$, for each $n \geq 0$. If $dist(A,B) = 0$, then T has a fixed point $x \in A\cap B$.*

Proof. Let $\varepsilon > 0$ be given. By Theorem 2.2, there exists N_1 such that $||x_{2n} - x_{2n+1}|| < \varepsilon$, for all $n \geq N_1$. By Theorem 2.8, there exists N_2 such that $||x_{2m} - x_{2n+1}|| < \varepsilon$, for all $m > n \geq N_2$. Take $N = \max\{N_1, N_2\}$. It follows that

$$||x_{2m} - x_{2n}|| \leq ||x_{2m} - x_{2n+1}|| + ||x_{2n+1} - x_{2n}|| < 2\varepsilon,$$

for all $m > n \geq N$. Thus $\{x_{2n}\}$ is a Cauchy sequence in A. Now, the completeness of X and closedness of A imply that $x_{2n} \to x \in A$. It follows from Theorem 2.3 that $||x - Tx|| = dist(A,B) = 0$. So X is a fixed point of T and hence $x \in F(T) \subseteq A\cap B$. □

The following results of Eldred and Veeramani [3] will be required in the sequel.

Lemma 2.10. *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*

(i) $||z_n - y_n|| \to dist(A,B)$,
(ii) *For every $\varepsilon > 0$ there exists N_0 such that for all $m > n \geq N_0$, $||x_m - y_n|| \leq dist(A,B) + \varepsilon$.*

Then, for every $\varepsilon > 0$ there exists N_1 such that for all $m > n \geq N_1$, $||x_m - z_n|| \leq \varepsilon$.

Lemma 2.11. *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*

(i) $||x_n - y_n|| \to dist(A,B)$,
(ii) $||z_n - y_n|| \to dist(A,B)$.

Then $||x_m - z_n|| \to 0$.

Theorem 2.12. *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space. Suppose that $T : A \cup B \to A \cup B$ is a cyclic (b, θ)-enriched contraction map, then there exists a unique best proximity point x in A (that is with $||x - T_\lambda x|| = dist(A, B)$). Further, if $x_0 \in A$ and $x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n = T_\lambda x_n$ with $\lambda = \dfrac{1}{b+1}$, then $\{x_{2n}\}$ converges to the best proximity point.*

Proof. As T is a cyclic (b, θ)-enriched contraction map, there exists $b \in [0, +\infty)$ and $\theta \in [0, b + 1)$ such that $||b(x - y) + Tx - Ty|| \leq \theta\, ||x - y|| + (1 - \theta)\ dist(A, B)$, for all $x \in A$, $y \in B$.

For $b = 0$, if we take $\lambda = 1$, and $c = \theta$, then for all $x \in A$, $y \in B$, we have $||Tx - Ty|| \leq \theta\, ||x - y|| + (1 - \theta)\ dist(A, B)$. The result follows from Theorem 3.10 of Eldred and Veeramani.

For $b > 0$, if we take $\lambda = \dfrac{1}{b+1}$, and $c = \lambda\theta$, then for all $x \in A$, $y \in B$, we have

$$||(\frac{1}{\lambda} - 1)(x - y) + Tx - Ty|| \leq \theta\, ||x - y|| + (1 - \theta)\ dist(A, B).$$

Hence $||T_\lambda x - T_\lambda y|| \leq c\, ||x - y|| + (\lambda - c)\ dist(A, B) < c\, ||x - y|| + (1 - c)\ dist(A, B)$.

Assume $dist(A, B) \neq 0$. Since $||x_{2n} - T_\lambda x_{2n}|| \to dist(A, B)$ and $||T_\lambda^2 x_{2n} - T_\lambda x_{2n}|| \to dist(A, B)$. By Lemma 2.11, $||x_{2n} - x_{2(n+1)}|| \to 0$.

Similarly, we can show that $||T_\lambda x_{2n} - T_\lambda x_{2(n+1)}|| \to 0$.

Now, we have to show that for every $\varepsilon > 0$ there exists N_0 such that for all $m > n \geq N_0$,

$$||x_{2m} - T_\lambda x_{2n}|| < dist(A, B) + \varepsilon.$$

Suppose, to the contrary, that there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ there exists $m_k > n_k \geq k$ for which

$$||x_{2m} - T_\lambda x_{2n}|| \geq dist(A, B) + \varepsilon,$$

this m_k can be chosen such that it is the least integer greater than n_k to satisfy the above inequality. Now,

$$\begin{aligned} dist(A, B) + \varepsilon &\leq ||x_{2m(k)} - T_\lambda x_{2n(k)}|| \\ &\leq ||x_{2m(k)} - x_{2m(k)-2}|| + ||x_{2m(k)-2} - T_\lambda x_{2n(k)}||. \end{aligned}$$

Hence $\lim\limits_{k\to\infty} ||x_{2m(k)} - T_\lambda x_{2n(k)}|| = dist(A, B) + \varepsilon$. Consequently, we get

$$\begin{aligned} ||x_{2m(k)} - T_\lambda x_{2n(k)}|| &\leq ||x_{2m(k)} - x_{2m(k)+2}|| + ||x_{2m(k)+2} - T_\lambda x_{2n(k)+2}|| \\ &\quad + ||T_\lambda x_{2n(k)+2} - T_\lambda x_{2n(k)}|| \\ &\leq ||x_{2m(k)} - x_{2m(k)+2}|| + c^2 ||x_{2m(k)} - T_\lambda x_{2n(k)}|| \\ &\quad + (1 - c^2) dist(A, B) + ||T_\lambda x_{2n(k)+2} - T_\lambda x_{2n(k)}||. \end{aligned}$$

Hence by taking $k \to \infty$, we have $dist(A, B) + \varepsilon \leq c^2(dist(A, B) + \varepsilon) + (1 - c^2)dist(A, B) = dist(A, b) + c^2\varepsilon$, which is a contradiction. Hence by Lemma 2.10,

$\{x_{2n}\}$ is a Cauchy sequence and converges to some $x \in A$. From Proposition 2.3, it follows that $||x - T_\lambda x|| = dist(A, B)$.

Suppose that $x, y \in A$ and $x \neq y$ such that $||x - T_\lambda x|| = dist(A, B)$ and $||y - T_\lambda y|| = dist(A, B)$. Therefore

$$
\begin{aligned}
||T_\lambda x - y|| &= ||T_\lambda x - T_\lambda^2 y|| \leq ||x - T_\lambda y|| \\
||T_\lambda y - x|| &= ||T_\lambda y - T_\lambda^2 x|| \leq ||y - T_\lambda x||,
\end{aligned}
$$

which implies that $||T_\lambda y - x|| = ||y - T_\lambda x||$. But since $||y - T_\lambda x|| > dist(A, B)$, it follows that $||T_\lambda y - x|| < ||y - T_\lambda x||$, a contradiction. Therefore $x = y$. Hence the result. □

Remark 2.13. Theorem 3.10 of Eldred et al. [3] and Theorem 8 of Al-Thagafi et al. [1] are special cases of Theorem 2.12.
If the convexity assumption is dropped from Theorem 2.12, then the convergence and uniqueness is not guaranteed even in finite dimensional spaces.

If $A = B$ in Theorem 2.12, then the existence of a fixed point for self-mapping can be obtained under weaker conditions.

Theorem 2.14. *Let A be a nonempty closed subset of a convex metric space (X, d, W) and $T : A \to A$ be an enriched contraction map. Then*

(i) *$F(T) = \{p\}$, for some $p \in A$,*
(ii) *the sequence $\{x_n\}$ obtained from Krasnoselskij iterative process*

$$x_{n+1} = W(x_n, Tx_n, \lambda) = T_\lambda x_n, n \geq 0, \tag{2.6}$$

converges to p, for any $x_0 \in X$.

Proof. As T is an enriched contraction, we have the mapping $T_\lambda : A \to A$ defined by $T_\lambda x = W(x, Tx, \lambda)$ satisfies

$$d(T_\lambda x, T_\lambda y) \leq c d(x, y), \tag{2.7}$$

for all $x, y \in X$. Hence by taking $x = x_n$ and $y = x_{n-1}$ in (2.7) and using Krasnoselskij iterative process $x_{n+1} = W(x_n, Tx_n, \lambda) = T_\lambda x_n, n \geq 0$, we get

$$d(x_{n+1}, x_n) \leq c^n d(x_1, x_0), n \geq 1.$$

As $c \in [0, 1)$, we get $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. For $m > n$, we have

$$
\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\
&\leq (c^n + c^{n+1} + \ldots + c^{m-1}) d(x_0, x_1) \\
&\leq \frac{c^n(1 - c^{m-n})}{1 - c} d(x_0, x_1).
\end{aligned}
$$

Therefore, $d(x_m, x_n) \to 0$, when $m, n \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence in A. Hence $\{x_n\}$ is convergent and denote

$$p = \lim_{n \to \infty} x_n.$$

Further, we get

$$\begin{aligned} d(x_{n+1}, T_\lambda p) &= d(W(x_n, Tx_n, \lambda), W(p, Tp, \lambda)) \\ &\leq cd(x_n, p). \end{aligned}$$

Taking $n \to \infty$, we have $T_\lambda p = p$. Therefore, $0 = d(p, T_\lambda p) = d(p, W(p, Tp, \lambda)) = (1-\lambda)d(p, Tp)$ and $d(p, Tp) = 0$. This completes the proof. □

Corollary 2.15. *Let A be a nonempty closed subset of a convex Banach space X and $T : A \to A$ be a (b, θ)-enriched contraction map. Then*

(i) $F(T) = \{p\}$, for some $p \in A$,

(ii) there exists $\lambda \in (0, 1]$ such that the sequence $\{x_n\}$ obtained from Krasnoselskij iterative process

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n = T_\lambda x_n, n \geq 0,$$

converges to p, for any $x_0 \in X$.

Proof. By taking $\lambda = \dfrac{1}{b+1}$ and using Theorem 2.14, we get the result. □

2.3. **Illustrations.** In this section, we provide some examples for the validity of our results.

Example 1. Consider $A = B = [0, 1] \subset X = \mathbb{R}$ with the usual metric wherein $W(x, y, \frac{1}{2}) = \frac{1}{2}x + \frac{1}{2}y$. Define $T : A \cup B \to A \cup B$ by

$$Tx = \begin{cases} \dfrac{2-x}{3}, & 0 \leq x \leq \dfrac{1}{2} \\ \dfrac{1}{2}, & \dfrac{1}{2} \leq x \leq 1. \end{cases}$$

Here $T_{\frac{1}{2}}x = W(x, Tx, \frac{1}{2}) = \frac{1}{2}x + \frac{1}{2}Tx$.

If $0 \leq x \leq \frac{1}{2}$, $Tx = \dfrac{2-x}{3}$. So, we have

$$\begin{aligned} d(T_{\frac{1}{2}}x, T_{\frac{1}{2}}y) &= d(W(x, Tx, \frac{1}{2}), W(y, Ty, \frac{1}{2})) \\ &= |\frac{1}{2}(x-y) + \frac{1}{2}(Tx - Ty)| \\ &= |\frac{1}{2}(x-y) + \frac{1}{2}\frac{y-x}{3}| \\ &= \frac{1}{2}|x-y||(1 - \frac{1}{3})| \\ &\leq \frac{1}{2}|x-y|. \end{aligned}$$

Also, if $\frac{1}{2} \leq x \leq 1$, $Tx = \frac{1}{2}$. So, we have

$$d(T_{\frac{1}{2}}x, T_{\frac{1}{2}}y) = d(W(x, Tx, \frac{1}{2}), W(y, Ty, \frac{1}{2}))$$

$$= |\frac{1}{2}(x-y) + \frac{1}{2}(Tx - Ty)|$$
$$= |\frac{1}{2}(x-y)|.$$

Thus all the conditions of Theorem 2.14 are satisfied.

Example 2. Consider $A = B = [1,2] \subset X = \mathbb{R}$ with the usual metric wherein $W(x, y, \frac{1}{2}) = \frac{1}{2}x + \frac{1}{2}y$. Define $T : A \cup B \to A \cup B$ by

$$Tx = \frac{1}{x}, x \in [1,2].$$

Here $T_{\frac{1}{2}}x = W(x, Tx, \frac{1}{2}) = \frac{1}{2}x + \frac{1}{2}Tx$. So, we have

$$\begin{aligned} d(T_{\frac{1}{2}}x, T_{\frac{1}{2}}y) &= d(W(x, Tx, \frac{1}{2}), W(y, Ty, \frac{1}{2})) \\ &= |\frac{1}{2}(x-y) + \frac{1}{2}(Tx - Ty)| \\ &= |\frac{1}{2}(x-y) + \frac{1}{2}\frac{y-x}{xy}| \\ &= \frac{1}{2}|x-y||(1 - \frac{1}{xy})| \\ &\leq \frac{1}{2}|x-y|. \end{aligned}$$

Thus all the conditions of Theorem 2.14 are satisfied.

Conclusions. This article focuses on best proximity point theorems for enriched contractions, which serve as non-self mapping analogues of contraction self-mappings. Also, some sufficient conditions are established for a non-self enriched contraction mapping to have a best proximity point.

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