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# ON THE $L^2$-ILLPOSED MIXED PROBLEM FOR THE WAVE EQUATIONS WITH OBLIQUE BOUNDARY CONDITION

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*Dedicated to the late Professor Wataru Takahashi and the late Professor Naoki Shioji*

Abstract. In the previous paper [1], we omitted the proof of the main theorems, but it is better to show it, so we shall introduce it in this paper. We consider the mixed problem with oblique boundary condition which is not well posed in the $L^2$ sense. First we introduce the space Y in which the equation is well posed. Next we consider the infinitesimal generator A which generates a $C_0$ -semigroup on Y.

## 1. Introduction

In the previous paper [1], we omitted the proof of the main theorems, but it is better to show it, so we shall introduce it in this paper. In what kinds of function spaces are nonparabolic equations well-posed? Here the well-posedness means that a family of solutions $\{u(t)\}$ defines a $C_0$-semigroup $\{T_t\} : \|T_t\| \leq Me^{\omega t}, T_t u(0) = u(t)$. We study this problem in [2].

This paper is concerned with "well posed function spaces" for hyperbolic equations of the simplest type. In [2] we mentioned this subject without proof.

We consider the following mixed problem

$$
(1.1)\qquad \begin{cases} \dfrac{\partial^2 u(t,\mathbf{x})}{\partial t^2} = \triangle u(t,\mathbf{x}), & (t,\mathbf{x}) \in (0,\infty)\times\Omega, \\ \left(\dfrac{\partial}{\partial x_1} + B(t,\mathbf{x})\right) u \Big|_{x_1=0} = 0, & \\ u(0,\mathbf{x}) = f(\mathbf{x}), & \mathbf{x}\in\Omega, \\ u_t(0,\mathbf{x}) = g(\mathbf{x}), & \mathbf{x}\in\Omega. \end{cases}
$$

Here $\Omega = \{\mathbf{x} = (x_1, \cdots, x_n) \mid x_1 > 0\} \subset \mathbb{R}^n$ and

$$
B(t,\mathbf{x})u(t,\mathbf{x}) \equiv b_0(t,\mathbf{x})\frac{\partial u(t,\mathbf{x})}{\partial t} + \sum_{j=2}^{n} \frac{\partial u(t,\mathbf{x})}{\partial x_j} + d(t,\mathbf{x})u(t,\mathbf{x}),
$$

where the coefficients are $C^\infty$-functions defined in a neighborhood of $x_1 = 0$.

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We study that the case of (1.1) is not well posed in $L^2$-sense (Cf. Remark 1). Then first we introduce the space $Y = D(\triangle_\sigma) \times \mathcal{M}$ in which (1.1) is wellposed, where $\mathcal{M}$ is a measure space. Next we consider the infinitesimal generator $A = \begin{pmatrix} 0 & I \\ \triangle_\sigma & 0 \end{pmatrix}$ which generates a $C_0$-semigroup on $Y$.

**Remark 1.** Many authors studied about $L^2$-well posedness for hyperbolic mixed problems of second order. (Cf. Ikawa [4], Miyatake [5, 6], Li [3] ,Taniguchi [7, 8]) In general (1.2) is not well-posed in $L^2$-sense (Cf. Miyatake [5, 6])

$$(1.2) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \triangle u, \\ \left(-\dfrac{\partial}{\partial x} + b\dfrac{\partial}{\partial y} - c\dfrac{\partial}{\partial t}\right) u \Big|_{x=0} = 0, \\ u(0,x,y) = f(x,y), \\ u_t(0,x,y) = g(x,y). \end{cases}$$

In fact the following three cases;

$$\begin{array}{lll} (i) & c > 0, & b \neq 0 \\ (ii) & c = 1, & |b| < 1 \\ (iii) & c \leq 0, & |b| > -c \end{array}$$

(1.2) is not $L^2$-well-posed. Therefore especially in the case that $c = 0$ it is not well-posed if $b \neq 0$.

## 2. Functional spaces

In this section we begin by introducing some terminology and notation which are required in subsequent sections. Let $\mathcal{M}$ denote a measure space. Let

$$\hat{\mathcal{M}} = \{\hat{u}; u \in \mathcal{M}\}, \qquad \Xi = \{u; |\xi|^{-1}\hat{u} \in \hat{\mathcal{M}}\}, \qquad \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$$

where $\hat{u}(\xi) = (\mathcal{F}u)(\xi)$ and $\mathcal{F}u$ means the Fourier transformation of $u$.
Let $X \equiv (X, \|\cdot\|_X)$ be the Banach space with norm $\|\cdot\|_X$, where $X =^t (\mathcal{M} \times \Xi)$ and

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_X = \left\| \left(\hat{u}(\cdot)^2 + |\xi|^{-2}\hat{v}(\cdot)^2\right)^{1/2} \right\|_{\hat{\mathcal{M}}} \qquad \text{for any} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in X.$$

Let $\triangle_N$ denote the Laplacian with Neumann condition $\dfrac{\partial}{\partial x_1} u \Big|_{x_1=0} = 0$.
The domain of $\triangle_N$ is

$$D(\triangle_N) = \left\{ u \in D(\triangle) \ ; \ \frac{\partial}{\partial x_1} u \Big|_{x_1=0} = 0 \right\}.$$

Let $\triangle_\sigma$ denote the Laplacian with boundary condition $\left(\frac{\partial}{\partial x_1} + B(t,\mathbf{x})\right) u \Big|_{x_1=0} = 0.$
The domain of $\triangle_\sigma$ is

$$D(\triangle_\sigma) = \left\{ u \in D(\triangle) \ ; \ \left(\frac{\partial}{\partial x_1} + B(t,\mathbf{x})\right) u \Big|_{x_1=0} = 0 \right\}.$$

Let $Y \equiv (Y, \| \cdot \|_\sigma)$ be a Banach space with norm $\| \cdot \|_\sigma$, where $Y =^t (D(\triangle_\sigma) \times \Xi)$ and

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\sigma := \left\| \begin{pmatrix} u - Su \\ v \end{pmatrix} \right\|_X \qquad \text{for} \qquad \text{all} \qquad \begin{pmatrix} u \\ v \end{pmatrix} \in Y.$$

Here $S$ satisfies the following two assumptions;

**Assumptions 1.** We assume that;
**(A.1)** $I - S : D(\triangle_\sigma) \longrightarrow D(\triangle_N).$
**(A.2)** There exists a consutant $C > 0$ such that

$$\| \triangle Su\|_{\mathcal{M}} \leq C \|u\|_{\mathcal{M}} \qquad for \qquad all \qquad u \in D(\triangle_\sigma).$$

**Remark 2.** In general the topology of finite measure space $(X, \mathcal{M}, \mu)$ is defined by the quasi-norm

$$\|\mu\|_{\mathcal{M}} = \inf_{\varepsilon > 0} tan^{-1} \left[\varepsilon + \mu\{x \in X; |\mu(x)| \geq \varepsilon\}\right]$$

(Cf.Yosida[9].)

## 3. Existence of $(I - \lambda\triangle_\sigma)^{-1}$

To show the existence of $(I - \lambda\triangle_\sigma)^{-1}$, it sufficient to show the following theorem.

**Theorem 3.1.** *Let $\mathcal{R}(I - \lambda\triangle_\sigma)$ be the range of $(I - \lambda\triangle_\sigma)$ then for some small $\lambda > 0$, we get*

$$\mathcal{R}(I - \lambda\triangle_\sigma) = \hat{\mathcal{M}}^{-1}.$$

*Proof.* It is sufficient to show that for any $f \in \hat{\mathcal{M}}^{-1}$, there exist $u \in D(\triangle_\sigma)$ such that $(I - \lambda\triangle_\sigma)u = f$.

Transformate $(I - \lambda\triangle_\sigma)u = f$, we have

$$\begin{aligned} u - \lambda[\triangle(u - Su) + \triangle Su] &= f \\ u - Su - \lambda \triangle (u - Su) &= f + \lambda \triangle Su - Su \\ (I - \lambda\triangle)(u - Su) &= f + \lambda \triangle Su - Su \end{aligned}$$

Note that for any $u \in D(\triangle_\sigma)$ it follows that $u - Su \in D(\triangle_N)$. Then we obtain that

$$u - Su = (I - \lambda\triangle_N)^{-1}[f + \lambda \triangle Su - Su].$$

Thus

$$\begin{aligned} u &= (I - \lambda\triangle_N)^{-1}[f + \lambda \triangle Su - Su] + Su \\ &= (I - \lambda\triangle_N)^{-1} f + Su + (I - \lambda\triangle_N)^{-1}[-Su + \lambda \triangle Su] \\ &= (I - \lambda\triangle_N)^{-1} f + Ku \end{aligned}$$

where

$$\begin{aligned} Ku &= [S-(I-\lambda\triangle_N)^{-1}(I-\lambda\triangle)S]u. \\ &= [I-(I-\lambda\triangle_N)^{-1}]Su+\lambda(I-\lambda\triangle_N)^{-1}(\triangle Su). \end{aligned}$$

Then we get

$$(I-K)u=(I-\lambda\triangle_N)^{-1}f.$$

Therefore if $\|K\|<1$ there exists $(I-K)^{-1}$ and we have

$$u=(I-Ku)^{-1}(I-\lambda\triangle_N)^{-1}f.$$

Note that

$$(I-\lambda\triangle_N)^{-1}\longrightarrow I \qquad \text{as} \qquad \lambda\to 0.$$

From the assumption (A.2), we obtain that $\|\triangle Su\|_{\mathcal{M}}\le C\|u\|_{\mathcal{M}}$. Then $\|K\|<1$ if $\lambda>0$ is small enough. It means that there exists $(I-\lambda\triangle_\sigma)^{-1}$. □

## 4. Systems of evolution equations

Putting $u_t=v$, let us rewriting the problem (1.1) in the following form:

$$(4.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}0 & I\\ \triangle_\sigma & 0\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}, \\ \begin{pmatrix}u(0)\\v(0)\end{pmatrix}=\begin{pmatrix}f\\g\end{pmatrix}. \end{cases}$$

Set

$$U=\begin{pmatrix}u\\v\end{pmatrix} \qquad \text{and} \qquad A=\begin{pmatrix}0 & I\\ \triangle_\sigma & 0\end{pmatrix}.$$

We shall formulate (4.1) as an abstract equation:

$$(4.2)\qquad \begin{cases} \dfrac{d}{dt}U=AU \qquad \text{in} \qquad Y, \\ U(0)=\begin{pmatrix}f\\g\end{pmatrix}. \end{cases}$$

**Theorem 4.1.** *A generates a $C_0$-semigroup $\{U(t)\}$ on $Y$.*

*Proof.* It sufficient to show that there exist $\lambda>0$ and $K'>0$ with $1-\lambda K'>0$ such taht

$$(4.3)\qquad \|(I-\lambda A)U\|_\sigma\ge(1-\lambda K')\|U\|_\sigma, \qquad \text{for any} \quad U\in Y.$$

Now we shall proof inequality (4.3).

$$\begin{aligned}\|(I-\lambda A)U\|_\sigma &= \left\|\begin{pmatrix} 1-0 & 0-\lambda \\ -\lambda \triangle_\sigma & 1-0\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix}\right\|_\sigma \\ &= \left\|\begin{pmatrix} u-\lambda v \\ v+\lambda \triangle_\sigma u\end{pmatrix}\right\|_\sigma \\ &= \left\|\begin{pmatrix} u-Su-\lambda(v-Sv) \\ v+\lambda \triangle_N (u-Su)+\lambda \triangle (Su)\end{pmatrix}\right\|_X \\ &\geq \left\|\begin{pmatrix} u-Su-\lambda v \\ v+\lambda \triangle_N (u-Su)\end{pmatrix}\right\|_X - \lambda\left\|\begin{pmatrix} Sv \\ \triangle(Su)\end{pmatrix}\right\|_X.\end{aligned}$$

From the definition of $\|\cdot\|_X$ we obtain that

$$\begin{aligned}&\left\|\begin{pmatrix} u-Su-\lambda v \\ v+\lambda \triangle_N (u-Su)\end{pmatrix}\right\|_X \\ &\quad= \left\|\left(\mathcal{F}\left(u-Su-\lambda v\right)^2+|\xi|^{-2}\mathcal{F}(v+\lambda \triangle_N (u-Su))^2\right)^{1/2}\right\|_{\hat{\mathcal{M}}} \\ &\quad= \left\|\left(\left(\widehat{u-Su}-\lambda\hat{v}\right)^2+|\xi|^{-2}\left(\hat{v}+\lambda|\xi|^2(\widehat{u-Su})\right)^2\right)^{1/2}\right\|_{\hat{\mathcal{M}}} \\ &\quad= \Bigg\|\Bigg[\left((\widehat{u-Su})^2-2\lambda(\widehat{u-Su})\hat{v}+\lambda^2\hat{v}^2\right) \\ &\qquad +|\xi|^{-2}\left(\hat{v}^2+2\lambda|\xi|^2(\widehat{u-Su})\hat{v}+\lambda^2|\xi|^4(\widehat{u-Su})^2\right)\Bigg]^{1/2}\Bigg\|_{\hat{\mathcal{M}}} \\ &\quad= \Bigg\|\Bigg[\left((\widehat{u-Su})^2-2\lambda(\widehat{u-Su})\hat{v}+\lambda^2\hat{v}^2\right) \\ &\qquad +|\xi|^{-2}\hat{v}^2+2\lambda(\widehat{u-Su})\hat{v}+\lambda^2|\xi|^2(\widehat{u-Su})^2\Bigg]^{1/2}\Bigg\|_{\hat{\mathcal{M}}} \\ &= \left\|(1+\lambda^2|\xi|^2)^{1/2}\left((\widehat{u-Su})^2+|\xi|^{-2}\hat{v}^2\right)^{1/2}\right\|_{\hat{\mathcal{M}}} \\ &\geq \left\|\begin{pmatrix} u-Su \\ v\end{pmatrix}\right\|_X = \left\|\begin{pmatrix} u \\ v\end{pmatrix}\right\|_\sigma\end{aligned}$$

Therefore we have

$$\|(I-\lambda A)U\|_\sigma \geq \left\|\begin{pmatrix} u \\ v\end{pmatrix}\right\|_\sigma - \lambda\left\|\begin{pmatrix} Sv \\ \triangle(Su)\end{pmatrix}\right\|_X.$$

From Section 3, there exists $K>0$ such that

$$\left\|\begin{pmatrix} Sv \\ \triangle(Su)\end{pmatrix}\right\|_X \leq K\left\|\begin{pmatrix} u \\ v\end{pmatrix}\right\|_\sigma$$

So that for some $K'>0$ we obtain (4.3). □

## 5. Application

As an application model of the abstract result obtained in the previous sections, let us consider the following mixed problem:

$$(5.1)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \triangle u \qquad \text{in} \quad (0,\infty)\times\Omega, \\ \left(\dfrac{\partial}{\partial x} + \sigma(y)\dfrac{\partial}{\partial y}\right) u \,\Big|_{x=0} = 0, \\ u(0,x,y) = f(x,y), \\ u_t(0,x,y) = g(x,y). \end{cases}$$

Here $\Omega = \{(x,y)\in\mathbb{R}^2 |\ x>0\}$. And we assume following conditions;

**Condition 1.** The function $\sigma$ in (5.1) satisfies following three conditions.

(i) There exists $r>0$ such that $\sigma(y)=0$ for any $|y|\geq r$.
(ii) There exists $\rho>0$ such that $|\sigma(y)|<\rho$, where $\rho$ is sufficiently small.
(iii) $\sigma$ is sufficiently smooth.

Let $\sigma$ be sufficiently small. Then we make $S$ (Cf. [1]).
As is well known that (5.1) is not well posed in $L^2$-sense (See, Remark 1).

See [1] for a detailed explanation of application model.

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