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CONVERGENCE TO A POINT OF COINCIDENCE FOR NONLINEAR MAPPINGS UNDER THE PRESENCE OF SUMMABLE ERRORS

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Abstract. In the present paper, we study an iterative process associated with a point of coincidence problem for two nonlinear mappings and show its convergence under the presence of summable computational errors.

1. Introduction

Since the seminal result of Banach [2] the fixed point theory of nonexpansive mappings has been a rapidly growing field of research. See [3, 10, 11, 13, 14, 19, 20, 23, 24, 25, 26, 27] and the reference mentioned therein. A significant progress has been done, in particular, in studies of common fixed point problems, which have important applications in engineering and medical sciences [8, 9, 12, 26, 27]. In [22] it was established the convergence of every inexact orbit of a strict contraction (that is, c-Lipschitz where $c \in (0, 1)$) mapping with summable errors. This result was generalized in [7] where it was shown that if any exact orbit of a nonexpansive mapping converges to its fixed point, then this convergence property also holds for its inexact orbits with summable errors. This result was obtained for a self-mapping of a complete metric space X. In the [28] the result of [7] was generalized for nonexpansive mappings which take a nonempty, closed subset K of the complete metric space X into X. In this paper we obtain an analog of these results for iterative processes associated with a point of coincidence problem for two nonlinear mappings and show their convergence under the presence of summable computational errors. The study of coincidence points of nonlinear mappings is an important topic of the fixed point theory [1, 4, 5, 16, 17, 18].

Assume that (X, ρ) is a complete metric space. For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \rho(x, y) \le r\}.$$

For each $x \in X$ and each set $A \subset X$ put

$$\rho(x, A) = \inf\{\rho(x, y) : \ y \in A\}.$$

For each mapping $S : X \to X$ set $S^0(x) = x$ for each $x \in X$ and for each integer $i \ge 0$ define $S^{i+1} = S \circ S^i$.

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In [7] it was studied the influence of errors on the convergence of orbits of nonexpansive mappings in metric spaces and it was obtained the following result (see also Theorem 2.72 of [24]).

Theorem 1.1. *Let $A : X \to X$ satisfy*

$$\rho(Ax, Ay) \le \rho(x, y) \text{ for all } x, y \in X,$$

let $F(A)$ be the set of all fixed points of A and let for each $x \in X$, the sequence $\{A^n x\}_{n=1}^{\infty}$ converges in (X, ρ).

Assume that $\{x_n\}_{n=0}^{\infty} \subset X$, $\{r_n\}_{n=0}^{\infty} \subset (0, \infty)$ satisfies

$$\sum_{n=0}^{\infty} r_n < \infty$$

and that

$$\rho(x_{n+1}, Ax_n) \le r_n,\ n = 0, 1, \ldots.$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of A in (X, ρ).

Theorem 1.1 found interesting applications and is an important ingredient in superiorization and perturbation resilience of algorithms. See [6, 8, 15, 21] and the references mentioned therein. The superiorization methodology works by taking an iterative algorithm, investigating its perturbation resilience, and then using proactively such perturbations in order to "force" the perturbed algorithm to do in addition to its original task something useful.

Assume that $S : X \to X$ and $T : X \to X$. If $x \in X$ and $S(x) = T(x)$, then the point x is called a coincidence point, while the point $y = T(x)$ is called a point of coincidence.

We associate with the coincidence point problem sequences

$$\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \subset X$$

satisfying

$$T(x_n) = y_n = S(x_{n+1})$$

for each integer $n \ge 0$. It is known that under certain assumptions the sequence $\{y_n\}_{n=0}^{\infty}$ converges to a point of coincidence. In the present paper we show that this property is stable under the presence of summable computational errors. It should be mentioned that the study of the equation above and the convergence of the sequence $\{y_n\}_{n=0}^{\infty}$ is now a growing area of research [1, 4, 5, 16, 17, 18]. Of course, our results are theoretical but they can find applications in superiorization and perturbation resilience of algorithms. See the discussion above.

2. The first result

Assume that (X, ρ) is a complete metric space, $S, T : X \to X$,

$$\rho(T(x), T(y)) \le \rho(S(x), S(y)) \text{ for each } x, y \in X, \tag{2.1}$$

$S(X)$ is closed and that

$$T(X) \subset S(X). \tag{2.2}$$

We assume that the following assumption holds.

(A) For each $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \subset X$ satisfying

$$T(x_n) = y_n = S(x_{n+1})$$

for each integer $n \geq 0$, there exists $\lim_{n\to\infty} y_n$.

Theorem 2.1. *Assume that* $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty} \subset X$, $\{r_i\}_{i=0}^{\infty} \subset (0, \infty)$,

$$\sum_{i=0}^{\infty} r_i < \infty$$

and that for each integer $i \geq 0$,

$$\rho(T(x_i), y_i) \leq r_i,\ \rho(S(x_{i+1}), y_i) \leq r_i. \tag{2.3}$$

Then there exist

$$y_* = \lim_{i\to\infty} y_i$$

and $x_* \in X$ *such that* $S(x_*) = y_* = T(x_*)$.

Proof. Let $\epsilon \in (0, 1)$. Clearly, there exists a natural number n_0 such that

$$\sum_{i=n_0}^{\infty} r_i < \epsilon/4. \tag{2.4}$$

There exist $\{\tilde{x}_n\}_{n=0}^{\infty}$, $\{\tilde{y}_n\}_{n=0}^{\infty} \subset X$ such that

$$\tilde{x}_{n_0} = x_{n_0}$$

and that for each integer $i \geq n_0$,

$$\tilde{y}_i = T(\tilde{x}_i),\ S(\tilde{x}_{i+1}) = \tilde{y}_i. \tag{2.5}$$

Assumption (A) implies that there exists $\lim_{i\to\infty} \tilde{y}_i$. Let $i \geq n_0$ be an integer. By (2.1), (2.3) and (2.5),

$$\begin{aligned}
\rho(\tilde{y}_{i+1}, y_{i+1}) &\leq S(y_{i+1}, T(x_{i+1})) + S(T(x_{i+1}), \tilde{y}_{i+1}) \\
&\leq r_{i+1} + \rho(T(x_{i+1}), T(\tilde{x}_{i+1})) \\
&\leq r_{i+1} + \rho(S(x_{i+1}), S(\tilde{x}_{i+1})) \\
&\leq r_{i+1} + \rho(S(x_{i+1}), y_i) + \rho(y_i, \tilde{y}_i) \\
&\leq r_i + r_{i+1} + \rho(y_i, \tilde{y}_i)
\end{aligned}$$

and

$$\rho(\tilde{y}_{i+1}, y_{i+1}) \leq \rho(y_i, \tilde{y}_i) + r_i. \tag{2.6}$$

In view of (2.5),

$$\rho(\tilde{y}_{n_0}, y_{n_0}) = \rho(T(x_{n_0}), y_{n_0}) \leq r_{n_0}. \tag{2.7}$$

By (2.4), (2.6) and (2.7),

$$\sum_{i=n_0}^{\infty} \rho(\tilde{y}_i, y_i) < 2 \sum_{i=n_0}^{\infty} r_i < \epsilon/2.$$

Since the sequence $\{\tilde{y}_i\}_{i=n_0}^{\infty}$ converges we conclude that for each pair of sufficiently large natural numbers i, j,

$$\rho(y_i, y_j) < \epsilon.$$

Since ϵ is any element from $(0, 1)$ $\{y_i\}_{i=0}^{\infty}$ is a Cauchy sequence and there exists

$$y_* = \lim_{n\to\infty} y_n. \tag{2.8}$$

By (2.3) and (2.8),

$$y_* = \lim_{n\to\infty} y_n = \lim_{n\to\infty} S(x_n).$$

There exists $x_* \in X$ such that

$$S(x_*) = y_*. \tag{2.9}$$

In view of (2.1), for each integer $n \geq 0$,

$$\rho(T(x_*), T(x_n)) \leq \rho(S(x_n), S(x_*)) \leq \rho(S(x_n), y_n) + \rho(y_n, y_*) \to 0$$

as $n \to \infty$. Thus

$$T(x_*) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} y_n = y_*$$

as $n \to \infty$. Theorem 2.1 is proved. □

It is not difficult to see that assumption (A) holds if $K = X$, $S : X \to X$, $c \in (0, 1)$, $Q : X \to X$ satisfies $\rho(Q(x), Q(y)) \leq c\rho(x, y)$, $x, y \in X$ and $T = Q \circ S$.

3. The second result

Assume that (X, ρ) is a complete metric space, K is a nonempty closed subset of X, $S, T : K \to X$,

$$\rho(T(x), T(y)) \leq \rho(S(x), S(y)) \text{ for each } x, y \in K, \tag{3.1}$$

$$K \cap T(K) \subset S(K). \tag{3.2}$$

Theorem 3.1. *Assume that the following assumption holds.*

(B) for each $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty} \subset K$ satisfying for each integer $i \geq 0$,

$$T(x_i) = y_i = S(x_{i+1})$$

there exists $\lim_{i\to\infty} y_i$.

Assume that $r \in (0, 1)$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty} \subset K$, $\{r_i\}_{i=0}^{\infty} \subset (0, \infty)$,

$$\sum_{i=0}^{\infty} r_i < \infty \tag{3.3}$$

and that for each integer $i \geq 0$,

$$\rho(T(x_i), y_i) \leq r_i, \ \rho(S(x_{i+1}), y_i) \leq r_i \tag{3.4}$$

and that for all sufficiently large natural numbers i,

$$B(y_i, r) \subset K. \tag{3.5}$$

Then there exists

$$y_* = \lim_{i\to\infty} y_i$$

and if $x_* \in X$ satisfies $S(x_*) = y_*$ (it exists if $S(K)$ is closed) then $T(x_*) = y_*$.

Proof. Let $\epsilon \in (0, r)$. By (3.3), there exists a natural number n_0 such that

$$\sum_{i=n_0}^{\infty} r_i < \epsilon/8, \tag{3.6}$$

$$B(y_i, r) \subset K \text{ for all integers } i \geq n_0. \tag{3.7}$$

Set

$$\tilde{x}_{n_0} = x_{n_0} \in K \tag{3.8}$$

and

$$\tilde{y}_{n_0} = T(\tilde{x}_{n_0}). \tag{3.9}$$

In view of (3.4), (3.8) and (3.9),

$$\rho(\tilde{y}_{n_0}, y_{n_0}) = \rho(\tilde{y}_{n_0}, T(x_{n_0})) \leq r_{n_0}. \tag{3.10}$$

By (3.6), (3.7) and (3.10),

$$\tilde{y}_{n_0} \in B(y_{n_0}, r_{n_0}) \subset K.$$

Assume that $n \geq n_0$ is an integer, (3.8), (3.9) hold, $\tilde{x}_i, \tilde{y}_i \in K$, $i = n_0, \dots, n$, for each integer $i \in \{n_0, \dots, n\}$,

$$\tilde{y}_i = T(\tilde{x}_i), \tag{3.11}$$

for each integer $i \in \{n_0, \dots, n\} \setminus \{n\}$,

$$S(\tilde{x}_{i+1}) = \tilde{y}_i, \tag{3.12}$$

$$\rho(y_n, \tilde{y}_n) \leq \sum \{2r_i : \ i \in \{n_0, \dots, n\} \setminus \{n\}\} + r_n. \tag{3.13}$$

(Note that in view of (3.8)-(3.10) our assumption holds for $n = n_0$.) By (3.6) and (3.13),

$$\rho(y_n, \tilde{y}_n) < r, \ \tilde{y}_n \in K. \tag{3.14}$$

Equations (3.2) and (3.11) imply that there exists

$$\tilde{x}_{n+1} \in K$$

such that

$$S(\tilde{x}_{n+1}) = \tilde{y}_n. \tag{3.15}$$

Set

$$\tilde{y}_{n+1} = T(\tilde{x}_{n+1}). \tag{3.16}$$

It follows from (3.1), (3.4), (3.13) and (3.16) that

$$\begin{aligned}
\rho(\tilde{y}_{n+1}, y_{n+1}) &\leq \rho(T(\tilde{x}_{n+1}), T(x_{n+1})) + \rho(y_{n+1}, T(x_{n+1})) \\
&\leq r_{n+1} + \rho(S(\tilde{x}_{n+1}), S(x_{n+1})) \\
&\leq r_{n+1} + \rho(\tilde{y}_n, y_n) + \rho(y_n, S(x_{n+1})) \\
&\leq r_{n+1} + r_n + \rho(y_n, \tilde{y}_n) \\
&\leq \sum_{i=n_0}^{n} 2r_i + r_{n+1}. \qquad (3.17)
\end{aligned}$$

In view of (3.6) and (3.17),

$$\tilde{y}_{n+1} \in B(y_{n+1}, r) \subset K.$$

Thus the assumption made for n also holds for $n+1$. Therefore by induction we constructed $\{\tilde{x}_n\}_{n=0}^{\infty}$, $\{\tilde{y}_n\}_{n=0}^{\infty} \subset K$ such that (3.8), (3.9) hold and that for each integer $n \geq n_0$,

$$\tilde{y}_n = T(\tilde{x}_n),\ S(\tilde{x}_{n+1}) = \tilde{y}_n,$$

$$\rho(y_n, \tilde{y}_n) \leq 2\sum_{i=n_0}^{n} r_i.$$

Assumption (B) and the relations above imply that for each $n \geq n_0$,

$$\rho(y_n, \tilde{y}_n) < \epsilon/4$$

and the sequence $\{\tilde{y}_i\}_{i=n_0}^{\infty}$ converges. This implies that for each pair of sufficiently large natural numbers i, j,

$$\rho(y_i, y_j) < \epsilon.$$

Since ϵ is any element from $(0, 1)$ $\{y_i\}_{i=0}^{\infty}$ is a Cauchy sequence and there exists

$$y_* = \lim_{n\to\infty} y_n \in K. \tag{3.18}$$

By (3.4) and (3.18),

$$y_* = \lim_{n\to\infty} S(x_n). \tag{3.19}$$

Assume that there exists $x_* \in K$ such that

$$S(x_*) = y_*. \tag{3.20}$$

In view of (3.1), (3.4), (3.19) and (3.20), for each integer $n \geq 0$,

$$\rho(T(x_*), T(x_n)) \leq \rho(S(x_n), S(x_*)) \leq \rho(S(x_n), y_*) \leq \rho(S(x_n), y_{n+1}) + \rho(y_{n+1}, y_*) \to 0$$

as $n \to \infty$. Thus

$$T(x_*) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} y_n = y_*.$$

Theorem 3.1 is proved. □

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