Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167
Volume 10, Number 3, 2024, 139–144

WEAK EXTENSION OF THE HAM-SANDWICH THEOREM: SOME RATIO NOT IN HALF

HIDEFUMI KAWASAKI

Abstract. The ham-sandwich theorem is famous as an application of Borsuk's antipodal theorem. It states that given n measurable sets $A_1, \dots, A_n$ with positive Lebesgue measure in $\mathbb{R}^n$, it is possible to divide each one of them in half by a hyperplane. Recently the author applied Borsuk's antipodal theorem to optimization theory in [3]. In this paper we extend the ham-sandwich theorem to some ratio not in half by using the technique of [3].

1. Introduction

Borsuk's antipodal theorem [1] is an important theorem of algebraic topology. It states that for any continuous mapping φ from the n-sphere S^n to the Euclidean space $\mathbb{R}^n$, there exists a point $\boldsymbol{u} \in S^n$ such that $\varphi(\boldsymbol{u}) = \varphi(-\boldsymbol{u})$. As for its applications, the ham-sandwich theorem, the necklace problem, and coloring of Kneser graph by Lovász [5] are well-known, see e.g. Matoušek [6].

The author [2, 3, 4] applied Borsuk's antipodal theorem to an n-tuple of parametric optimization problems with parameter $\boldsymbol{u} \in S^n$, and presented antipodal theorems for them.

Now we explain our notations. For any $\boldsymbol{u} = (u_1, \dots, u_{n+1}) \in S^n$, we set $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ and $\boldsymbol{u} = (u, u_{n+1})$. Then u is an element of the n-disk D^n. We assign to $\boldsymbol{u} \in S^n$ a hyperplane $H_{\boldsymbol{u}} = \{x \in \mathbb{R}^n \mid \langle u, x\rangle = u_{n+1}\}$ and two closed half-spaces:

$$H^+_{\boldsymbol{u}} = \{x \in \mathbb{R}^n \mid \langle u, x\rangle \geq u_{n+1}\},\ H^-_{\boldsymbol{u}} = \{x \in \mathbb{R}^n \mid \langle u, x\rangle \leq u_{n+1}\},$$

where $\langle u, x\rangle$ denotes the inner product $u_1x_1 + \cdots + u_nx_n$.

Next, let us review the ham-sandwich theorem. Let $A_i \subset \mathbb{R}^n$ $(i = 1, \dots, n)$ be measurable sets with positive Lebesgue measure. Define $\varphi_i(\boldsymbol{u}) = \mu(A_i \cap H^+_{\boldsymbol{u}})$. Then $\varphi = (\varphi_1, \dots, \varphi_n)$ is a continuous mapping from S^n to $\mathbb{R}^n$. By applying Borsuk's antipodal theorem to φ, there exists $\boldsymbol{u} \in S^n$ such that $\mu(A_i \cap H^+_{\boldsymbol{u}}) = \mu(A_i \cap H^+_{-\boldsymbol{u}})$. Since $H^+_{-\boldsymbol{u}} = H^-_{\boldsymbol{u}}$, we have

$$\mu(A_i \cap H^+_{\boldsymbol{u}}) = \mu(A_i \cap H^-_{\boldsymbol{u}}) \quad (i = 1, \dots, n), \tag{1.1}$$

which is the ham-sandwich theorem. If we denote the ratio of the division by

2020 *Mathematics Subject Classification.* 90C31, 54C05.

Key words and phrases. Borsuk's antipodal theorem, ham-sandwich theorem.

This research is supported by JSPS KAKENHI Grant Number 20K03751 and the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

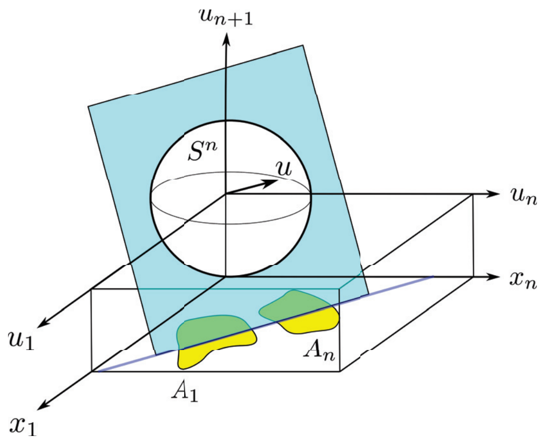

FIGURE 1. Place $A_1, \ldots, A_n$ on the hyperplane $u_{n+1} = -1$ and divide them in half by a hyperplane passing through the origin of $\mathbb{R}^{n+1}$ whose normal vector is $\boldsymbol{u} = (u, u_{n+1}) \in S^n$. Then the intersection of two hyperplanes is $\{(x, -1) \mid \langle u, x \rangle = u_{n+1}\}$.

$$\rho_i(\boldsymbol{u}) := \frac{\mu(A_i \cap H_{\boldsymbol{u}}^+)}{\mu(A_i)}, \tag{1.2}$$

then (1.1) implies $\rho_i(\boldsymbol{u}) = 1/2$. In this paper we show that there exists $\boldsymbol{u} \in S^n$ such that

$$\frac{1}{2} < \rho_1(\boldsymbol{u}) = \cdots = \rho_n(\boldsymbol{u}) < 1.$$

2. Weak extension of the ham-sandwich theorem

In this section, we will extend the ham-sandwich theorem. We start with a general setting. Let $f_i(x, u)$ $(i = 1, \ldots, n)$ be real-valued continuous functions defined on $\mathbb{R}^n \times D^n$. We assume that $f_i(x, u) = f_i(x, -u)$ for any $(x, u) \in \mathbb{R}^n \times D^n$. For any $\gamma_i \in \mathbb{R}$ and $\boldsymbol{u} = (u, u_{n+1}) \in S^n$, we define

$$\nu_i(\boldsymbol{u}) := \int_{A_i \cap H_{\boldsymbol{u}}^+} f_i(x, u)dx - \gamma_i u_{n+1}. \tag{2.1}$$

Then $\nu := (\nu_1, \ldots, \nu_n) : S^n \to \mathbb{R}^n$ is continuous. Taking $f_i(x, u) = 1$ and $\gamma_i = 0$ in (2.1), we see that $\nu_i(\boldsymbol{u}) = \mu(A_i \cap H_{\boldsymbol{u}}^+)$. The term $-\gamma_i u_{n+1}$ is important in this paper. It comes from [3], where we used

$$\varphi_i(\boldsymbol{u}) := \max_{x \in A_i} f_i(x, u) - \gamma_i u_{n+1}.$$

By applying Borsuk's antipodal theorem to ν, we obtain the following.

Theorem 2.1. (a) *For any $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$, there exists $\boldsymbol{u} = (u, u_{n+1}) \in S^n$ such that*

$$\int_{A_i \cap H_{\boldsymbol{u}}^+} f_i(x, u)dx - \int_{A_i \cap H_{\boldsymbol{u}}^-} f_i(x, u)dx = 2\gamma_i u_{n+1} \ (i = 1, \ldots, n). \tag{2.2}$$

(b) *There exists* $\boldsymbol{u} \in S^n$ *such that* $\int_{A_i \cap H_{\boldsymbol{u}}^+} f_i(x,u)dx = \int_{A_i \cap H_{\boldsymbol{u}}^-} f_i(x,u)dx$ *for any* $i = 1, \ldots, n$.

(c) *If there is no* $\boldsymbol{v} = (v, 0) \in S^n$ *such that*

$$\int_{A_i \cap H_{\boldsymbol{v}}^+} f_i(x,v)dx = \int_{A_i \cap H_{\boldsymbol{v}}^-} f_i(x,v)dx \ (i = 1, \ldots, n), \tag{2.3}$$

then for any $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$, *there exists* $\boldsymbol{u} \in S^n$ *such that* $u_{n+1} > 0$ *and*

$$\int_{A_i \cap H_{\boldsymbol{u}}^+} f_i(x,u)dx - \int_{A_i \cap H_{\boldsymbol{u}}^-} f_i(x,u)dx \ (i = 1, \ldots, n) \tag{2.4}$$

are proportionate to γ_i $(i = 1, \ldots, n)$.

Proof. (a) By Borsuk's antipodal theorem, there exists $\boldsymbol{u} \in S^n$ such that $\nu(\boldsymbol{u}) = \nu(-\boldsymbol{u})$. Hence

$$\begin{aligned} \int_{A_i \cap H_{\boldsymbol{u}}^+} f_i(x,u)dx - \gamma_i u_{n+1} &= \int_{A_i \cap H_{-\boldsymbol{u}}^+} f_i(x,-u)dx + \gamma_i u_{n+1} \\ &= \int_{A_i \cap H_{\boldsymbol{u}}^-} f_i(x,u)dx + \gamma_i u_{n+1}. \end{aligned}$$

(b) is a direct consequence of (a) for $\gamma_i = 0$.

By the assumption of (c), we have $u_{n+1} \neq 0$. Therefore (c) follows from (a). When u_{n+1} is negative, it suffices to take $-\boldsymbol{u}$ instead of $\boldsymbol{u}$. □

By taking $f_i(x,u) = 1/\mu(A_i)$ in Theorem 2.1, we obtain the following. (b) is nothing but the ham-sandwich theorem. (a) and (c) are new results.

Theorem 2.2. (a) *For any* $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$, *there exists* $\boldsymbol{u} = (u, u_{n+1}) \in S^n$ *such that*

$$\rho_i(\boldsymbol{u}) - \frac{1}{2} = \gamma_i u_{n+1}. \tag{2.5}$$

In particular when $\gamma_i \neq -1/2$, *it holds that* $-1 < u_{n+1} < 1$.

(b) *There exists a hyperplane that divides each* A_i *in half.*

(c) *If there is no hyperplane passing the origin of* $\mathbb{R}^n$ *that divides each* A_i *in half, then for any* $\gamma_i \neq -1/2$, *there exists* $\boldsymbol{u} \in S^n$ *such that* $0 < u_{n+1} < 1$ *and*

$$\rho_1(\boldsymbol{u}) - \frac{1}{2} : \cdots : \rho_n(\boldsymbol{u}) - \frac{1}{2} = \gamma_1 : \cdots : \gamma_n. \tag{2.6}$$

In particular, for any $\gamma_1 = \cdots = \gamma_n = c$ *with* $0 < c \leq 1/2$, *there exists* $\boldsymbol{u} \in S^n$ *such that* $0 < u_{n+1} < 1$ *and*

$$\frac{1}{2} < \rho_1(\boldsymbol{u}) = \cdots = \rho_n(\boldsymbol{u}) < \frac{1}{2} + c \ (\leq 1). \tag{2.7}$$

Proof. When $f_i(x,u) = 1/\mu(A_i)$, (2.2) reduces to

$$2\gamma_i u_{n+1} = \frac{\mu(A_i \cap H_{\boldsymbol{u}}^+)}{\mu(A_i)} - \frac{\mu(A_i \cap H_{\boldsymbol{u}}^-)}{\mu(A_i)} = 2\rho_i(\boldsymbol{u}) - 1,$$

which implies (2.5). If $u_{n+1} = 1$, then since $A_i \cap H_{\boldsymbol{u}}^+$ is empty, we have $\rho_i(\boldsymbol{u}) = 0$. Hence, we see from (2.5) that $\gamma_i = -1/2$, which contradicts the assumption of (a). If $u_{n+1} = -1$, then since $A_i \cap H_{\boldsymbol{u}}^+ = A_i$, we have $\rho_i(\boldsymbol{u}) = 1$. Hence, we see from (2.5) that $\gamma_i = -1/2$, which contradicts the assumption of (a). Therefore we have $-1 < u_{n+1} < 1$.

(b) follows from (a) by taking $\gamma_i = 0$. (c) If $u_{n+1} = 0$ in (2.5), then hyperplane $H_{\boldsymbol{u}}$ passes through the origin, and $\rho_1(\boldsymbol{u}) = \cdots = \rho_n(\boldsymbol{u}) = 1/2$, which contradicts the assumption of (c). Therefore $u_{n+1} \neq 0$. So (2.5) implies (2.6). When u_{n+1} is negative, it suffices to take $-\boldsymbol{u}$ instead of $\boldsymbol{u}$. In particular, if we take $\gamma_1 = \cdots = \gamma_n = c$ with $0 < c \leq 1/2$, we get (2.7) from $0 < u_{n+1} < 1$. □

Remark 2.3. We see from (2.5) that

$$|\gamma_i u_{n+1}| = \left| \rho_i(\boldsymbol{u}) - \frac{1}{2} \right| \leq \frac{1}{2}$$

for any i. Hence, if some $|\gamma_j|$ is exceptionally large, $|u_{n+1}|$ becomes small. So $\rho_i(\boldsymbol{u})$ $(i \neq j)$ are approximately equal to 1/2.

When we take $\gamma_1 = \cdots = \gamma_n = 1/2$, we see from (2.5) that

$$\rho_i(\boldsymbol{u}) = \frac{u_{n+1} + 1}{2}.$$

Since u_{n+1} is unknown, $\rho_1(\boldsymbol{u}) = \cdots = \rho_n(\boldsymbol{u})$ are also unknown. This is the reason why the title is considered "weak extension".

Example 2.4. This example indicates the possibility of extending the ham-sandwich theorem. Given ratio $0 \leq \rho \leq 1$, we compute $\ell : \langle u, x \rangle = u_3$ that divides two triangles with $\rho_1(\boldsymbol{u}) = \rho_2(\boldsymbol{u}) = \rho$. Line ℓ passing through $(-2, a)$ and $(2, b)$ is

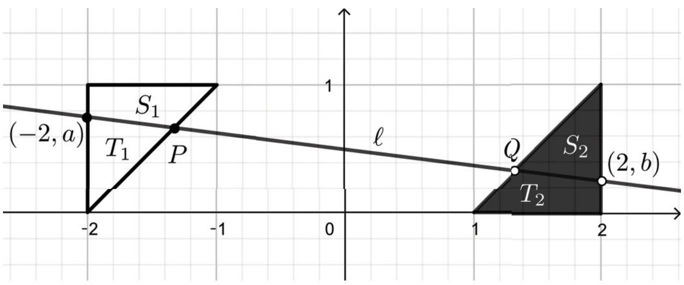

FIGURE 2. $2\mu(S_1) = 2\mu(S_2) = \rho$.

$(a - b)x_1 + 4x_2 = 2(a + b)$. Let P and Q be the intersections of ℓ and hypotenuses $x_2 = x_1 + 2$ and $x_2 = x_1 - 1$, respectively. Then their x_1-coordinates are

$$x_1 = \frac{4a}{a - b + 4} - 2, \ \frac{4 - 4b}{a - b + 4},$$

respectively. Hence the areas of triangle S_1 and S_2 are given by

$$\mu(S_1) = \frac{1}{2} - T_1 = \frac{1}{2}\left(1 - \frac{4a^2}{a - b + 4}\right), \quad \mu(S_2) = \frac{1}{2}(1 - b)\frac{4 - 4b}{a - b + 4},$$

respectively. Therefore $\mu(S_1) = \mu(S_2)$ if and only if $4a^2 + 4b^2 - a - 7b = 0$, which is equivalent to

$$\left(a - \frac{1}{8}\right)^2 + \left(b - \frac{7}{8}\right)^2 = \frac{25}{32}.$$

Hence, we may put

$$a - \frac{1}{8} = \frac{5}{4\sqrt{2}} \cos\theta, \quad b - \frac{7}{8} = \frac{5}{4\sqrt{2}} \sin\theta.$$

Therefore, the equation of ℓ is as follows.

$$\left\{ \frac{5}{4\sqrt{2}}(\cos\theta - \sin\theta) - \frac{3}{4} \right\} x_1 + 4x_2 = \frac{5}{2\sqrt{2}}(\sin\theta + \cos\theta) + 2.$$

Normalizing the coefficients, we obtain $\boldsymbol{u} \in S^2$. Since the area of the triangles is $1/2$, the ratio of the division is

$$\rho_1(\boldsymbol{u}) = 2S_1 = 1 - \frac{\left(\frac{5}{4\sqrt{2}}\cos\theta + \frac{1}{8}\right)^2}{\frac{5}{4\sqrt{2}}(\cos\theta - \sin\theta) + \frac{3}{4}} = 1 - \frac{1}{8} \cdot \frac{(5\sqrt{2}\cos\theta + 1)^2}{5\sqrt{2}(\cos\theta - \sin\theta) + 6}.$$

3. Concluding remarks

In the ham-sandwich theorem, we placed $A_1, \ldots, A_n$ on the hyperplane $u_{n+1} = -1$ and divided them by a hyperplane passing through the origin of $\mathbb{R}^{n+1}$ whose normal vector is $\boldsymbol{u} = (u, u_{n+1}) \in S^n$ (Figure 3 Left). On the other hand, in Theorem 2.2, we place A_i on the hyperplane $u_{n+1} = -\gamma_i - 1$ (Figure 3 Right). This is why Theorem 2.2 is richer than the ham-sandwich theorem.

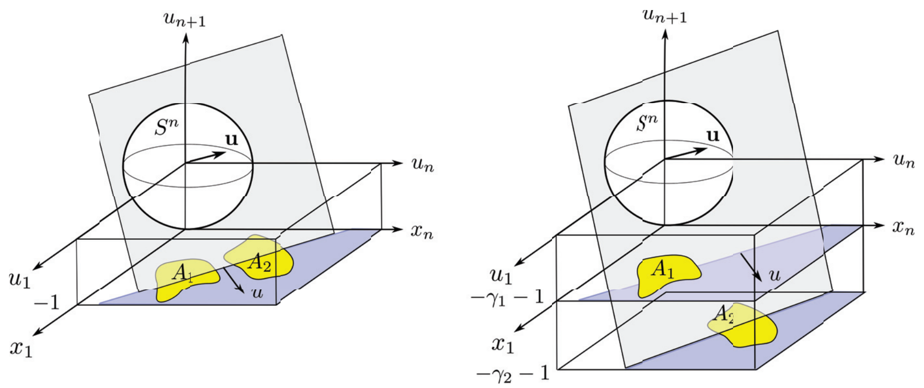

FIGURE 3. Left: $\gamma_1 = \gamma_2 = 0$. Right: $\gamma_1 < \gamma_2$.

4. Acknowledgements

The author would like to thank the referee for careful reading.

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Manuscript received 28 December 2024
revised 18 March 2025

H. Kawasaki
Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan
E-mail address: `kawasaki.hidefumi.245@m.kyushu-u.ac.jp`