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INTEGRATION BY PARTS FOR OPERATOR VALUED MCSHANE INTEGRABLE FUNCTIONS

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Abstract. In this article, we introduce McShane integration theory in the class of operator valued functions. We extend this theory to McShane-Stieltjes integral of operator values functions, McShane-Stieljes type integrals of operator valued functions with values in bounded linear operators of Hilbert Spaces. Finally, we extend the McShane integral in this setting and prove a version of integration by parts theorem.

1. Introduction

McShane established the Lebesgue integral's equality with a modified Henstock integral in the late 1960's. McShane expanded the class of tagged partitions with more flexible, it is not mandatory that tagged of a certain interval need not to belong in the mentioned interval. In this approaches it is complicated to find integral of a given function. It can be seen that a function is McShane integrable if and only if the absolute values of the function is also McShane integrable. This condition makes the McShane integral equal to the Lebesgue integral. (see [6]). The McShane integral was the sole emphasis of Gordon's work, and he developed its features for the situation where the function has values in a Banach space (see [7]). Additionally note the references [4, 14] below. The McShane-Stieltjes integral, which is an extension of the McShane integral, was first discussed by Park (see [13]). The multilinear McShane-Stieltjes integrals of vector-valued functions formed on compact intervals in $\mathbb{R}^n$ have recently been studied by Halilović et al. (see [8]). Schwabik et al. [16] introduced Stieltjes integral. Integration by parts results concerning Stieltjes integrals for functions with values in Banach spaces were presented. Fundamental results concerning Stieltjes integrals for functions with values in Banach spaces are presented by Schwabik et al. in [15]. A bilinear form was used to defined a Stieltjes type integral by them. Becerra et al. adapted the approach of [16] to defined Henstock-Stieltjes integral for vector-valued functions in [3]. Becerra et al. established certain integration by parts theorems for the Henstock integral and a Riesz-type theorem also see [2]. One can see[12] for Henstock-Kurzweil-Stieltjes integrals and [10, 11] for recent work of McShane integrals.
Operator-valued functions is that these functions need not have Lebesgue (like) integral (see [9]). However, they always have a Henstock-Kurzweil integral. One can see [17] for detailed of Henstock-Kurzweil integrals. One must be able to approximate

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the integral using straightforward functions in either the strong sense (Bochner) or the weak sense (Gelfand-Pettis) in order to define it constructively. However, in any case, each simple function must either be countably valued and strongly measurable, or countably valued and weakly measurable. Bagheri-Bardi [1] discussed operator-valued measurable functions. Gill et al. introduced Henstock-Kurzweil integration of Operator-Valued Functions (see [5, Chapter 4]).

We were inspired by the work of Becerra et al. [3] for Henstock integral to think for McShane and strong McShane integrable function.

This article is organized into four sections. As a first step, Section 2 contains some preliminary material that will be helpful in the next sections. In Section 3, we introduce the McShane integral of operator-valued functions ϕ, defined on a measure space (Ω, Σ, μ), $\Omega = [u, v] \subset \mathbb{R}$ with values in $L(H)$. The Bochner integral of operator-valued functions is McShane integral of operator-valued functions in $[u, v]$, as shown. We introduce the McShane-Stieltjes integral of the operator-valued function ϕ with values in $L(H)$ in Section 4. In the follow-up, we develop a bilinear bounded operator definition for the McShane-Stieltjes integral in approach to [15], and established a number of significant results. Subsequently, our settings are used to prove an integration by parts theorem.

2. Preliminaries

Throughout the article we denote (Ω, Σ, μ), $\Omega = [u, v] \subset \mathbb{R}$ be a measure space with Lebesgue measure μ, H for Hilbert space. $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ symbolised for Banach spaces. The space of bounded operators on H is denoted by $L(H)$. Recalling almost separably valued, strongly measurable, and Bochner integrals as follows:

Definition 2.1. ([5], Definition 2.1) Let $\phi : \Omega \to H$ be a given function and $E \subset \Omega$ with $\mu(E) = 0$.

(1) The given function ϕ is called almost surely separably valued (or essential separably valued) on E if $\phi(\Omega \backslash E) \subset H$ is separable.
(2) The given function ϕ is called strongly measurable if we can find a sequence of countably valued function (ϕ_n) converges almost surely to ϕ.
(3) The given function ϕ is called Bochner integrable if $||\phi||_H$ is Lebesgue integrable.

In our work, the Hilbert space H is not separable unless we mention the separability of H . Let us consider $\mathcal{A} : \Omega \to H$ be a family of operator-valued functions. In our assumption $\Omega \subset [u, v]$ need not be almost separable valued so Ω is not necessarily strongly nor weakly measurable. In Section 3 and in Section 4, ϕ is operator-valued function on $[u, v]$.

Theorem 2.2. *([7], Theorem 16) Let $\phi : [u, v] \to \mathcal{X}$ be a given function. If ϕ is Bochner integrable on $[u, v]$, then it is McShane integrable on $[u, v]$.*

We recall bounded variation, strongly bounded variation, and strongly absolutely continuous as follows:

Definition 2.3. [3, 14]

(1) Let $\psi : [u,v] \to \mathbb{R}$ be a given function and $E \subset [u,v]$. The given function ψ is of bounded variation (in short BV) on E if

$$\sup \sum_i |\psi(d_i) - \psi(c_i)| < \infty$$

whenever the supremum is considering over all finite sequences of non-overlapping intervals $\{[c_i, d_i]\}$ containing end points are in E.

(2) Let $\phi : [u,v] \to \mathcal{X}$ be a given function. The given function ϕ is strongly bounded variation (BV) on $E \subset [u,v]$ if

$$V(\phi, E) = \sup \left\{ \sum_i \Big\| \phi(d_i) - \phi(c_i) \Big\|_{\mathcal{X}} \right\} < \infty$$

whenever the supremum is consider over all finite sequences $\{[ci, di]\}$ of non-overlapping intervals whose end points are in E.

(3) The given function $\phi : [u,v] \to \mathcal{X}$ is said to be strongly continuous on $E \subset [u,v]$ if for every $\epsilon > 0$ there exists $\nu > 0$ such that

$$\sum_i ||\phi(d_i) - \phi(c_i)||_{\mathcal{X}} < \epsilon$$

whenever $\{[c_i, d_i]\}$ is a finite sequence of non-overlapping intervals containing the end points are in E with $\sum\limits_i (d_i - c_i) < \nu$.

We recall $B : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is a bounded bilinear operator if B is a linear in each variable and there exists $M > 0$ such that

$$\big\|B(x,y)\big\|_{\mathcal{Z}} \leq M \big\|x\big\|_{\mathcal{X}} \big\|y\big\|_{\mathcal{Y}}$$

It is easy to prove

$$\big\|B\big\| = inf \Big\{ M > 0 : \big\|B(x,y)\big\|_{\mathcal{Z}} \leq M \big\|x\big\|_{\mathcal{X}} \big\|y\big\|_{\mathcal{Y}} \Big\}$$

is a norm of the operator B. If $\phi : [u,v] \to \mathcal{X}$ and $\Phi : [u,v] \to \mathcal{Y}$ be two vector-valued functions. Consider $P = \{t_0, x_1, t_1, x_2, t_2, ..., x_n, t_n\}$ be a M- partition of $[u,v]$. We denote the Riemann-Stieltjes sum as $S(\phi, \Phi, P) = \sum\limits_{i=1}^{n} B\bigg(\phi(x_i), \Phi(t_i) - \Phi(t_{i-1}) \bigg)$. Remembering strongly bounded variation with reference to the bilinear operator $B : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ as follows:

Definition 2.4 ([3], Definition 6)**.** Let $\Phi : [u,v] \to \mathcal{Y}$ be a given function. Let us consider $P = \{t_0, t_1, ..., t_n\}$ be a M-partition of $[u,v]$. Then

$$V_u^v(\Phi, P) = \sup \left\{ \sum_{i=1}^{n} \Big\| B(x_i, \Phi(t_i) - \Phi(t_{i-1}) \Big\|_{\mathcal{Z}} \right\}$$

whenever the supremum is considering over all possible elements $x_i \in \mathcal{X}, i = 1, 2, ..., n$ along with $||x_i|| \leq 1$. The strongly bounded variation of ϕ on $[u,v]$ is

$$sup \left\{ V_u^v(\phi, \Phi) \right\}$$

Next we recall following results of vector-valued McShane integrable function.

Theorem 2.5 ([14], Theorem 7.4.9)**.** *Let $\phi : [u,v] \to \mathcal{X}$ be a given McShane integrable on $[u,v]$. Consider the primitive of ϕ be $\mathcal{F}$. Then $\mathcal{F}$ is differentiable almost everywhere and $\mathcal{F}'(t) = \phi(t)$ a.e. on $[u,v]$.*

Theorem 2.6 ([14], Theorem 4.4.11)**.** *If the given function $\phi : [u,v] \to \mathcal{X}$ is McShane integrable on $[u,v]$ then $\mathcal{F}$ is strongly absolutely continuous on $[u,v]$ where $\mathcal{F}$ is the primitive of ϕ on $[u,v]$.*

3. McShane integration of Operator-Valued Functions

In this Section, we described McShane integrals of operator-valued functions. Assuming that H is a separable Hilbert Space, $L(H)$ is the class of bounded linear operators on H . Let us denote a given family of operators as $\mathcal{A}(t)$ on $[u,v]$ where $[u,v] \subset \mathbb{R}$, $\mathcal{A}(t) \in L(H)$ for each $t \in [u,v]$.

Let us define $\delta : [u,v] \to (0,\infty)$ be a gauge. Let us consider a free tagged partition

$$P = \{t_0, \xi_1, t_1, \xi_2, ..., \xi_n, t_n\}$$

with $u = t_0 \le \xi_1 \le t_1 \le ... \le \xi_n \le t_n = v$ which is sub-ordinate to δ. The free tagged partition P is called M-partition for δ if for $0 \le i \le n, (t_{i-1}, t_i) \subset \delta(\xi_i)$. The Riemann sum with M-partition P is written as

$$S(\mathcal{A}, P) = \sum_{i=1}^{n} \mathcal{A}(\xi_i)(t_i - t_{i-1})$$

Lemma 3.1. *Let us define $\delta_1 : [u,v] \to (0,\infty)$ and $\delta_2 : [u,v] \to (0,\infty)$ with $\delta_1(t) \le \delta_2(t)$. If P_1 is a M-partition subordinate to δ_1, it is also a M-partition subordinate to δ_2.*

Proof. The proof follows from the definition of M-partition, so we have omitted the proof. □

We now specify our primary definition as follows:

Definition 3.2. We called the given family of operators, $\mathcal{A}(t), t \in [u,v]$ as McShane integrable if there is an operator Δ in $L(H)$ in such a way that for each > 0, there exists a M-partition P such that

$$\left\| \sum_{i=1}^{n} \mathcal{A}(\xi_i)(t_i - t_{i-1}) - \Delta \right\| < \epsilon$$

whenever P is sub-ordinate to δ.

In this occasion,$\Delta = (M) \int_u^v \mathcal{A}(t)dt$, M denoted the set of all McShane integrable operators on $[u,v]$. In equivalence to the Definition 3.2, we can define operator-valued strong McShane integral as below.

Definition 3.3. A family $\mathcal{A}(t), t \in [u,v]$ is said to have McShane integral if there exists $\mathcal{F} : [u,v] \to H$ such that for every $\epsilon > 0$ there exists a M-partition P such that

$$\sum_{i=1}^{n} \left\|\mathcal{A}(\xi_i)(t_i - t_{i-1}) - \mathcal{F}(t_i) - \mathcal{F}(t_{i-1})\right\|_H < \epsilon$$

whenever P is sub-ordinate to δ.

In this case, $(M)\int_u^v \mathcal{A}(t) = \mathcal{F}(v) - \mathcal{A}(u)$

Remark 3.4. We can find linearity, integrability over sub intervals, and the continuity of $\mathcal{F} : [u,v] \to H$, called primitive, given by $\mathcal{F}(t) = \int_a^t \mathcal{A}(t)dt, t \in [u,v]$. It is not hard to see the space of operator-valued McShane integrals in sense of Definition 3.2 and the space of operator-valued McShane integrals in sense of Definition 3.3 are not exactly same. In this article, we are not interested to investigate the inclusion relationship of above mentioned spaces.

We denote $M([u,v],H)$ the space of all McShane integrals of operator-valued function ϕ from $[u,v]$ on H . It is not hard to prove

$$||\phi||_A = \sup\{\left\|(M)\int_u^v \phi\right\|_H : u \le t \le v\}$$

a norm on $M([u,v],H)$. We call it the Alexiewicz norm. One can see [14] for McShane integral of Banach-valued functions. The following theorem shows the uniqueness of the operator-valued McShane integral.

Theorem 3.5. *The McShane integral of a family $\mathcal{A}(t), t \in [u,v]$ is unique on $[u,v]$.*

Proof. Suppose $\mathcal{A}(t), t \in [u,v]$ is McShane integrable on $[u,v]$. If possible suppose Δ_1, Δ_2 are the McShane integrals of $\mathcal{A}(t), t \in [u,v]$ with $\Delta_1 \neq \Delta_2$. Since Δ_1, Δ_2 are McShane integrals of $\mathcal{A}(t), t \in [u,v]$ so for all $\epsilon > 0$, there exists a $M - \delta$ partition P such that

$$\left\|\sum_{i=1}^{n} \mathcal{A}(\xi_i)(t_i - t_{i-1}) - \Delta_1\right\| < \frac{\epsilon}{2} \tag{3.1}$$

and

$$\left\|\sum_{i=1}^{n} \mathcal{A}(\xi_i)(t_i - t_{i-1}) - \Delta_2\right\| < \frac{\epsilon}{2} \tag{3.2}$$

whenever the free tagged partition P is sub-ordinate to δ. From (3.1) and (3.2), $||\Delta_1 - \Delta_2|| < \epsilon$. The arbitrary nature of ϵ gives $\Delta_1 = \Delta_2$. □

Next theorem gives the linearity nature of operator-valued McShane integrable functions.

Theorem 3.6. *Suppose for all $t \in [u,v]$, the given family of operators $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ are McShane integrable. Then*

(1) $\mathcal{A}_1(t)+\mathcal{A}_2(t)$ is also McShane integrable and

$$(M)\int_u^v \Big(\mathcal{A}_1(t)+\mathcal{A}_2(t)\Big)dt = (M)\int_u^v \mathcal{A}_1(t)dt + (M)\int_u^v \mathcal{A}_2(t)dt.$$

(2) For $\alpha \in \mathbb{R}$, $\alpha\mathcal{A}(t)$ is McShane integrable with

$$(M)\int_u^v \alpha\mathcal{A}(t)dt = \alpha(M)\int_u^v \mathcal{A}(t)dt$$

Proof. For (1) : Let $\epsilon > 0$ be given and suppose Δ_1, Δ_2 are McShane integrals of $\mathcal{A}_1(t), \mathcal{A}_2(t)$, t$\in$ $[u,v]$ respectively. Since $\mathcal{A}_1(t)$ is McShane integrable, consider a gauge δ_1 on $[u,v]$ such that

$$\Big\|S(\mathcal{A}_1, P_1) - \Delta_1\Big\| < \frac{\epsilon}{2} \tag{3.3}$$

for each M-partition P_1 which is sub-ordinate to δ_1 on $[u,v]$. Similarly, there exists a positive function δ_2 on $[u,v]$ so that for every M-partition P_2 of $[u,v]$ which is sub-ordinate to δ_2 we have,

$$\Big\|S(\mathcal{A}_2, P_2) - \Delta_2\Big\| < \frac{\epsilon}{2} \tag{3.4}$$

Let $\delta = \max\{\delta_1, \delta_2\}$, Lemma 3.1, confirm us that $P = P_1 \cup P_2$ is sub-ordinate to δ which is M-partition also. Now, using (3.3) and (3.4) we can find,

$$\Big\|S\Big(\mathcal{A}_1+\mathcal{A}_2, P\Big) - \Big(\Delta_1+\Delta_2\Big)\Big\| < \epsilon$$

Consequently, $\mathcal{A}_1+\mathcal{A}_2$ is McShane integrable on $[u,v]$ and

$$(M)\int_u^v \Big(\mathcal{A}_1(t)+\mathcal{A}_1(t)\Big)dt = (M)\int_u^v (\mathcal{A}_1(t)dt + (M)\int_u^v (\mathcal{A}_2(t)dt.$$

For (2) : The proof is similar to (1). ☐

Theorem 3.7. *Let $\mathcal{A} : [u,v] \to H$. If $\mathcal{A} = 0$ a.e. in $[u,v]$, then $\mathcal{A}$ is McShane integrable on $[u,v]$ and $(M)\int_u^v \mathcal{A} = 0$.*

Proof. Let $\epsilon > 0$ be chosen. Let $\overline{\mathcal{A}} = \{t \in [u,v] : \mathcal{A}(t) = 0\}$ for each $n \in \mathbb{N}$. Let us set $\overline{\mathcal{A}}_n = \{t \in \overline{\mathcal{A}} : n-1 \leq ||\mathcal{A}(t)|| < n\}$. Since $\mu(\overline{\mathcal{A}}) = 0$, we have also $\mu(\overline{\mathcal{A}}_n) = 0$ for $n \in \mathbb{N}$. So we can construct an open set $\overline{G}_n$ such that $\mu(\overline{G}_n) < \epsilon$. Let $\delta : [u,v] \to (0,\infty)$ be a positive function with

$$\delta(t) = \begin{cases} 1; & t \in [u,v]\backslash\overline{\mathcal{A}} \\ 0; & otherwise \end{cases}$$

If we fix P be a M-partition sub-ordinate to δ then,

$$\begin{aligned}\left\|\sum_{i=1}^{n}\mathcal{A}(\xi_i)(t_i-t_{i-1})\right\| &\le \left\|\sum_{n=1}^{\infty}\sum_{\xi_i\in\overline{\mathcal{A}}_n}^{p}\mathcal{A}(\xi_i)(t_i-t_{i-1})\right\| \\ &\le \sum_{n=1}^{\infty}\left\|\sum_{\xi_i\in\overline{\mathcal{A}}_n}^{p}\mathcal{A}(\xi_i)(t_i-t_{i-1})\right\| \\ &< \sum_{n=1}^{\infty} n\sum_{\xi_i\in\overline{\mathcal{A}}_n}(t_i-t_{i-1}) \\ &< \sum_{n=1}^{\infty} n\mu(\overline{G}_n) < \epsilon\end{aligned}$$

So, $\mathcal{A}(t)$ is McShane integrable and $(M)\int_u^v(\mathcal{A}(t)dt = 0$ □

Directly from the Definition 3.2, we can proof Cauchy's criterion for operator-valued McShane integral. Here is the Cauchy's criterion for operator-valued McShane integral.

Theorem 3.8. *A given family of operator $\mathcal{A}(t), t \in [u,v]$ is McShane integrable on $[u,v]$ if and only if*

$$\left\|S(\mathcal{A},P_1) - S(\mathcal{A},P_2)\right\| < \epsilon$$

whenever for each $\epsilon > 0$ there exists a gauge δ on $[u,v]$ such that P_1 and P_2 are free tagged partitions of $[u,v]$ that are sub-ordinate to δ.

The following theorem can be proven using a method similar to [[14], Theorem 7.4.9].

Theorem 3.9. *Let a given family of operators $\mathcal{A}(t), t \in [u,v]$ be McShane integrable on $[a,b]$. If $\mathcal{F}$ be it's primitive then $\mathcal{F}$ is differentiable a.e. and $\mathcal{F}'(t) = \mathcal{A}(t)$ a.e. on $[u,v]$.*

The following theorem gives the nature of a primitive function of operator-valued on $[u,v]$.

Theorem 3.10. *If $\mathcal{A}(t)$ is McShane integrable on $[u,v]$ with the primitive $\mathcal{F}$, then $\mathcal{F}$ is strongly absolutely continuous on $[u,v]$.*

Proof. Proof is similar to the [[14], Theorem 7.4.11] so, we omit the proof. □

Next theorem assure that [[7], Theorem 16] for McShane integrals of Banach valued-functions also holds for operators on $[u,v]$.

Theorem 3.11. *Let $\mathcal{A}(t), t \in [u,v]$ be a given family of operators. If $\mathcal{A}(t)$ is Bochner integrable on $[u,v]$, then $\mathcal{A}(t)$ is McShane integrable on $[u,v]$ with*

$$(B)\int_u^v(\mathcal{A}(t)dt = (M)\int_u^v \mathcal{A}(t)dt.$$

Proof. Let E be a measurable subset of $[u, v]$, $\chi_E(t)$ be the characteristic function of E. Let us fix, $\mathcal{A}(t) = \mathcal{A}\chi_E(t)$. Let F be a compact subset of E and $\epsilon > 0$ be chosen. Let $D \subset [u, v]$ is an open set containing E in a way that $\mu(D\backslash F) < \frac{\epsilon}{||\mathcal{A}||}$. Consider a gauge function $\delta : [u, v] \to (0, \infty)$ defined as

$$\delta(t) = \begin{cases} d(t, [u, v]\backslash D) & \text{if} \quad t \in E \\ d(t, F) & \text{if} \quad t \in [u, v]\backslash E \end{cases}$$

where $d(t, [u, v]\backslash D)) = \inf\{|t - x| : x \in [u, v]\backslash D\}$ and $d(t, F) = \inf\{|t - x| : x \in F\}$. Suppose $P = \{t_0, \xi_1, t_1, \xi_2, t_2, \xi_3, t_3, ..., \xi_n, t_n\}$ be M-partition that is sub-ordinate to δ. If $\xi_i \in E, 1 \leq i \leq n$, then $(t_{i-1}, t_i) \subset D$ so that

$$\left\| \sum_{i=1}^{n} \mathcal{A}(\xi_i)(t_i - t_{i-1}) - \mathcal{A}\mu(D) \right\| = \left\| \mathcal{A} \right\| \left[\mu(D) - \sum_{\xi \in E} (t_i - t_{i-1}) \right]. \tag{3.5}$$

If $\xi_i \notin E$ then (t_{i-1}, t_i) and F are disjoint and

$$\left\| \sum_{i=1}^{n} \mathcal{A}(\xi_i)(t_i - t_{i-1}) - \mathcal{A}\mu(F) \right\| = \left\| \mathcal{A} \right\| \left[\sum_{\xi \notin E} (t_i - t_{i-1}) - \mu(F) \right] \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\begin{aligned} \left\| \sum_{i=1}^{n} \mathcal{A}(\xi_i)(t_i - t_{i-1}) - \mathcal{A}\mu(F) \right\| &= \left\| \mathcal{A} \right\| \left[\sum_{\xi \in E} (t_i - t_{i-1}) - \mu(E) \right] \\ &\leq \left\| \mathcal{A} \right\| \left[\mu(D) - \mu(E) \right] \\ &\leq \left\| \mathcal{A} \right\| \mu(D\backslash F) < \epsilon \end{aligned}$$

Now, Bochner integrability of $\mathcal{A}$ with $\mathcal{A}(t) = \sum\limits_{j=1}^{\infty} \mathcal{A}_j \chi_{E_j}(t)$ implies

$$(B) \int_u^v \mathcal{A}(t)dt = \sum_{j=1}^{\infty} \mathcal{A}_j \mu(E_j)$$

and

$$(L) \int_u^v ||\mathcal{A}(t)||dt = \sum_{j=1}^{\infty} ||\mathcal{A}_j|| \mu(E_j)$$

From the definition of Bochner integral, $\mathcal{A}(t)$ is uniformly measurable on $[u, v]$, and there exists a sequence of countably valued operators$(\mathcal{A}_j(t))$ in $L(H)$, converges to $\mathcal{A}(t)$ in the sense of uniform operator topology such that

$$\lim_{j\to\infty} (L) \int_u^v ||\mathcal{A}_j(t) - \mathcal{A}(t)||dt = 0$$

and

$$(B) \int_u^v \mathcal{A}(t)dt = \lim_{j\to\infty} (B) \int_u^v \mathcal{A}_j(t)dt.$$

Countably valued properties of $\mathcal{A}_j(t)$ gives,

$$(M)\int_u^v \mathcal{A}_j(t)dt = (B)\int_u^v \mathcal{A}_j(t)dt$$

So,

$$(B)\int_u^v \mathcal{A}(t)dt = \lim_{j\to\infty}(M)\int_u^v \mathcal{A}_j(t)dt$$

Finally, since every Lebesgue integral is McShane integral (see [7]), $f_j(t) = ||\mathcal{A}_j(t) - \mathcal{A}(t)||$ has McShane integral and hence

$$\lim_{j\to\infty}(M)\int_u^v \mathcal{A}_j(t)dt = 0$$

Hence,

$$\left\|(B)\int_u^v \mathcal{A}(t)dt = (M)\int_u^v \mathcal{A}_m(t)dt\right\| < \epsilon$$

whenever m is large. Consequently, $(B)\int_u^v \mathcal{A}(t)dt = (M)\int_u^v \mathcal{A}(t)dt$ □

Corollary 3.12. *Every operator valued McShane integrable functions are operator valued Henstock-Kurzweil integrable functions and the integrals are equal.*

4. Operator-valued McShane-Stieltjes integrals

In this section, we defined McShane-Stieljes integrals for operator valued functions. We extend McShane-Stieltjes type integrals for bounded bilinear operators. We begin this section with definition of McShane-Stieltjes integral of operator valued function as follows:

Definition 4.1. Let $\psi : [u, v] \to \mathbb{R}$ be an increasing function. If $\phi : [u, v] \to H$ is an operator valued function, Consider

$$P = \{t_0, \xi_1, t_1, \xi_2, t_2, ..., \xi_n, t_n\}$$

be a M-partition that is subordinate to a gauge δ. We denote the Riemann-Stieltjes sum as

$$S(\phi, \psi, P) = \sum_{i=1}^{n} \phi(\xi_i)[\psi(t_i) - \psi(t_{i-1})].$$

Then ϕ is McShane-Stieltjes integrable with respect to ψ on $[u, v]$ with McShane-Stieltjes integral I_{MS} if for each chosen $\epsilon > 0$ there exists a gauge δ such that

$$\left\|S(\phi, \psi, P) - I_{MS}\right\|_H < \epsilon.$$

whenever P is a M-partition that is subordinate to δ.

The following theorem follows from the definition of McShane-Stieltjes integrals for operator valued functions.

Definition 4.2. An operator valued function $\phi : [u, v] \to H$ is McShane-Stieltjes integrable with respect to an increasing function $\psi : [u, v] \to \mathbb{R}$ if and only if for each $\epsilon > 0$ there exists a gauge $\delta : [u, v] \to (0, \infty)$ such that

$$\left\| S(\phi, \psi, P_1) - S(\phi, \psi, P_2) \right\| < \epsilon$$

whenever P_1, P_2 are M-partition that are sub-ordinate to δ.

Theorem 4.3. *Let $\phi : [u, v] \to H$ be of BV on $[u, v]$. If ψ is continuous on $[u, v]$, then*

$$(M) \int_u^v \psi(s) d\phi(s)$$

exists.

Proof. Since ψ is continuous and $\phi(t)$, is of BV on $[u, v]$. Let us define the Riemann-Stieltjes sum $S(\phi, \psi, P)$ by

$$S(\phi, \psi, P) = \sum_P \psi(t_i)[\phi(v_i) - \phi(u_i)].$$

where P is M-partition that is sub-ordinate to δ.

The uniform continuity of ψ is obvious. Let $\epsilon > 0$ be chosen. Then there exists a $\delta > 0$, such that $|\psi(t) - \psi(r)| < \epsilon$ whenever $|t - r| < \delta$. If P_1, P_2 are M-partitions that are sub-ordinate to δ with max $\{b_i - a_i\} = |P| < \frac{\delta}{2}$. Let H' be the dual space of H , for each $\phi(.) \in H$, we consider $\mathcal{J}\psi(\phi) = (\phi, \psi)$. Using triangle inequality, we have

$$\left\| \mathcal{J}_\psi \Big(S(\phi, \psi, P_1) - S(\phi, \psi, P_2) \Big) \right\| \leq \epsilon BV_u^v \{\mathcal{J}_H(\phi)\}$$

for each linear functionals $\mathcal{J}_H(.) \in H'$. Now

$$\begin{aligned} BV_u^v \{\mathcal{J}_H(\phi) &\leq BV_u^v [Re\{\mathcal{J}_H(\phi)] + BV_u^v [Im\{\mathcal{J}_H(\phi)\}] \\ &\leq \sup_P \left\| \mathcal{J}_H \left\{ \sum \Big[\psi(v_i) - \psi(u_i) \Big] \right\} \right\| \end{aligned}$$

where the sup is over partitions of $[u, v]$. By definition and Theorem 4.2, there is an M such that $BV_u^v\{\mathcal{J}_H(\phi)\} \leq M||\mathcal{J}_H||$. It follow that

$$\begin{aligned} \left\| S(\phi, \psi, P_1) - S(\phi, \psi, P_2) \right\| &= \sup_{||\mathcal{J}_H||=1} |\mathcal{J}_H(S(\phi, \psi, P_1) - S(\phi, \psi, P_2))| \\ &\leq M\epsilon \end{aligned}$$

Hence $(M) \int_u^v \psi(s) d\phi(s)$ exists.

Theorem 4.4. *Let $\phi : [u, v] \to H$ be of BV on $[u, v]$. Consider an increasing function $\psi : [u, v] \to \mathbb{R}$ is defined. Let $\mathcal{A}$ be a closed dense linear operator on H , $\phi \in D(\mathcal{A})$ and $\mathcal{A}\phi(t) = f(t)$ is BV, then*

$$\mathcal{A} \int_u^v \psi(t) d\phi(t) = \int_u^v \psi(t) df(t).$$

Proof. Since $\mathcal{A}$ is linear, For any M-partition P, it is easy to see $\mathcal{A}S(\phi,\psi,P) = S(\mathcal{A}\phi,\psi,P)$. Also, we can find

$$\lim_{||P||\to 0} S(\phi,\psi,P) = \int_u^v \psi(t)d\phi(t) \tag{4.1}$$

Now from the linearlity of $\mathcal{A}$, bounded variation nature of $\mathcal{A}\phi$ with (4.1) we can find

$$\begin{aligned}\mathcal{A}\lim_{||P||\to 0} S(\phi,\psi,P) &= \lim_{||P||\to 0} S(\mathcal{A}\phi,\psi,P)\\ &= \int_u^v \psi(t)d\mathcal{A}\phi(t)\\ &= \int_u^v \psi(t)df(t)\end{aligned}$$

Consequently,

$$\mathcal{A}\int_u^v \psi(t)d\phi(t) = \int_u^v \psi(t)df(t)$$

□

4.1. **McShane-Stieltjes integral with respect to bilinear bounded operator.** In this Section, we extend operator-valued McShane-Stieltjes integral with respect to a bounded bilinear operator to find "Integration by parts" theorem for operator-valued McShane integral. Let H_1, H_2 , and H_3 are Hilbert spaces. Let us denote their respective norms as $||.||_{H_1}, ||.||_{H_2}$, and $||.||_{H_3}$. Duality of H_1', H_2' , and H_3' are H_1', H_2' , and H_3' respectively. Let us denote $B : H_1 \times H_2 \to H_3$ as a bounded bilinear operator which we defined as follows:

Definition 4.5. The given bilinear operator $B : H_1 \times H_2 \to H_3$ is said to be a bounded bilinear operator if B is a linear in each variable and there exists $M > 0$ such that

$$\big\|B(h_1,h_2)\big\|_{H_3} \le M\big\|h_1\big\|_{H_1}\big\|h_2\big\|_{H_2}.$$

It is easy to prove

$$||B|| = \inf\{M > 0 : ||B(h_1,h_2)||_{H_3} \le M||h_1||_{H_1}||h_2||_{H_2}\}$$

is a norm of the operator B.

Let $\phi : [u,v] \to H_1, \Phi : [u,v] \to H_2$ be two operator-valued functions. Let $P = \{t_0, x_1, t_1, x_2, ..., x_n, t_n\}$ be a M-partition of $[u,v]$. Then we can defined Riemann-Stieltjes sum $S(\phi,\psi,P) = \sum_{i=1}^{n} B\Big(\phi(x_i).\Phi(t_i) - \Phi(t_{i-1})\Big)$. Now we define Stieltjes type integrals of operator valued function on $[u,v]$ as below

Definition 4.6. An operator $I_{MS} \in H_3$ is said to be McShane-Stieltjes integral of $\phi : [u,v] \to H_1$ with respect to $\Phi : [u,v] \to H_2$ if for every $\epsilon > 0$ there exists a gauge δ on $[u,v]$ such that

$$\Big\|\sum_{i=1}^{n} B\Big(\phi(\xi_i),\Phi(t_i) - \Phi(t_{i-1})\Big) - I_{MS}\Big\|_{H_3} < \epsilon$$

for each M-partition $P = \{t_0, \xi_1, t_1, \xi_2, t_2, ..., \xi_n, t_n\}$ of $[u, v]$.

It is not hard to see operator-valued McShane-Stieltjes integral with respect to bilinear bounded operator of Definition 4.6 is unique. From the definition, we can immediately verify the properties of linear and integrability over sub-intervals for the operator-valued McShane integral in the context of bilinear bounded operator. We defined bounded variation and strong bounded variation of the bilinear operator $B : H_1 \times H_2 \to H_3$ as below:

Definition 4.7. (1) Let $\Phi : [u, v] \to H_2$ be a McShane integrable operator valued function. Let us consider

$$P = \{t_0, \xi_1, t_1, \xi_2, t_2, ..., \xi_n, t_n\}$$

be a M-partition of $[u, v]$, then

$$V_u^v(\Phi, P) = \sup\left\{\sum_{i=1}^{n} \left\|B\Big(x_i, \Phi(t_i) - \Phi(t_{i-1})\Big)\right\|_{H_3}\right\}$$

where the supremum is taken over all elements $x_i \in H_1, i = 1, 2, ..., n$ with $||x_i||_{H_1} \leq 1$.

(2) The given bilinear operator $B : H_1 \times H_2 \to H_3$ is said to be strong bounded variation if

$$sBV_u^v(\Phi) = sup\,\{V_u^v(\Phi, P)\}$$

whenever the supremum is consider over all partitions of $[u, v]$.

From the definition, we can find the Cauchy's criterion type result as below.

Proposition 4.8. *Let $\phi : [u, v] \to H_1$ be give with respect to $\Phi : [u, v] \to H_2$. If $B_1 : H_1 \times H_2 \to H_3$ and $B_2 : H_1 \times H_2 \to H_3$ are defined. Then for every $\epsilon > 0$*

$$\left\|B_1\Big(\phi(\xi_i), \Phi(t_i) - \Phi(t_{i-1})\Big) - B_2\Big(\phi(\xi_i), \Phi(t_i) - \Phi(t_{i-1})\Big)\right\| < \epsilon. \tag{4.2}$$

Next, we are presenting convergence results of operator-valued McShane integral.

Theorem 4.9. *Let ϕ, $\phi_n : [u, v] \to H_1$ and $\Phi : [u, v] \to H_2$ are given. If $sBV_u^v(\Phi) < \infty$ the McShane-Stieltjes integral $(MS) \int_u^v B(\phi_n, d\Phi)$ exists and the sequence (ϕ_n) converges uniformly to ϕ on $[u, v]$ then the integral $(MS) \int_u^v B(\phi, d\Phi)$ exists and*

$$(MS) \int_u^v B(\phi_n, d\Phi) = \lim_{n\to\infty} (MS) \int_u^v B(\phi, d\Phi).$$

Proof. Since $\phi_n \to \phi$ uniformly on $[u, v]$ then for a chosen $\epsilon > 0$ there exists $N_0 > 0$ such that

$$||\phi_n(t) - \phi_m(t)||_{H1} < \epsilon$$

for every $n, m > N_0$. From [[14], Theorem 9] we can have

$$M(I_{MS}) = (M)\int_E B(\phi_n, d\Phi)$$

for every $E \subset [u,v]$ and $\lim_{n\to\infty} M_n(I_{MS}) = M(I_{MS})$. So,

$$||M_n(I_{MS}) - M(I_{MS})||_{H_3} < \epsilon$$

for every $n > N_1, N_1 \in \mathbb{N}$. Let $m > \max N_0, N_1$. Since

$$(MS)\int_u^v B(\phi_m, d\Phi) = M_m(v) - M_m(u)$$

exists, then

$$\sum_{i=1}^{n} \left\| B(\phi_m(\xi_i), \Phi(t_i) - \psi(t_{i-1}) - (MS)\int_{t_{i-1}}^{t_1} B(\phi_m, d\Phi) \right\|_{H_3} < \epsilon$$

So, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \left\| B(\phi(\xi_i), \Phi(t_i) - \Phi(t_{i-1}) - \left[(MS)(t_i)(MS)(t_{i-1})\right] \right\|_{H_3} \\
&\leq \sum_{i=1}^{n} \Big\| B(\phi(\xi_i), \Phi(t_i) - \Phi(t_{i-1}) - B(\phi_m(\xi_i), \Phi(t_i)) \\
&- \Phi(t_{i-1}))\Big\|_{H_3} + \sum_{i=1}^{n} \left\| B(\phi_m(\xi_i), \Phi(t_i) - \Phi(t_{i-1}) - (MS)\int_{t_{i-1}}^{t_1} B(\phi_m, d\Phi) \right\|_{H_3} \\
&+ \sum_{i=1}^{n} \left\| (MS)\int_{t_{i-1}}^{t_1} B(\phi_m, d\Phi) - \left[(MS)(t_i) - (MS)(t_{i-1})\right] \right\|_{H_3} \\
&\leq \sum_{\phi(\xi_i) \neq \phi_m(\xi_i)} \left\| B(\frac{\left[\phi(\xi_i) - \phi_m(\xi_i)\right] \left\| \phi(\xi_i) - \phi_n(\xi_i) \right\|_{H_1}}{\left\| \phi(\xi_i) - \phi_m(\xi_i) \right\|_{H_1}}, \Phi(t_i) - \Phi(t_{i-1}) \right\|_{H_3} + \epsilon \\
&\leq \left(\sup_{t\in[u,v]} \left\| \phi(t) - \phi_m(t) \right\|_{H_1} \right) sBV_u^v(\Phi) + \epsilon \\
&< \epsilon
\end{aligned}$$

Hence $(MS)\int_u^v B(\phi, d\Phi)$ exists and

$$(MS)\int_u^v B(\phi_n, d\Phi) = \lim_{n\to\infty} (MS)\int_u^v B(\phi, d\Phi).$$

We need the following Lemma to find "Integration by parts" theorem for operator-valued McShane integral on $[u, v]$. □

Lemma 4.10. *Let $\phi : [u, v] \to H_1$ is McShane integrable operator valued function on $[u, v]$.Consider $\mathcal{F}$ is a regulated primtive of ϕ and $\Phi : [u, v] \to H_2$ is defined with $sBV_u^v(\Phi) < \infty$ then $(MS) \int_u^v B(\mathcal{F}, d\Phi)$ exists.*

Proof. Given $\mathcal{F}$ is regulated function. Then there exists a sequence $\mathcal{F}_n : [u, v] \to H_1$, $n = 1, 2, ...$ of step functions which converges uniformly to $\mathcal{F}$. See [15, 16] for the detailed of regulated function. Also the limit of $(\mathcal{F}_n)$ is of the uniform limit of step function (see [14], Proposition 2). Now from [[3], Theorem 24] we can conclude $(MS) \int_u^v B(\mathcal{F}, d\Phi)$ exists.

The following theorem is our main result of this article.

Theorem 4.11. *(Integration by parts) Let $\phi : [u, v] \to H_1$ is McShane integrable operator valued function on $[u, v]$. Consider $\mathcal{F}$ is a regulated primtive of ϕ. If $\Phi : [u, v] \to H_2$ is defined with $sBV_u^v(\Phi) < \infty$ Then $(M) \int_u^v B(\phi, \Phi)$ exists for a M-partition P that is sub-ordinate to δ and*

$$(M) \int_u^v B(\phi, \Phi) = B\Big(\mathcal{F}(v), \Phi(v)\Big) - (MS) \int_u^v B\Big(\mathcal{F}, d\Phi\Big).$$

Proof. Because of the fact that ϕ is a McShane integral of an operator-valued function on $[u, v]$ with its primitive $\mathcal{F}$. There exists a M-partition $P_1 = \{t_0, \xi_1, t_1, \xi_2, t_2, ..., \xi_n, t_n\}$ for a chosen $\epsilon > 0$ that is subordinate to a gauge δ_1 such that

$$\sum_{i=1}^{n} \Big\| \phi(\xi_i)(t_i - t_{i-1}) - \Big[\mathcal{F}(t_i) - \mathcal{F}(t_{i-1})\Big] \Big\|_{H_1} < \epsilon$$

Since $\mathcal{F}$ is a regulated primitive of ϕ with $sBV_u^v(\Phi) < \infty$. By the Lemma 4.10, $(MS) \int_u^v B(\mathcal{F}, d\Phi)$ exists. In this situation there exists a gauge δ_2 such that

$$\sum_{j=1}^{m} \Big\| B\Big(\mathcal{F}(\gamma_j), \Phi(t_j) - \Phi(t_{j-1})\Big) - (MS) \int_{t_{j-1}}^{t_j} B(\mathcal{F}, d\Phi) \Big\|_{H_3} < \epsilon$$

whenever $P_2 = \{t_0, \gamma_1, t_1, \gamma_2, t_2, ..., \gamma_m, t_m\}$ is a M-partition that is subordinate to δ_2 implies

$$\Big\| \sum_{i=1}^{n} \phi(\xi_i)(t_i - t_{i-1}) - \Big[\mathcal{F}(t_i) - \mathcal{F}(t_{i-1})\Big] \Big\|_{H_1} < \epsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$, $P = P_1 \cup P_2$. Then

$$\sum_{k=1}^{l} \left\| B\Big(\phi(t_k), \Phi(t_k)\Big)(t_k - t_{k-1}) - \Big[B\Big(\mathcal{F}(t_k), \Phi(t_k)\Big) - B\Big(\mathcal{F}(t_{k-1}), \Phi(t_{k-1})\Big)\Big] - (MS)\int_{t_{k-1}}^{t_k} B(\mathcal{F}, d\Phi) \right\|_{H_3}$$

$$\leq \sum_{k=1}^{l} \left\| B(\phi(t_k)(t_k - t_{k-1}) - \Big[\mathcal{F}(t_k) - \mathcal{F}(t_{k-1})\Big], \Phi(t_k) \right\|_{H_3}$$

$$+ \sum_{k=1}^{l} \left\| B\Big(\mathcal{F}(t_k), \Phi(t_k) - \Phi(t_{k-1})\Big) - (MS)\int_{t_{k-1}}^{t_k} B(\mathcal{F}, d\Phi) \right\|_{H_3} < \epsilon s BV_u^v(\Phi) + \epsilon < \epsilon$$

Hence,

$$(M)\int_u^v B(\phi, \Phi) = B(\mathcal{F}(v), \Phi(v)) - (MS)\int_u^v B(\mathcal{F}, d\Phi).$$

□

Remark 4.12. The fundamental theorem of calculus does not apply to Theorem 4.11 because Φ is not necessarily differentiable.

5. Conclusions

In this article, we introduced integration theory of McShane integrals for operator-valued function. We extend Stieltjes type integrals of McShane integrals for operator-valued functions. Finally, we extend McShane-Stieltjes integrals via bilinear operators to execute an integration by parts theorem. Interested researcher can work on the validity of the fundamental theorem of calculus to the Theorem 4.11.

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