Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167

Volume 10, Number 2, 2024, 89–109

SERIES OF SETS AND FUZZY SETS

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Abstract. In this study, we propose series of sets and fuzzy sets and investigate their properties. We derive the properties of addition, scalar multiplication, and orderings for these sets. The relationships between our definitions and existing definitions are also investigated.

1. Introduction

Kurano et al. [7] considered an infinite-horizon Markov decision process (MDP) as a maximization problem of the discounted total reward that is a series of compact convex sets, which represent rewards, defined by the limit of a sequence with respect to the Pompeiu-Hausdorff distance. Here, we propose another series of sets that are not necessarily compact convex and investigate their properties. We then prove that our definition is equivalent to the definition with respect to the Pompeiu-Hausdorff distance under some conditions.

Carrero-Vera et al. [1], Cruz-Suárez et al. [2], Kurano et al. [8], and Semmouri et al. [11] considered an infinite-horizon MDP as a maximization problem of the discounted total reward that is a series of fuzzy numbers, which represent rewards, defined by the limit of a sequence with respect to the distance between these numbers, which is defined using the Pompeiu-Hausdorff distances between the level sets of these numbers. Kurano et al. [9] considered an infinite-horizon MDP as a maximization problem of the discounted total reward that is a series of compact convex fuzzy sets, which represent rewards, defined by the limit of a sequence with respect to the distance between these sets, which is defined using the Pompeiu-Hausdorff distances between the level sets of the compact convex fuzzy sets. Here, we propose another series of fuzzy sets that are not necessarily compact convex and investigate their properties.

Stojaković et al. [12, 13] proposed a series of fuzzy sets based on Zadeh's extension principle and investigated their properties. While their definition differs from ours, we prove that the two definitions are equivalent. Stojaković et al. [12, 13] mainly derived the properties of the level sets of a series. Here, we derive other properties of a series of fuzzy sets.

It is expected that the properties derived in this study will be useful for an infinite-horizon MDP or dynamic programming as a maximization problem of the

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2020 *Mathematics Subject Classification.* 03E72.

Key words and phrases. Series of sets, series of fuzzy sets, (fuzzy) convergence assumption, (fuzzy) absolute convergence assumption, Painlevé-Kuratowski convergence, Pompeiu-Hausdorff convergence.

discounted total reward in which the rewards are given as compact sets or compact fuzzy sets and are not necessarily convex.

The remainder of this article is organized as follows. Section 2 presents definitions of operations and orderings for sets. Section 3 presents definitions of the limit of a sequence of sets. Section 4 describes the proposed series of sets and their properties. Section 5 presents definitions of operations and orderings for fuzzy sets. Section 6 presents some properties for a generator of a fuzzy set. Section 7 presents definitions of the limit of a sequence of fuzzy sets. Section 8 describes the proposed series of fuzzy sets and their properties. Finally, Section 9 gives the conclusions.

2. Operations and orderings for sets

For $a, b \in \mathbb{R}$, we set $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $]a,b] = \{x \in \mathbb{R} : a < x \leq b\}$, $[a,b[= \{x \in \mathbb{R} : a \leq x < b\}$, and $]a,b[= \{x \in \mathbb{R} : a < x < b\}$. In addition, we set $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq \boldsymbol{0}\}$ and $\mathbb{R}^n_- = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \leq \boldsymbol{0}\}$. For $S \subset \mathbb{R}^n$, $\mathrm{int}(S)$ and $\mathrm{cl}(S)$ denote the interior and closure of S, respectively. Its characteristic function $c_S : \mathbb{R}^n \to \{0,1\}$ is defined as

$$c_S(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in S, \\ 0 & \text{if } \boldsymbol{x} \notin S \end{cases}$$

for each $\boldsymbol{x} \in \mathbb{R}^n$. Let $\|\cdot\|, \|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}_+$ be Euclidean and Chebyshev norms on $\mathbb{R}^n$, respectively, and $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the canonical inner product on $\mathbb{R}^n$. For $S \subset \mathbb{R}^n$, define

$$|S| = \sup_{\boldsymbol{x} \in S} \|\boldsymbol{x}\| \tag{2.1}$$

where $\sup \emptyset = 0$. Let $\mathcal{P}(\mathbb{R}^n)$ be the set of all subsets of $\mathbb{R}^n$ and $\mathcal{C}(\mathbb{R}^n)$ be the set of all nonempty compact subsets of $\mathbb{R}^n$. We define addition and scalar multiplication on $\mathcal{P}(\mathbb{R}^n)$ as follows:

$$A + B = \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in A, \boldsymbol{y} \in B\}, \tag{2.2}$$

$$\mu A = \{\mu \boldsymbol{x} : \boldsymbol{x} \in A\} \tag{2.3}$$

for $A, B \in \mathcal{P}(\mathbb{R}^n)$ and $\mu \in \mathbb{R}$. For $A, B \in \mathcal{P}(\mathbb{R}^n)$, we define their orderings as follows:

$$A \leq B \overset{\text{def}}{\Leftrightarrow} B \subset A + \mathbb{R}^n_+, A \subset B + \mathbb{R}^n_-, \tag{2.4}$$

$$A < B \overset{\text{def}}{\Leftrightarrow} B \subset A + \mathrm{int}(\mathbb{R}^n_+), A \subset B + \mathrm{int}(\mathbb{R}^n_-). \tag{2.5}$$

Thus, $\leq$ is a preorder relation on $\mathcal{P}(\mathbb{R}^n)$ (a binary relation on $\mathcal{P}(\mathbb{R}^n)$ that is reflexive and transitive) and $<$ is a strict partial order relation on $\mathcal{C}(\mathbb{R}^n)$ (a binary relation on $\mathcal{C}(\mathbb{R}^n)$ that is irreflexive and transitive). See [5, Proposition 3.3] or [6, Theorem 6.6] regarding the irreflexivity of the strict partial order relation $<$ on $\mathcal{C}(\mathbb{R}^n)$.

For $A, B, C, D \in \mathcal{P}(\mathbb{R}^n)$, it holds that (see [5, Proposition 3.4] or [6, Theorem 6.7])

$$A \leq B, C \leq D \Rightarrow A + C \leq B + D, \tag{2.6}$$

$$A \leq B, C < D \Rightarrow A + C < B + D. \tag{2.7}$$

3. Limit of sequence of sets

Let $\mathbb{N}$ be the set of all natural numbers. Then, we set

$$\begin{aligned}
\mathcal{N}_\infty &= \{N \subset \mathbb{N} : \mathbb{N} \setminus N \text{ finite}\} \\
&= \{ \text{ subsequences of } \mathbb{N} \text{ containing all } k \text{ beyond some } k_0 \}, \\
\mathcal{N}_\infty^\sharp &= \{N \subset \mathbb{N} : N \text{ infinite}\} \\
&= \{ \text{ all subsequences of } \mathbb{N}\}.
\end{aligned}$$

A subsequence of a sequence $\{x_k\}_{k\in\mathbb{N}}$ is represented as $\{x_k\}_{k\in N}$ for some $N \in \mathcal{N}_\infty^\sharp$. We write $\lim_k, \lim_{k\to\infty}$, or $\lim_{k\in\mathbb{N}}$ when $k \to \infty$ in $\mathbb{N}$, but $\lim_{k\in N}$ or $\lim_{k \underset{N}{\to} \infty}$ in the case of the convergence of $\{x_k\}_{k\in N}$ for some $N \in \mathcal{N}_\infty^\sharp$ or $N \in \mathcal{N}_\infty$.

Definition 3.1. ([10, Definition 4.1]) For a sequence $\{C_k\}_{k\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$, its lower limit is defined as the set

$$\liminf_{k\to\infty} C_k = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \exists N \in \mathcal{N}_\infty, \exists \boldsymbol{x}_k \in C_k (k \in N) \text{ with } \boldsymbol{x}_k \underset{N}{\to} \boldsymbol{x} \right\},$$

and its upper limit is defined as the set

$$\limsup_{k\to\infty} C_k = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \exists N \in \mathcal{N}_\infty^\sharp, \exists \boldsymbol{x}_k \in C_k (k \in N) \text{ with } \boldsymbol{x}_k \underset{N}{\to} \boldsymbol{x} \right\}.$$

The limit of $\{C_k\}_{k\in\mathbb{N}}$ is said to exist if $\liminf_{k\to\infty} C_k = \limsup_{k\to\infty} C_k$. Then, the limit is defined as the set

$$\lim_{k\to\infty} C_k = \limsup_{k\to\infty} C_k = \liminf_{k\to\infty} C_k.$$

When $C = \lim_k C_k$ exists in the sense of Definition 3.1, it is said that the sequence $\{C_k\}_{k\in\mathbb{N}}$ converges to C as *Painlevé-Kuratowski convergence*, and we write

$$C_k \to C.$$

The distance between a point $\boldsymbol{x} \in \mathbb{R}^n$ and a set $C \subset \mathbb{R}^n$ is defined as

$$d_C(\boldsymbol{x}) = \inf_{\boldsymbol{y}\in C} \|\boldsymbol{x} - \boldsymbol{y}\|.$$

We also write this distance as $d(\boldsymbol{x}, C)$, where $d_C(\boldsymbol{x}) = \infty$ for $C = \emptyset$.

For nonempty closed sets $C, D \subset \mathbb{R}^n$, the *Pompeiu-Hausdorff distance* between C and D is defined as

$$\boldsymbol{d}_\infty(C, D) = \sup_{\boldsymbol{x}\in\mathbb{R}^n} |d_C(\boldsymbol{x}) - d_D(\boldsymbol{x})|.$$

The supremum could equally be taken just over $C \cup D$, yielding the alternative formula

$$\boldsymbol{d}_\infty(C, D) = \inf\{\eta \geq 0 : C \subset D + \eta\mathbb{B}, D \subset C + \eta\mathbb{B}\}$$

where $\mathbb{B} = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leq 1\}$.

A sequence $\{C_k\}_{k\in\mathbb{N}}$ of nonempty closed subsets of $\mathbb{R}^n$ is said to converge to a nonempty closed set $C \subset \mathbb{R}^n$ with respect to the Pompeiu-Hausdorff distance when $\boldsymbol{d}_\infty(C_k, C) \to 0$.

When $C_k, C \subset X$ for some bounded set $X \subset \mathbb{R}^n$, Pompeiu-Hausdorff convergence and Painlevé-Kuratowski convergence are equivalent, where these sets are nonempty and closed. However, they are not equivalent without this boundedness restriction. For example, see Rockafellar [10] for details.

4. Series of sets

Let $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n) \setminus \{\emptyset\}$. When

$$\sum_{t=1}^{\infty} \boldsymbol{x}_t = \lim_{m\to\infty} \sum_{t=1}^{m} \boldsymbol{x}_t$$

converges for any $\boldsymbol{x}_t \in S_t, t \in \mathbb{N}$, it is said that $\{S_t\}$ satisfies the *convergence assumption (CA)*. Then, we define a series of sets as follows:

$$\sum_{t=1}^{\infty} S_t = \left\{ \sum_{t=1}^{\infty} \boldsymbol{x}_t : \boldsymbol{x}_t \in S_t, t \in \mathbb{N} \right\}. \tag{4.1}$$

When

$$\sum_{t=1}^{\infty} |S_t| < \infty, \tag{4.2}$$

it is said that $\{S_t\}$ satisfies the *absolute convergence assumption (ACA)*, where $|S_t|$ is defined in (2.1). For example, if $\bigcup_{t=1}^{\infty} S_t$ is bounded, then $\{\gamma^{t-1}S_t\}_{t\in\mathbb{N}}$ satisfies ACA for any $\gamma \in [0, 1[$. If $\{S_t\}$ satisfies ACA, then it satisfies CA. However, the converse does not hold in general. The following example demonstrates that the converse does not hold.

Example 4.1. (i) If we set $S_1 = \mathbb{R}^n \in \mathcal{P}(\mathbb{R}^n) \setminus \{\emptyset\}$ and $S_t = \{\boldsymbol{0}\} \in \mathcal{P}(\mathbb{R}^n) \setminus \{\emptyset\}$ for each $t \geq 2$, then $\{S_t\}$ satisfies CA, but not ACA. In this case, $\sum_{t=1}^{\infty} S_t = \mathbb{R}^n$.

(ii) If we set $S_t = \left\{ \frac{(-1)^{t-1}}{t} \right\} \in \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ for each $t \in \mathbb{N}$, then $\{S_t\}$ satisfies CA, but not ACA. In this case, $\sum_{t=1}^{\infty} S_t = \{\log 2\}$.

Lemma 4.2. *Let $\boldsymbol{k} \in \mathbb{R}^n_+ \setminus \{\boldsymbol{0}\}$ and $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{S}_{\boldsymbol{k}}(\mathbb{R}^n)$, where*

$$\mathcal{S}_{\boldsymbol{k}}(\mathbb{R}^n) = \left\{ A \in \mathcal{P}(\mathbb{R}^n) : \max_{\boldsymbol{x}\in A} \langle \boldsymbol{k}, \boldsymbol{x} \rangle \text{ and } \min_{\boldsymbol{x}\in A} \langle \boldsymbol{k}, \boldsymbol{x} \rangle \text{ exist} \right\}.$$

If $\{S_t\}_{t\in\mathbb{N}}$ satisfies CA, then

$$\sum_{t=1}^{\infty} S_t = \left\{ \sum_{t=1}^{\infty} \boldsymbol{x}_t : \boldsymbol{x}_t \in S_t, t \in \mathbb{N} \right\} \in \mathcal{S}_{\boldsymbol{k}}(\mathbb{R}^n).$$

Proof. For each $t \in \mathbb{N}$, there exist $\boldsymbol{x}_t^1, \boldsymbol{x}_t^2 \in S_t$ such that $\langle \boldsymbol{k}, \boldsymbol{x}_t^1 \rangle = \max_{\boldsymbol{x}\in S_t} \langle \boldsymbol{k}, \boldsymbol{x} \rangle$ and $\langle \boldsymbol{k}, \boldsymbol{x}_t^2 \rangle = \min_{\boldsymbol{x}\in S_t} \langle \boldsymbol{k}, \boldsymbol{x} \rangle$. Then, $\sum_{t=1}^{\infty} \boldsymbol{x}_t^1, \sum_{t=1}^{\infty} \boldsymbol{x}_t^2 \in \sum_{t=1}^{\infty} S_t$. Fix any $\boldsymbol{x} \in \sum_{t=1}^{\infty} S_t$. There exist $\boldsymbol{x}_t \in S_t, t \in \mathbb{N}$ such that $\boldsymbol{x} = \sum_{t=1}^{\infty} \boldsymbol{x}_t$. Because

$$\left\langle \boldsymbol{k}, \sum_{t=1}^{m} \boldsymbol{x}_t^1 \right\rangle = \sum_{t=1}^{m} \langle \boldsymbol{k}, \boldsymbol{x}_t^1 \rangle = \sum_{t=1}^{m} \max_{\boldsymbol{x}\in S_t} \langle \boldsymbol{k}, \boldsymbol{x} \rangle \geq \sum_{t=1}^{m} \langle \boldsymbol{k}, \boldsymbol{x}_t \rangle = \left\langle \boldsymbol{k}, \sum_{t=1}^{m} \boldsymbol{x}_t \right\rangle,$$

$$\left\langle \boldsymbol{k}, \sum_{t=1}^{m} \boldsymbol{x}_t^2 \right\rangle = \sum_{t=1}^{m} \langle \boldsymbol{k}, \boldsymbol{x}_t^2 \rangle = \sum_{t=1}^{m} \min_{\boldsymbol{x}\in S_t} \langle \boldsymbol{k}, \boldsymbol{x} \rangle \leq \sum_{t=1}^{m} \langle \boldsymbol{k}, \boldsymbol{x}_t \rangle = \left\langle \boldsymbol{k}, \sum_{t=1}^{m} \boldsymbol{x}_t \right\rangle$$

for any $m \in \mathbb{N}$, it follows that

$$\left\langle \boldsymbol{k}, \sum_{t=1}^{\infty} \boldsymbol{x}_t^1 \right\rangle \geq \left\langle \boldsymbol{k}, \sum_{t=1}^{\infty} \boldsymbol{x}_t \right\rangle = \langle \boldsymbol{k}, \boldsymbol{x} \rangle, \quad \left\langle \boldsymbol{k}, \sum_{t=1}^{\infty} \boldsymbol{x}_t^2 \right\rangle \leq \left\langle \boldsymbol{k}, \sum_{t=1}^{\infty} \boldsymbol{x}_t \right\rangle = \langle \boldsymbol{k}, \boldsymbol{x} \rangle$$

as $m \to \infty$ by the continuity of the inner product. Therefore, we have

$$\left\langle \boldsymbol{k}, \sum_{t=1}^{\infty} \boldsymbol{x}_t^1 \right\rangle = \max_{\boldsymbol{x}\in\sum_{t=1}^{\infty} S_t} \langle \boldsymbol{k}, \boldsymbol{x} \rangle, \quad \left\langle \boldsymbol{k}, \sum_{t=1}^{\infty} \boldsymbol{x}_t^2 \right\rangle = \min_{\boldsymbol{x}\in\sum_{t=1}^{\infty} S_t} \langle \boldsymbol{k}, \boldsymbol{x} \rangle.$$

by the arbitrariness of $\boldsymbol{x} \in \sum_{t=1}^{\infty} S_t$. □

The following proposition presents the properties of addition, scalar multiplication, and orderings for a series of sets.

Proposition 4.3. *Let $\{S_t\}_{t\in\mathbb{N}}, \{T_t\}_{t\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n) \setminus \{\emptyset\}$ satisfy CA and let $\beta \in \mathbb{R}$. The following statements then hold.*

(i) $\{S_t + T_t\}$ satisfies CA and

$$\sum_{t=1}^{\infty} (S_t + T_t) = \sum_{t=1}^{\infty} S_t + \sum_{t=1}^{\infty} T_t.$$

(ii) $\{\beta S_t\}$ satisfies CA and

$$\sum_{t=1}^{\infty} \beta S_t = \beta \sum_{t=1}^{\infty} S_t.$$

(iii) If $S_t \leq T_t$ for any $t \in \mathbb{N}$, then

$$\sum_{t=1}^{\infty} S_t \leq \sum_{t=1}^{\infty} T_t.$$

(iv) If $S_t \leq T_t$ for any $t \in \mathbb{N}$ and if there exists $m \in \mathbb{N}$ such that $S_m < T_m$, then

$$\sum_{t=1}^{\infty} S_t < \sum_{t=1}^{\infty} T_t.$$

Proof. (i) First, fix any $\boldsymbol{x}_t \in S_t + T_t$ $(t \in \mathbb{N})$. For each $t \in \mathbb{N}$, there exist $\boldsymbol{y}_t \in S_t$ and $\boldsymbol{z}_t \in T_t$ such that $\boldsymbol{x}_t = \boldsymbol{y}_t + \boldsymbol{z}_t$. Because $\{S_t\}$ and $\{T_t\}$ satisfy CA, it follows that

$$\sum_{t=1}^{\infty} \boldsymbol{x}_t = \sum_{t=1}^{\infty} (\boldsymbol{y}_t + \boldsymbol{z}_t) = \sum_{t=1}^{\infty} \boldsymbol{y}_t + \sum_{t=1}^{\infty} \boldsymbol{z}_t.$$

Therefore, $\{S_t + T_t\}$ satisfies CA by the arbitrariness of $\boldsymbol{x}_t \in S_t + T_t$ $(t \in \mathbb{N})$.

Next, let $\boldsymbol{x} \in \sum_{t=1}^{\infty} (S_t + T_t)$. There exist $\boldsymbol{y}_t \in S_t, \boldsymbol{z}_t \in T_t$ $(t \in \mathbb{N})$ such that $\boldsymbol{x} = \sum_{t=1}^{\infty} (\boldsymbol{y}_t + \boldsymbol{z}_t)$. It then follows that

$$\boldsymbol{x} = \sum_{t=1}^{\infty} (\boldsymbol{y}_t + \boldsymbol{z}_t) = \sum_{t=1}^{\infty} \boldsymbol{y}_t + \sum_{t=1}^{\infty} \boldsymbol{z}_t \in \sum_{t=1}^{\infty} S_t + \sum_{t=1}^{\infty} T_t.$$

Therefore, we have $\sum_{t=1}^{\infty} (S_t + T_t) \subset \sum_{t=1}^{\infty} S_t + \sum_{t=1}^{\infty} T_t$.

Next, let $\boldsymbol{x} \in \sum_{t=1}^{\infty} S_t + \sum_{t=1}^{\infty} T_t$. There exist $\boldsymbol{y} \in \sum_{t=1}^{\infty} S_t$ and $\boldsymbol{z} \in \sum_{t=1}^{\infty} T_t$ such that $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$, and there exist $\boldsymbol{y}_t \in S_t, \boldsymbol{z}_t \in T_t$ $(t \in \mathbb{N})$ such that $\boldsymbol{y} = \sum_{t=1}^{\infty} \boldsymbol{y}_t$ and $\boldsymbol{z} = \sum_{t=1}^{\infty} \boldsymbol{z}_t$. Because

$$\boldsymbol{y}_t + \boldsymbol{z}_t \in S_t + T_t \quad (t \in \mathbb{N}),$$

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z} = \sum_{t=1}^{\infty} \boldsymbol{y}_t + \sum_{t=1}^{\infty} \boldsymbol{z}_t = \sum_{t=1}^{\infty} (\boldsymbol{y}_t + \boldsymbol{z}_t) \in \sum_{t=1}^{\infty} (S_t + T_t),$$

we have $\sum_{t=1}^{\infty} (S_t + T_t) \supset \sum_{t=1}^{\infty} S_t + \sum_{t=1}^{\infty} T_t$.

(ii) First, fix any $\boldsymbol{x}_t \in \beta S_t$ $(t \in \mathbb{N})$. For each $t \in \mathbb{N}$, there exists $\boldsymbol{y}_t \in S_t$ such that $\boldsymbol{x}_t = \beta \boldsymbol{y}_t$. Because $\{S_t\}$ satisfies CA, it follows that

$$\sum_{t=1}^{\infty} \boldsymbol{x}_t = \sum_{t=1}^{\infty} \beta \boldsymbol{y}_t = \beta \sum_{t=1}^{\infty} \boldsymbol{y}_t.$$

Therefore, $\{\beta S_t\}$ satisfies CA by the arbitrariness of $\boldsymbol{x}_t \in \beta S_t$ $(t \in \mathbb{N})$.

Next, let $\boldsymbol{x} \in \sum_{t=1}^{\infty} \beta S_t$. There exist $\boldsymbol{y}_t \in S_t$ $(t \in \mathbb{N})$ such that $\boldsymbol{x} = \sum_{t=1}^{\infty} \beta \boldsymbol{y}_t$. It then follows that

$$\boldsymbol{x} = \sum_{t=1}^{\infty} \beta \boldsymbol{y}_t = \beta \sum_{t=1}^{\infty} \boldsymbol{y}_t \in \beta \sum_{t=1}^{\infty} S_t.$$

Therefore, we have $\sum_{t=1}^{\infty} \beta S_t \subset \beta \sum_{t=1}^{\infty} S_t$.

Next, let $\boldsymbol{x} \in \beta \sum_{t=1}^{\infty} S_t$. There exists $\boldsymbol{y} \in \sum_{t=1}^{\infty} S_t$ such that $\boldsymbol{x} = \beta \boldsymbol{y}$, and there exist $\boldsymbol{y}_t \in S_t$ $(t \in \mathbb{N})$ such that $\boldsymbol{y} = \sum_{t=1}^{\infty} \boldsymbol{y}_t$. Because

$$\beta \boldsymbol{y}_t \in \beta S_t, \quad t \in \mathbb{N},$$

$$\boldsymbol{x} = \beta \boldsymbol{y} = \beta \sum_{t=1}^{\infty} \boldsymbol{y}_t = \sum_{t=1}^{\infty} \beta \boldsymbol{y}_t \in \sum_{t=1}^{\infty} \beta S_t,$$

we have $\sum_{t=1}^{\infty} \beta S_t \supset \beta \sum_{t=1}^{\infty} S_t$.

(iii) In order to show that

$$\sum_{t=1}^{\infty} T_t \subset \sum_{t=1}^{\infty} S_t + \mathbb{R}^n_+, \quad \sum_{t=1}^{\infty} S_t \subset \sum_{t=1}^{\infty} T_t + \mathbb{R}^n_-,$$

we show that

$$\boldsymbol{y} \in \sum_{t=1}^{\infty} T_t \Rightarrow \exists \boldsymbol{x} \in \sum_{t=1}^{\infty} S_t \text{ s.t. } \boldsymbol{x} \leq \boldsymbol{y},$$

$$\boldsymbol{x} \in \sum_{t=1}^{\infty} S_t \Rightarrow \exists \boldsymbol{y} \in \sum_{t=1}^{\infty} T_t \text{ s.t. } \boldsymbol{x} \leq \boldsymbol{y}.$$

First, let $\boldsymbol{y} \in \sum_{t=1}^{\infty} T_t$. There exist $\boldsymbol{y}_t \in T_t$ $(t \in \mathbb{N})$ such that $\boldsymbol{y} = \sum_{t=1}^{\infty} \boldsymbol{y}_t$. For each $t \in \mathbb{N}$, because $T_t \subset S_t + \mathbb{R}^n_+$, there exists $\boldsymbol{x}_t \in S_t$ such that $\boldsymbol{x}_t \leq \boldsymbol{y}_t$. If we put $\boldsymbol{x} = \sum_{t=1}^{\infty} \boldsymbol{x}_t \in \sum_{t=1}^{\infty} S_t$, then we have $\boldsymbol{x} = \sum_{t=1}^{\infty} \boldsymbol{x}_t \leq \sum_{t=1}^{\infty} \boldsymbol{y}_t = \boldsymbol{y}$.

Next, let $\boldsymbol{x} \in \sum_{t=1}^{\infty} S_t$. There exist $\boldsymbol{x}_t \in S_t$ $(t \in \mathbb{N})$ such that $\boldsymbol{x} = \sum_{t=1}^{\infty} \boldsymbol{x}_t$. For each $t \in \mathbb{N}$, because $S_t \subset T_t + \mathbb{R}^n_-$, there exists $\boldsymbol{y}_t \in T_t$ such that $\boldsymbol{x}_t \leq \boldsymbol{y}_t$. If we put $\boldsymbol{y} = \sum_{t=1}^{\infty} \boldsymbol{y}_t \in \sum_{t=1}^{\infty} T_t$, then we have $\boldsymbol{x} = \sum_{t=1}^{\infty} \boldsymbol{x}_t \leq \sum_{t=1}^{\infty} \boldsymbol{y}_t = \boldsymbol{y}$.

(iv) It is easy to see that

$$\sum_{t=1}^{\infty} S_t = \sum_{t=1}^{m} S_t + \sum_{t=m+1}^{\infty} S_t, \quad \sum_{t=1}^{\infty} T_t = \sum_{t=1}^{m} T_t + \sum_{t=m+1}^{\infty} T_t.$$

Because $S_t \leq T_t (t \in \mathbb{N})$ and $S_m < T_m$ from the assumption, it follows that

$$\sum_{t=1}^{m} S_t < \sum_{t=1}^{m} T_t, \quad \sum_{t=m+1}^{\infty} S_t \leq \sum_{t=m+1}^{\infty} T_t$$

from (2.6), (2.7), and (iii) of this proposition. Therefore, we have

$$\sum_{t=1}^{\infty} S_t = \sum_{t=1}^{m} S_t + \sum_{t=m+1}^{\infty} S_t < \sum_{t=1}^{m} T_t + \sum_{t=m+1}^{\infty} T_t = \sum_{t=1}^{\infty} T_t$$

from (2.7). □

For a sequence of sets in a series, the following proposition shows that the sequence approaches the origin.

Proposition 4.4. *Let $\{S_t\}_{t \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n) \setminus \{\emptyset\}$. If $\{S_t\}_{t \in \mathbb{N}}$ satisfies CA, then for any $\varepsilon > 0$, there exists $t_0 \in \mathbb{N}$ such that $S_t \subset \mathbb{B}_\varepsilon$ for any $t \geq t_0$, where $\mathbb{B}_\varepsilon = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_\infty < \varepsilon\}$.*

Proof. Suppose that for some $\varepsilon_0 > 0$ and any $t \in \mathbb{N}$, there exists $k_t \geq t$ such that $S_{k_t} \not\subset \mathbb{B}_{\varepsilon_0}$. We derive the following contradiction. From the above assumption, there exists $N_0 \in \mathcal{N}^\sharp_\infty$ such that $S_t \not\subset \mathbb{B}_{\varepsilon_0}$ for any $t \in N_0$. Fix any $\boldsymbol{x}_t = (x_{t1}, x_{t2}, \ldots, x_{tn}) \in S_t \setminus \mathbb{B}_{\varepsilon_0}$ for each $t \in N_0$, and fix any $\boldsymbol{x}_t = (x_{t1}, x_{t2}, \ldots, x_{tn}) \in S_t$ for each $t \in \mathbb{N} \setminus N_0$.

For each $t \in N_0$, because $\boldsymbol{x}_t = (x_{t1}, x_{t2}, \ldots, x_{tn}) \notin \mathbb{B}_{\varepsilon_0}$, there exists $i_t \in \{1, 2, \ldots, n\}$ such that $|x_{ti_t}| \geq \varepsilon_0$. Thus, there exist $i_0 \in \{1, 2, \ldots, n\}$ and $N_1 \in \mathcal{N}^\sharp_\infty, N_1 \subset N_0$ such that $|x_{ti_0}| \geq \varepsilon_0$ for any $t \in N_1$. It then follows that $x_{ti_0} \not\to 0$ as $t \to \infty$. Therefore, because $\sum_{t=1}^{\infty} x_{ti_0}$ does not converge, $\sum_{t=1}^{\infty} \boldsymbol{x}_t$ does not converge. This contradicts that $\{S_t\}$ satisfies CA. □

For a series of sets, the following proposition shows that the sum of the latter part of the series approaches the origin.

Proposition 4.5. *Let* $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n) \setminus \{\emptyset\}$. *If* $\{S_t\}_{t\in\mathbb{N}}$ *satisfies ACA, then for any* $\varepsilon > 0$, *there exists* $m_0 \in \mathbb{N}$ *such that* $\sum_{t=m}^{\infty} S_t \subset \mathbb{B}_\varepsilon$ *for any* $m \geq m_0$, *where* $\mathbb{B}_\varepsilon = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| < \varepsilon\}$.

Proof. First, fix any $\varepsilon > 0$. Because $\sum_{t=1}^{\infty} |S_t| < \infty$, there exists $m_0 \in \mathbb{N}$ such that $\sum_{t=m}^{\infty} |S_t| < \varepsilon$ for any $m \geq m_0$.

Next, fix any $m \geq m_0$ and any $\boldsymbol{x} \in \sum_{t=m}^{\infty} S_t$. There exist $\boldsymbol{x}_t \in S_t, t \geq m$ such that $\boldsymbol{x} = \sum_{t=m}^{\infty} \boldsymbol{x}_t$. It then follows that

$$\|\boldsymbol{x}\| \leq \sum_{t=m}^{\infty} \|\boldsymbol{x}_t\| \leq \sum_{t=m}^{\infty} |S_t| < \varepsilon.$$

Thus, $\boldsymbol{x} \in \mathbb{B}_\varepsilon$. Therefore, we have $\sum_{t=m}^{\infty} S_t \subset \mathbb{B}_\varepsilon$ for any $m \geq m_0$. □

Proposition 4.6. *Let* $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n) \setminus \{\emptyset\}$. *If* $\{S_t\}_{t\in\mathbb{N}}$ *satisfies CA, then* $\{cl(S_t)\}_{t\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n) \setminus \{\emptyset\}$ *also satisfies CA.*

Proof. Fix any $\varepsilon > 0$ and any $\overline{\boldsymbol{x}}_t = (\overline{x}_{t1}, \overline{x}_{t2}, \ldots, \overline{x}_{tn}) \in \mathrm{cl}(S_t)$ $(t \in \mathbb{N})$. In addition, fix any $\gamma \in]0, 1[$ with $\frac{\gamma}{1-\gamma} < \frac{\varepsilon}{2}$.

For each $t \in \mathbb{N}$, because $\overline{\boldsymbol{x}}_t = (\overline{x}_{t1}, \overline{x}_{t2}, \ldots, \overline{x}_{tn}) \in \mathrm{cl}(S_t)$, there exists $\boldsymbol{x}_t = (x_{t1}, x_{t2}, \ldots, x_{tn}) \in S_t$ such that $|x_{ti} - \overline{x}_{ti}| < \gamma^t$ $(i = 1, 2, \ldots, n)$. Because $\sum_{t=1}^{\infty} x_{ti}$ $(i = 1, 2, \ldots, n)$ converge, $\{\sum_{t=1}^{m} x_{ti}\}_{m\in\mathbb{N}}$ $(i = 1, 2, \ldots, n)$ are Cauchy sequences. There exists $t_0 \in \mathbb{N}$ such that $\left|\sum_{t=s+1}^{u} x_{ti}\right| < \frac{\varepsilon}{2}$ $(i = 1, 2, \ldots, n)$ for any $u > s \geq t_0$.

For each $t \in \mathbb{N}$, because $x_{ti} - \gamma^t < \overline{x}_{ti} < x_{ti} + \gamma^t$ $(i = 1, 2, \ldots, n)$, it follows that

$$\begin{aligned} -\varepsilon &< \sum_{t=s+1}^{u} x_{ti} - \frac{\gamma}{1-\gamma} \\ &< \sum_{t=s+1}^{u} x_{ti} - \sum_{t=s+1}^{u} \gamma^t \\ &< \sum_{t=s+1}^{u} \overline{x}_{ti} \\ &< \sum_{t=s+1}^{u} x_{ti} + \sum_{t=s+1}^{u} \gamma^t \\ &< \sum_{t=s+1}^{u} x_{ti} + \frac{\gamma}{1-\gamma} \\ &< \varepsilon \quad (i = 1, 2, \ldots, n) \end{aligned}$$

for any $u > s \geq t_0$ and that $\left|\sum_{t=s+1}^{u} \overline{x}_{ti}\right| < \varepsilon$ $(i = 1, 2, \ldots, n)$. By the arbitrariness of $\varepsilon > 0$, $\{\sum_{t=1}^{m} \overline{x}_{ti}\}_{m\in\mathbb{N}}$ $(i = 1, 2, \ldots, n)$ are Cauchy sequences. By the completeness of $\mathbb{R}$, $\sum_{t=1}^{\infty} \overline{x}_{ti}$ $(i = 1, 2, \ldots, n)$ converge. Consequently, $\sum_{t=1}^{\infty} \overline{\boldsymbol{x}}_t$ converges. Therefore, $\{\mathrm{cl}(S_t)\}$ satisfies CA. □

Proposition 4.7. *Let $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n)\setminus\{\emptyset\}$. If $\{S_t\}_{t\in\mathbb{N}}$ satisfies CA and $S_t (t \in \mathbb{N})$ are bounded, then $\displaystyle\sum_{t=1}^{\infty} S_t$ is nonempty and bounded.*

Proof. From the assumption, we have $\sum_{t=1}^{\infty} S_t \neq \emptyset$. We show that $\sum_{t=1}^{\infty} \mathrm{cl}(S_t) \supset \sum_{t=1}^{\infty} S_t$ is bounded. From Proposition 4.6, $\{\mathrm{cl}(S_t)\}_{t\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}^n)$ satisfies CA. For each $i \in \{1, 2, \ldots, n\}$, we put $\boldsymbol{e}_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{in}) \in \mathbb{R}^n_+ \setminus \{\boldsymbol{0}\}$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

From Lemma 4.2, $\min_{\boldsymbol{x}'\in\sum_{t=1}^{\infty}\mathrm{cl}(S_t)}\langle \boldsymbol{e}_i, \boldsymbol{x}'\rangle$ and $\max_{\boldsymbol{x}'\in\sum_{t=1}^{\infty}\mathrm{cl}(S_t)}\langle \boldsymbol{e}_i, \boldsymbol{x}'\rangle$ exist for each $i \in \{1, 2, \ldots, n\}$. Because

$$\min_{\boldsymbol{x}'\in\sum_{t=1}^{\infty}\mathrm{cl}(S_t)}\langle \boldsymbol{e}_i, \boldsymbol{x}'\rangle \leq \langle \boldsymbol{e}_i, \boldsymbol{x}\rangle = x_i \leq \max_{\boldsymbol{x}'\in\sum_{t=1}^{\infty}\mathrm{cl}(S_t)}\langle \boldsymbol{e}_i, \boldsymbol{x}'\rangle \quad (i = 1, 2, \ldots, n)$$

for any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \sum_{t=1}^{\infty}\mathrm{cl}(S_t)$, $\sum_{t=1}^{\infty}\mathrm{cl}(S_t)$ is bounded. Therefore, $\sum_{t=1}^{\infty} S_t \subset \sum_{t=1}^{\infty}\mathrm{cl}(S_t)$ is bounded. □

Proposition 4.8. *Let $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}^n)$. If $\{S_t\}_{t\in\mathbb{N}}$ satisfies ACA, then $\displaystyle\sum_{t=1}^{\infty} S_t \in \mathcal{C}(\mathbb{R}^n)$.*

Proof. From Proposition 4.7, $\sum_{t=1}^{\infty} S_t$ is nonempty and bounded. We show that $\sum_{t=1}^{\infty} S_t$ is closed. Fix any convergent sequence $\{\boldsymbol{y}_k\}_{k\in\mathbb{N}} \subset \sum_{t=1}^{\infty} S_t$ and let $\boldsymbol{y}_k \to \boldsymbol{y}_0 \in \mathbb{R}^n$. We show that $\boldsymbol{y}_0 \in \sum_{t=1}^{\infty} S_t$. For each $k \in \mathbb{N}$, there exist $\boldsymbol{x}_{kt} \in S_t$ $(t \in \mathbb{N})$ such that $\boldsymbol{y}_k = \sum_{t=1}^{\infty} \boldsymbol{x}_{kt}$. Because $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}^n)$, the following statements hold.

- There exists $N_1 \in \mathcal{N}^\sharp_\infty$ such that $\boldsymbol{x}_{k1} \underset{k\in N_1}{\to} \boldsymbol{x}_{01}$ for some $\boldsymbol{x}_{01} \in S_1$.
- There exists $N_2 \in \mathcal{N}^\sharp_\infty$ with $N_2 \subset N_1$ such that $\boldsymbol{x}_{k2} \underset{k\in N_2}{\to} \boldsymbol{x}_{02}$ for some $\boldsymbol{x}_{02} \in S_2$.

$$\vdots$$

- There exists $N_t \in \mathcal{N}^\sharp_\infty$ with $N_t \subset N_{t-1}$ such that $\boldsymbol{x}_{kt} \underset{k\in N_t}{\to} \boldsymbol{x}_{0t}$ for some $\boldsymbol{x}_{0t} \in S_t$.

$$\vdots$$

Put $\overline{\boldsymbol{y}}_0 = \sum_{t=1}^{\infty} \boldsymbol{x}_{0t} \in \sum_{t=1}^{\infty} S_t$. Now, fix any $\varepsilon > 0$. From Proposition 4.5, there exists $m_0 \in \mathbb{N}$ such that $\left\|\sum_{t=m_0+1}^{\infty} \boldsymbol{x}_{kt}\right\| < \frac{\varepsilon}{4}$ $(k \in \mathbb{N} \cup \{0\})$. Because $\sum_{t=1}^{m_0} \boldsymbol{x}_{kt} \to \sum_{t=1}^{m_0} \boldsymbol{x}_{0t}$ and $\boldsymbol{y}_k \to \boldsymbol{y}_0$ as $k \to \infty$, there exists $k_0 \in \mathbb{N}$ such that $\|\sum_{t=1}^{m_0} \boldsymbol{x}_{k_0t} - \sum_{t=1}^{m_0} \boldsymbol{x}_{0t}\| < \frac{\varepsilon}{4}$ and $\|\boldsymbol{y}_{k_0} - \boldsymbol{y}_0\| < \frac{\varepsilon}{4}$. For the above m_0 and k_0, it follows that

$$\|\overline{\boldsymbol{y}}_0 - \boldsymbol{y}_0\| = \left\|\overline{\boldsymbol{y}}_0 - \sum_{t=1}^{m_0} \boldsymbol{x}_{0t} + \sum_{t=1}^{m_0} \boldsymbol{x}_{0t} - \sum_{t=1}^{m_0} \boldsymbol{x}_{k_0t}\right.$$

$$
\begin{aligned}
&\quad + \sum_{t=1}^{m_0} \boldsymbol{x}_{k_0 t} - \boldsymbol{y}_{k_0} + \boldsymbol{y}_{k_0} - \boldsymbol{y}_0 \Bigg\| \\
\leq\; & \left\| \overline{\boldsymbol{y}}_0 - \sum_{t=1}^{m_0} \boldsymbol{x}_{0t} \right\| + \left\| \sum_{t=1}^{m_0} \boldsymbol{x}_{0t} - \sum_{t=1}^{m_0} \boldsymbol{x}_{k_0 t} \right\| \\
&+ \left\| \sum_{t=1}^{m_0} \boldsymbol{x}_{k_0 t} - \boldsymbol{y}_{k_0} \right\| + \| \boldsymbol{y}_{k_0} - \boldsymbol{y}_0 \| \\
=\; & \left\| \sum_{t=m_0+1}^{\infty} \boldsymbol{x}_{0t} \right\| + \left\| \sum_{t=1}^{m_0} \boldsymbol{x}_{0t} - \sum_{t=1}^{m_0} \boldsymbol{x}_{k_0 t} \right\| \\
&+ \left\| \sum_{t=1}^{m_0} \boldsymbol{x}_{k_0 t} - \sum_{t=1}^{\infty} \boldsymbol{x}_{k_0 t} \right\| + \| \boldsymbol{y}_{k_0} - \boldsymbol{y}_0 \| \\
=\; & \left\| \sum_{t=m_0+1}^{\infty} \boldsymbol{x}_{0t} \right\| + \left\| \sum_{t=1}^{m_0} \boldsymbol{x}_{0t} - \sum_{t=1}^{m_0} \boldsymbol{x}_{k_0 t} \right\| \\
&+ \left\| \sum_{t=m_0+1}^{\infty} \boldsymbol{x}_{k_0 t} \right\| + \| \boldsymbol{y}_{k_0} - \boldsymbol{y}_0 \| \\
<\; & \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}
$$

By the arbitrariness of $\varepsilon > 0$, we have $\boldsymbol{y}_0 = \overline{\boldsymbol{y}}_0 \in \sum_{t=1}^{\infty} S_t$. □

The following proposition shows that our series of sets is equivalent to the limit of partial sums of a series as Painlevé-Kuratowski convergence under some conditions.

Proposition 4.9. *Let* $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}^n)$*. If* $\{S_t\}_{t\in\mathbb{N}}$ *satisfies ACA, then*

$$\sum_{t=1}^{\infty} S_t = \lim_{m\to\infty} \sum_{t=1}^{m} S_t$$

where the limit on the right-hand side means Painlevé-Kuratowski convergence.

Proof. First, let $\boldsymbol{y} \in \sum_{t=1}^{\infty} S_t$. There exist $\boldsymbol{x}_t \in S_t$ $(t \in \mathbb{N})$ such that $\boldsymbol{y} = \sum_{t=1}^{\infty} \boldsymbol{x}_t$. Because $\sum_{t=1}^{m} \boldsymbol{x}_t \in \sum_{t=1}^{m} S_t$ $(m \in \mathbb{N})$ and $\sum_{t=1}^{m} \boldsymbol{x}_t \to \sum_{t=1}^{\infty} \boldsymbol{x}_t = \boldsymbol{y}$, it follows that $\boldsymbol{y} \in \liminf_{m\to\infty} \sum_{t=1}^{m} S_t$. Therefore, we have $\sum_{t=1}^{\infty} S_t \subset \liminf_{m\to\infty} \sum_{t=1}^{m} S_t$.

Next, let $\boldsymbol{y} \in \limsup_{m\to\infty} \sum_{t=1}^{m} S_t$. There exist $N \in \mathcal{N}_\infty^\sharp$ and $\boldsymbol{y}_m \in \sum_{t=1}^{m} S_t$ $(m \in N)$ such that $\boldsymbol{y}_m \underset{m\in N}{\to} \boldsymbol{y}$. Set $N = \{m_1, m_2, \ldots\}$ with $m_1 < m_2 < \cdots$. For each $k \in \mathbb{N}$, there exist $\boldsymbol{x}_t^{m_k} \in S_t, t = 1, 2, \ldots, m_k$ such that $\boldsymbol{y}_{m_k} = \sum_{t=1}^{m_k} \boldsymbol{x}_t^{m_k}$. For each $k \in \mathbb{N}$, fix any $\boldsymbol{x}_t^{m_k} \in S_t$ $(t \geq m_k + 1)$, and set $\overline{\boldsymbol{y}}_k = \sum_{t=1}^{\infty} \boldsymbol{x}_t^{m_k} \in \sum_{t=1}^{\infty} S_t$. Because $\sum_{t=1}^{\infty} S_t \in \mathcal{C}(\mathbb{R}^n)$ from Proposition 4.8, suppose that $\overline{\boldsymbol{y}}_k \to \overline{\boldsymbol{y}}_0$ for some $\overline{\boldsymbol{y}}_0 \in \sum_{t=1}^{\infty} S_t$ without loss of generality. If this is not so, we can reset $N = \{m_{k_1}, m_{k_2}, \ldots\}$ for a convergent subsequence $\overline{\boldsymbol{y}}_{k_1}, \overline{\boldsymbol{y}}_{k_2}, \ldots$ of $\{\overline{\boldsymbol{y}}_k\}$. Now, fix any $\varepsilon > 0$. Because $\overline{\boldsymbol{y}}_k \to \overline{\boldsymbol{y}}_0$, there exists $k_1 \in \mathbb{N}$ such that $\|\overline{\boldsymbol{y}}_k - \overline{\boldsymbol{y}}_0\| < \frac{\varepsilon}{2}$ $(k \geq k_1)$. From Proposition 4.5, there exists $m_0 \in \mathbb{N}$ such that $\left\|\sum_{t=m+1}^{\infty} \boldsymbol{x}_t^{m_k}\right\| < \frac{\varepsilon}{2}$ $(k \in \mathbb{N},\, m \geq m_0)$. Thus, there exists

$k_2 \in \mathbb{N}$ such that

$$\|\boldsymbol{y}_{m_k} - \overline{\boldsymbol{y}}_k\| = \left\| \sum_{t=1}^{m_k} \boldsymbol{x}_t^{m_k} - \sum_{t=1}^{\infty} \boldsymbol{x}_t^{m_k} \right\| = \left\| \sum_{t=m_k+1}^{\infty} \boldsymbol{x}_t^{m_k} \right\| < \frac{\varepsilon}{2} \quad (k \geq k_2).$$

Put $k_0 = \max\{k_1, k_2\}$. It then follows that

$$\begin{aligned} \|\boldsymbol{y}_{m_k} - \overline{\boldsymbol{y}}_0\| &= \|\boldsymbol{y}_{m_k} - \overline{\boldsymbol{y}}_k + \overline{\boldsymbol{y}}_k - \overline{\boldsymbol{y}}_0\| \\ &\leq \|\boldsymbol{y}_{m_k} - \overline{\boldsymbol{y}}_k\| + \|\overline{\boldsymbol{y}}_k - \overline{\boldsymbol{y}}_0\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (k \geq k_0). \end{aligned}$$

Thus, it follows that $\boldsymbol{y}_{m_k} \to \overline{\boldsymbol{y}}_0$. By the uniqueness of the limit, it follows that $\boldsymbol{y} = \overline{\boldsymbol{y}}_0 \in \sum_{t=1}^{\infty} S_t$. Therefore, we have $\limsup_{m\to\infty} \sum_{t=1}^{m} S_t \subset \sum_{t=1}^{\infty} S_t$.

Because

$$\sum_{t=1}^{\infty} S_t \subset \liminf_{m\to\infty} \sum_{t=1}^{m} S_t \subset \limsup_{m\to\infty} \sum_{t=1}^{m} S_t \subset \sum_{t=1}^{\infty} S_t,$$

we have

$$\sum_{t=1}^{\infty} S_t = \liminf_{m\to\infty} \sum_{t=1}^{m} S_t = \limsup_{m\to\infty} \sum_{t=1}^{m} S_t = \lim_{m\to\infty} \sum_{t=1}^{m} S_t.$$

□

Example 4.10. Set $A = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_\infty < 1\}$ and let $\gamma \in [0,1[$. If we set $S_t = \gamma^{t-1} A \in \mathcal{P}(\mathbb{R}^n) \setminus \{\emptyset\}$ for each $t \in \mathbb{N}$, then $\{S_t\}$ satisfies ACA. However, the assumption of Proposition 4.9 is not satisfied because $\{S_t\} \not\subset \mathcal{C}(\mathbb{R}^n)$. In this case, $\sum_{t=1}^{\infty} S_t = \left\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_\infty < \frac{1}{1-\gamma}\right\}$ and $\lim_{m\to\infty} \sum_{t=1}^{m} S_t = \left\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_\infty \leq \frac{1}{1-\gamma}\right\}$, where the limit means Painlevé-Kuratowski convergence. Therefore, we have $\sum_{t=1}^{\infty} S_t \neq \lim_{m\to\infty} \sum_{t=1}^{m} S_t$.

From Proposition 4.9 and the statements about the Pompeiu-Hausdorff distance in the last part of Section 3, the following proposition can be obtained.

Proposition 4.11. *Let $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}^n)$. If $\{S_t\}_{t\in\mathbb{N}}$ satisfies ACA, then $\sum_{t=1}^{\infty} S_t$ is both the limit as Painlevé-Kuratowski convergence and the limit with respect to the Pompeiu-Hausdorff distance.*

5. Operations and orderings for fuzzy sets

For notational convenience, we identify a fuzzy set $\widetilde{a}$ on $\mathbb{R}^n$ with its membership function $\widetilde{a} : \mathbb{R}^n \to [0,1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all fuzzy sets on $\mathbb{R}^n$.

Let $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$. For $\alpha \in]0,1]$, $[\widetilde{a}]_\alpha = \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{a}(\boldsymbol{x}) \geq \alpha\}$ is called the *α-level set* of $\widetilde{a}$. The set $\mathrm{supp}(\widetilde{a}) = \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{a}(\boldsymbol{x}) > 0\}$ is called the *support* of $\widetilde{a}$. The fuzzy set $\widetilde{a}$ is said to be *normal* if $[\widetilde{a}]_1 \neq \emptyset$. The fuzzy set $\widetilde{a}$ is said to be *compact* if $[\widetilde{a}]_\alpha$ is compact for any $\alpha \in]0,1]$. Let $\mathcal{FN}(\mathbb{R}^n)$ be the set of all normal fuzzy sets on $\mathbb{R}^n$ and let $\mathcal{FC}(\mathbb{R}^n)$ be the set of all normal compact fuzzy sets on $\mathbb{R}^n$.

Addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ by Zadeh's extension principle are defined as follows (see [4, 14]). For $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$ and $\mu \in \mathbb{R}$, $\widetilde{a} + \widetilde{b}, \mu\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ are defined as

$$(\widetilde{a} + \widetilde{b})(\boldsymbol{x}) = \sup_{\boldsymbol{x}=\boldsymbol{y}+\boldsymbol{z}} \min\{\widetilde{a}(\boldsymbol{y}), \widetilde{b}(\boldsymbol{z})\}, \tag{5.1}$$

$$(\mu\widetilde{a})(\boldsymbol{x}) = \sup_{\boldsymbol{x}=\mu\boldsymbol{y}} \widetilde{a}(\boldsymbol{y}) \tag{5.2}$$

for each $\boldsymbol{x} \in \mathbb{R}^n$. For $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$, we define their orderings as follows:

$$\widetilde{a} \leq \widetilde{b} \overset{\text{def}}{\Leftrightarrow} [\widetilde{a}]_\alpha \leq [\widetilde{b}]_\alpha \text{ for any } \alpha \in \,]0,1]\,, \tag{5.3}$$

$$\widetilde{a} < \widetilde{b} \overset{\text{def}}{\Leftrightarrow} [\widetilde{a}]_\alpha < [\widetilde{b}]_\alpha \text{ for any } \alpha \in \,]0,1]\,. \tag{5.4}$$

Thus, $\leq$ is a preorder relation on $\mathcal{F}(\mathbb{R}^n)$ (a binary relation on $\mathcal{F}(\mathbb{R}^n)$ that is reflexive and transitive) and $<$ is a strict partial order relation on $\mathcal{FC}(\mathbb{R}^n)$ (a binary relation on $\mathcal{FC}(\mathbb{R}^n)$ that is irreflexive and transitive). See [5, Proposition 5.2] or [6, Theorem 8.9] regarding the irreflexivity of the strict partial order relation $<$ on $\mathcal{FC}(\mathbb{R}^n)$.

For finitely many $\widetilde{a}, \widetilde{b}, \ldots, \widetilde{c} \in \mathcal{FC}(\mathbb{R}^n)$, $\beta, \gamma, \ldots, \delta \in \mathbb{R}$ and $\mu \in \mathbb{R}$, $\alpha \in \,]0,1]$, it holds that ([5, Propositions 4.1 and 4.4] or [6, Theorems 8.1 and 8.5])

$$\mu(\widetilde{a} + \widetilde{b} + \cdots + \widetilde{c}) = \mu\widetilde{a} + \mu\widetilde{b} + \cdots + \mu\widetilde{c}, \tag{5.5}$$

$$[\beta\widetilde{a} + \gamma\widetilde{b} + \cdots + \delta\widetilde{c}]_\alpha = \beta[\widetilde{a}]_\alpha + \gamma[\widetilde{b}]_\alpha + \cdots + \delta[\widetilde{c}]_\alpha. \tag{5.6}$$

6. Generator of fuzzy set

Set

$$\mathcal{Q}(\mathbb{R}^n) = \{\{S_\alpha\}_{\alpha\in]0,1]} : S_\alpha \subset \mathbb{R}^n, \alpha \in \,]0,1]\}, \tag{6.1}$$

$$\begin{aligned} \mathcal{S}(\mathbb{R}^n) = \big\{\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{Q}(\mathbb{R}^n) :& \\ \beta, \gamma \in \,]0,1]\,, \beta \leq \gamma \text{ imply } S_\beta \supset S_\gamma&\big\}. \end{aligned} \tag{6.2}$$

We then define $M : \mathcal{Q}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ as

$$M\left(\{S_\alpha\}_{\alpha\in]0,1]}\right) = \sup_{\alpha\in]0,1]} \alpha c_{S_\alpha} \tag{6.3}$$

for each $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{Q}(\mathbb{R}^n)$. For $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{Q}(\mathbb{R}^n)$ and $\boldsymbol{x} \in \mathbb{R}^n$, it can be easily seen that

$$M\left(\{S_\alpha\}_{\alpha\in]0,1]}\right)(\boldsymbol{x}) = \sup_{\alpha\in]0,1]} \alpha c_{S_\alpha}(\boldsymbol{x}) = \sup\{\alpha \in \,]0,1] : \boldsymbol{x} \in S_\alpha\} \tag{6.4}$$

where $\sup \emptyset = 0$.

When $\widetilde{a} = M(\{S_\alpha\}_{\alpha\in]0,1]})$ for $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ and $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{Q}(\mathbb{R}^n)$, $\widetilde{a}$ is called the fuzzy set generated by $\{S_\alpha\}_{\alpha\in]0,1]}$ and $\{S_\alpha\}_{\alpha\in]0,1]}$ is called a *generator* of $\widetilde{a}$. Notably, the generator of $\widetilde{a}$ is not unique in general.

A fuzzy set $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ can be represented as

$$\widetilde{a} = M\left(\{[\widetilde{a}]_\alpha\}_{\alpha\in]0,1]}\right) = \sup_{\alpha\in]0,1]} \alpha c_{[\widetilde{a}]_\alpha}. \tag{6.5}$$

This is known as the *decomposition* or *representation theorem* (see, for example, [4] or [6, Theorem 7.1]).

Let $\{S_\alpha\}_{\alpha\in]0,1]}, \{T_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$ and let $\widetilde{a} = M(\{S_\alpha\}_{\alpha\in]0,1]}) \in \mathcal{F}(\mathbb{R}^n)$, $\widetilde{b} = M(\{T_\alpha\}_{\alpha\in]0,1]}) \in \mathcal{F}(\mathbb{R}^n)$. In addition, let $\mu \in \mathbb{R}$. It then holds that ([5, Proposition 4.2] or [6, Theorem 8.2])

$$\widetilde{a} + \widetilde{b} = M\left(\{S_\alpha + T_\alpha\}_{\alpha\in]0,1]}\right) = \sup_{\alpha\in]0,1]} \alpha c_{S_\alpha+T_\alpha}, \tag{6.6}$$

$$\mu\widetilde{a} = M\left(\{\mu S_\alpha\}_{\alpha\in]0,1]}\right) = \sup_{\alpha\in]0,1]} \alpha c_{\mu S_\alpha}. \tag{6.7}$$

7. Limit of sequence of fuzzy sets

The following definition is a fuzzified version of Definition 3.1.

Definition 7.1. Let $\{\widetilde{s}_k\}_{k\in\mathbb{N}} \subset \mathcal{F}(\mathbb{R}^n)$ and

$$L_\alpha = \liminf_{k\to\infty} [\widetilde{s}_k]_\alpha, \quad U_\alpha = \limsup_{k\to\infty} [\widetilde{s}_k]_\alpha$$

for each $\alpha \in]0,1]$. The lower limit of $\{\widetilde{s}_k\}_{k\in\mathbb{N}}$ is defined as the fuzzy set

$$\liminf_{k\to\infty} \widetilde{s}_k = \sup_{\alpha\in]0,1]} \alpha c_{L_\alpha} = M(\{L_\alpha\}_{\alpha\in]0,1]}),$$

and the upper limit of $\{\widetilde{s}_k\}_{k\in\mathbb{N}}$ is defined as the fuzzy set

$$\limsup_{k\to\infty} \widetilde{s}_k = \sup_{\alpha\in]0,1]} \alpha c_{U_\alpha} = M(\{U_\alpha\}_{\alpha\in]0,1]}).$$

The limit of $\{\widetilde{s}_k\}_{k\in\mathbb{N}}$ is said to exist if $\liminf_{k\to\infty} \widetilde{s}_k = \limsup_{k\to\infty} \widetilde{s}_k$. Then, the limit is defined as the fuzzy set

$$\lim_{k\to\infty} \widetilde{s}_k = \liminf_{k\to\infty} \widetilde{s}_k = \limsup_{k\to\infty} \widetilde{s}_k.$$

For sets $S_k \subset \mathbb{R}^n (k \in \mathbb{N})$, put $L = \liminf_{k\to\infty} S_k, U = \limsup_{k\to\infty} S_k$, and put $T = \lim_{k\to\infty} S_k$ if the limit of $\{S_k\}_{k\in\mathbb{N}}$ exists. It can be seen that $\liminf_{k\to\infty} c_{S_k} = c_L, \limsup_{k\to\infty} c_{S_k} = c_U$, and that $\lim_{k\to\infty} c_{S_k} = c_T$ if the limit of $\{S_k\}_{k\in\mathbb{N}}$ exists. Thus, the lower limit, upper limit, and limit of a sequence of fuzzy sets are generalizations of those of a sequence of sets.

8. Series of fuzzy sets

Let $\{\widetilde{a}_t\}_{t\in\mathbb{N}} \subset \mathcal{FN}(\mathbb{R}^n)$. When $\{[\widetilde{a}_t]_\alpha\}_{t\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n) \setminus \{\emptyset\}$ satisfies CA for any $\alpha \in \,]0,1]$, it is said that $\{\widetilde{a}_t\}$ satisfies the *fuzzy convergence assumption (FCA)*. Then, we define a series of fuzzy sets as follows:

$$\sum_{t=1}^{\infty} \widetilde{a}_t = M\left(\left\{\sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha\right\}_{\alpha\in]0,1]}\right). \tag{8.1}$$

When

$$\sum_{t=1}^{\infty} |[\widetilde{a}_t]_\alpha| < \infty \text{ for any } \alpha \in \,]0,1]\,, \tag{8.2}$$

it is said that $\{\widetilde{a}_t\}$ satisfies the *fuzzy absolute convergence assumption (FACA)*, where $|[\widetilde{a}_t]_\alpha|$ is defined in (2.1). If $\{\widetilde{a}_t\}$ satisfies FACA, then it satisfies FCA.

Let $\{\widetilde{a}_t\}_{t\in\mathbb{N}} \subset \mathcal{FN}(\mathbb{R}^n)$. For $\alpha, \beta \in \,]0,1]$ with $\alpha < \beta$, if $\{[\widetilde{a}_t]_\alpha\}$ satisfies CA, then $\{[\widetilde{a}_t]_\beta\}$ also satisfies CA. When $\{\mathrm{supp}(\widetilde{a}_t)\}_{t\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n) \setminus \{\emptyset\}$ satisfies CA, it is said that $\{\widetilde{a}_t\}$ satisfies the *fuzzy support convergence assumption (FSCA)*. If $\{\widetilde{a}_t\}$ satisfies FSCA, then it satisfies FCA. However, the converse does not hold in general. The following example demonstrates that the converse does not hold.

Example 8.1. Let $\widetilde{a} \in \mathcal{FC}(\mathbb{R})$ with $\widetilde{a}(x) = e^{-|x|}$ for each $x \in \mathbb{R}$ and let $\gamma \in \,]0,1[$. Set $\widetilde{a}_t = \gamma^{t-1}\widetilde{a}$ $(t \in \mathbb{N})$. For each $\alpha \in \,]0,1]$, it follows that

$$[\widetilde{a}]_\alpha = [\log\alpha, -\log\alpha],$$

$$[\widetilde{a}_t]_\alpha = [\gamma^{t-1}\widetilde{a}]_\alpha = \gamma^{t-1}[\widetilde{a}]_\alpha = [\gamma^{t-1}\log\alpha, -\gamma^{t-1}\log\alpha] \quad (t \in \mathbb{N}).$$

Thus, $\{\widetilde{a}_t\}_{t\in\mathbb{N}} \subset \mathcal{FC}(\mathbb{R})$ satisfies FACA. Therefore, $\{\widetilde{a}_t\}$ satisfies FCA, but not FSCA because $\mathrm{supp}(\widetilde{a}_t) = \mathbb{R}$ for any $t \in \mathbb{N}$. In this case, because

$$\begin{aligned}
\sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha &= \sum_{t=1}^{\infty}[\gamma^{t-1}\log\alpha, -\gamma^{t-1}\log\alpha] \\
&= \left[\frac{1}{1-\gamma}\log\alpha, -\frac{1}{1-\gamma}\log\alpha\right] \\
&= \frac{1}{1-\gamma}[\log\alpha, -\log\alpha] \\
&= \frac{1}{1-\gamma}[\widetilde{a}]_\alpha \\
&= \left[\frac{1}{1-\gamma}\widetilde{a}\right]_\alpha
\end{aligned}$$

for each $\alpha \in \,]0,1]$, we have

$$\sum_{t=1}^{\infty} \widetilde{a}_t = M\left(\left\{\sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha\right\}_{\alpha\in]0,1]}\right)$$

$$
\begin{aligned}
&= M\left(\left\{\left[\frac{1}{1-\gamma}\widetilde{a}\right]_\alpha\right\}_{\alpha\in]0,1]}\right) \\
&= \frac{1}{1-\gamma}\widetilde{a} \\
&\in \mathcal{FC}(\mathbb{R})
\end{aligned}
$$

from the decomposition theorem (6.5).

The following proposition presents the properties of addition and scalar multiplication for a series of fuzzy sets.

Proposition 8.2. *Let $\{\widetilde{a}_t\}_{t\in\mathbb{N}}, \{\widetilde{b}_t\}_{t\in\mathbb{N}} \subset \mathcal{FC}(\mathbb{R}^n)$ satisfy FCA and let $\beta \in \mathbb{R}$.*

(i) $\{\widetilde{a}_t + \widetilde{b}_t\}$ satisfies FCA and

$$\sum_{t=1}^{\infty}(\widetilde{a}_t + \widetilde{b}_t) = \sum_{t=1}^{\infty}\widetilde{a}_t + \sum_{t=1}^{\infty}\widetilde{b}_t.$$

(ii) $\{\beta\widetilde{a}_t\}$ satisfies FCA and

$$\sum_{t=1}^{\infty}\beta\widetilde{a}_t = \beta\sum_{t=1}^{\infty}\widetilde{a}_t.$$

Proof. (i) Fix any $\alpha \in]0,1]$. Because $\{[\widetilde{a}_t]_\alpha\}$ and $\{[\widetilde{b}_t]_\alpha\}$ satisfy CA, $\{[\widetilde{a}_t + \widetilde{b}_t]_\alpha\} = \{[\widetilde{a}_t]_\alpha + [\widetilde{b}_t]_\alpha\}$ also satisfies CA from Proposition 4.3 (i). Therefore, $\{\widetilde{a}_t + \widetilde{b}_t\}$ satisfies FCA by the arbitrariness of $\alpha \in]0,1]$.

Fix any $\alpha \in]0,1]$ again. Because $\{[\widetilde{a}_t]_\alpha\}$ and $\{[\widetilde{b}_t]_\alpha\}$ satisfy CA, it follows that

$$\sum_{t=1}^{\infty}[\widetilde{a}_t + \widetilde{b}_t]_\alpha = \sum_{t=1}^{\infty}([\widetilde{a}_t]_\alpha + [\widetilde{b}_t]_\alpha) = \sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha + \sum_{t=1}^{\infty}[\widetilde{b}_t]_\alpha$$

from Proposition 4.3 (i). From the decomposition theorem (6.5) and (6.6),

$$
\begin{aligned}
\sum_{t=1}^{\infty}(\widetilde{a}_t + \widetilde{b}_t) &= M\left(\left\{\sum_{t=1}^{\infty}[\widetilde{a}_t + \widetilde{b}_t]_\alpha\right\}_{\alpha\in]0,1]}\right) \\
&= M\left(\left\{\sum_{t=1}^{\infty}([\widetilde{a}_t]_\alpha + [\widetilde{b}_t]_\alpha)\right\}_{\alpha\in]0,1]}\right) \\
&= M\left(\left\{\sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha + \sum_{t=1}^{\infty}[\widetilde{b}_t]_\alpha\right\}_{\alpha\in]0,1]}\right) \\
&= M\left(\left\{\sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha\right\}_{\alpha\in]0,1]}\right) + M\left(\left\{\sum_{t=1}^{\infty}[\widetilde{b}_t]_\alpha\right\}_{\alpha\in]0,1]}\right) \\
&= \sum_{t=1}^{\infty}\widetilde{a}_t + \sum_{t=1}^{\infty}\widetilde{b}_t.
\end{aligned}
$$

(ii) Fix any $\alpha \in\,]0,1]$. Because $\{[\widetilde{a}_t]_\alpha\}$ satisfies CA, $\{[\beta\widetilde{a}_t]_\alpha\} = \{\beta[\widetilde{a}_t]_\alpha\}$ also satisfies CA from Proposition 4.3 (ii). Therefore, $\{\beta\widetilde{a}_t\}$ satisfies FCA by the arbitrariness of $\alpha \in\,]0,1]$.

Fix any $\alpha \in\,]0,1]$ again. Because $\{[\widetilde{a}_t]_\alpha\}$ satisfies CA, it follows that

$$\sum_{t=1}^{\infty}[\beta\widetilde{a}_t]_\alpha = \sum_{t=1}^{\infty}\beta[\widetilde{a}_t]_\alpha = \beta\sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha$$

from Proposition 4.3 (ii). From the decomposition theorem (6.5) and (6.7),

$$\begin{aligned}\sum_{t=1}^{\infty}\beta\widetilde{a}_t &= M\left(\left\{\sum_{t=1}^{\infty}[\beta\widetilde{a}_t]_\alpha\right\}_{\alpha\in]0,1]}\right)\\ &= M\left(\left\{\sum_{t=1}^{\infty}\beta[\widetilde{a}_t]_\alpha\right\}_{\alpha\in]0,1]}\right)\\ &= M\left(\left\{\beta\sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha\right\}_{\alpha\in]0,1]}\right)\\ &= \beta M\left(\left\{\sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha\right\}_{\alpha\in]0,1]}\right)\\ &= \beta\sum_{t=1}^{\infty}\widetilde{a}_t.\end{aligned}$$

□

The following proposition shows that our series for fuzzy sets is equivalent to that obtained using Zadeh's extension principle.

Proposition 8.3. *Let* $\{\widetilde{a}_t\}_{t\in\mathbb{N}} \subset \mathcal{FN}(\mathbb{R}^n)$. *If* $\{\widetilde{a}_t\}_{t\in\mathbb{N}}$ *satisfies FCA, then*

$$\left(\sum_{t=1}^{\infty}\widetilde{a}_t\right)(\boldsymbol{x}) = \sup\left\{\inf_{t\in\mathbb{N}}\widetilde{a}_t(\boldsymbol{x}_t) : \sum_{t=1}^{\infty}\boldsymbol{x}_t = \boldsymbol{x}\right\} \tag{8.3}$$

for any $\boldsymbol{x} \in \mathbb{R}^n$.

Proof. Fix any $\boldsymbol{x} \in \mathbb{R}^n$ and set

$$\begin{aligned}f(\boldsymbol{x}) &= \left(\sum_{t=1}^{\infty}\widetilde{a}_t\right)(\boldsymbol{x})\\ &= M\left(\left\{\sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha\right\}_{\alpha\in]0,1]}\right)(\boldsymbol{x})\\ &= \sup\left\{\alpha \in\,]0,1] : \boldsymbol{x} \in \sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha\right\},\end{aligned}$$

$$g(\boldsymbol{x}) \quad = \quad \sup\left\{\inf_{t\in\mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t) : \sum_{t=1}^{\infty} \boldsymbol{x}_t = \boldsymbol{x}\right\}.$$

We show that $f(\boldsymbol{x}) = g(\boldsymbol{x})$.

First, we show that

$$f(\boldsymbol{x}) > 0 \Leftrightarrow g(\boldsymbol{x}) > 0.$$

Suppose that

$$f(\boldsymbol{x}) = \sup\left\{\alpha \in\,]0,1] : \boldsymbol{x} \in \sum_{t=1}^{\infty} [\widetilde{a}_t]_\alpha\right\} > 0.$$

There exists $\alpha_0 \in\,]0,1]$ such that $\boldsymbol{x} \in \sum_{t=1}^{\infty}[\widetilde{a}_t]_{\alpha_0}$ and there exist $\boldsymbol{x}_t^0 \in [\widetilde{a}_t]_{\alpha_0}$ $(t \in \mathbb{N})$ such that $\boldsymbol{x} = \sum_{t=1}^{\infty} \boldsymbol{x}_t^0$. Because $\widetilde{a}_t(\boldsymbol{x}_t^0) \geq \alpha_0 > 0$ $(t \in \mathbb{N})$, it follows that $\inf_{t\in\mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t^0) \geq \alpha_0 > 0$. Therefore, we have

$$g(\boldsymbol{x}) = \sup\left\{\inf_{t\in\mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t) : \sum_{t=1}^{\infty} \boldsymbol{x}_t = \boldsymbol{x}\right\} \geq \inf_{t\in\mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t^0) \geq \alpha_0 > 0.$$

Suppose that

$$g(\boldsymbol{x}) = \sup\left\{\inf_{t\in\mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t) : \sum_{t=1}^{\infty} \boldsymbol{x}_t = \boldsymbol{x}\right\} > 0.$$

There exist $\boldsymbol{x}_t^0 \in \mathbb{R}^n (t \in \mathbb{N})$ such that $\boldsymbol{x} = \sum_{t=1}^{\infty} \boldsymbol{x}_t^0$ and $\inf_{t\in\mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t^0) > 0$. Put $\alpha_0 = \inf_{t\in\mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t^0) > 0$. Because $\widetilde{a}_t(\boldsymbol{x}_t^0) \geq \alpha_0 > 0$ $(t \in \mathbb{N})$, it follows that $\boldsymbol{x}_t^0 \in [\widetilde{a}_t]_{\alpha_0}$ $(t \in \mathbb{N})$ and $\boldsymbol{x} = \sum_{t=1}^{\infty} \boldsymbol{x}_t^0 \in \sum_{t=1}^{\infty}[\widetilde{a}_t]_{\alpha_0}$. Therefore, we have

$$f(\boldsymbol{x}) = \sup\left\{\alpha \in\,]0,1] : \boldsymbol{x} \in \sum_{t=1}^{\infty} [\widetilde{a}_t]_\alpha\right\} \geq \alpha_0 > 0.$$

Therefore, it is proved that

$$f(\boldsymbol{x}) > 0 \Leftrightarrow g(\boldsymbol{x}) > 0,$$

and we have

$$f(\boldsymbol{x}) = 0 \Leftrightarrow g(\boldsymbol{x}) = 0.$$

Next, suppose that

$$f(\boldsymbol{x}) = \sup\left\{\alpha \in\,]0,1] : \boldsymbol{x} \in \sum_{t=1}^{\infty} [\widetilde{a}_t]_\alpha\right\} = \alpha_0 > 0.$$

It follows that $\boldsymbol{x} \in \sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha$ $(\alpha \in\,]0,\alpha_0[)$ and $\boldsymbol{x} \notin \sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha$ $(\alpha \in\,]\alpha_0, 1])$. For any $\alpha \in\,]0,\alpha_0[$, there exist $\boldsymbol{x}_t^\alpha \in [\widetilde{a}_t]_\alpha$ $(t \in \mathbb{N})$ such that $\boldsymbol{x} = \sum_{t=1}^{\infty} \boldsymbol{x}_t^\alpha$. It holds that $\boldsymbol{x} \neq \sum_{t=1}^{\infty} \boldsymbol{x}_t^\alpha$ for any $\alpha \in\,]\alpha_0, 1]$ and $\boldsymbol{x}_t^\alpha \in [\widetilde{a}_t]_\alpha$ $(t \in \mathbb{N})$. For any $\alpha \in\,]0,\alpha_0[$, it follows that $\inf_{t\in\mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t^\alpha) \geq \alpha$ because $\widetilde{a}_t(\boldsymbol{x}_t^\alpha) \geq \alpha$ $(t \in \mathbb{N})$. Because

$$g(\boldsymbol{x}) = \sup\left\{\inf_{t\in\mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t) : \sum_{t=1}^{\infty} \boldsymbol{x}_t = \boldsymbol{x}\right\} \geq \inf_{t\in\mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t^\alpha) \geq \alpha$$

for any $\alpha \in]0, \alpha_0[$, it follows that

$$g(\boldsymbol{x}) = \sup \left\{ \inf_{t \in \mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t) : \sum_{t=1}^{\infty} \boldsymbol{x}_t = \boldsymbol{x} \right\} \geq \alpha_0.$$

For any $\alpha \in]\alpha_0, 1]$, if $\boldsymbol{x} = \sum_{t=1}^{\infty} \boldsymbol{x}_t$, then there exists $t_0 \in \mathbb{N}$ such that $\boldsymbol{x}_{t_0} \notin [\widetilde{a}_{t_0}]_\alpha (\Leftrightarrow \widetilde{a}_{t_0}(\boldsymbol{x}_{t_0}) < \alpha)$, and then $\inf_{t \in \mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t) < \alpha$. Because

$$g(\boldsymbol{x}) = \sup \left\{ \inf_{t \in \mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t) : \sum_{t=1}^{\infty} \boldsymbol{x}_t = \boldsymbol{x} \right\} \leq \alpha$$

for any $\alpha \in]\alpha_0, 1]$, it follows that

$$g(\boldsymbol{x}) = \sup \left\{ \inf_{t \in \mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t) : \sum_{t=1}^{\infty} \boldsymbol{x}_t = \boldsymbol{x} \right\} \leq \alpha_0.$$

Therefore, we have

$$g(\boldsymbol{x}) = \sup \left\{ \inf_{t \in \mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t) : \sum_{t=1}^{\infty} \boldsymbol{x}_t = \boldsymbol{x} \right\} = \alpha_0 = f(\boldsymbol{x}).$$

□

Remark 8.4. In Stojaković and Stojaković [12, 13], a series of $\{\widetilde{a}_t\}_{t \in \mathbb{N}} \subset \mathcal{FN}(\mathbb{R}^n)$, which is not necessary to satisfy FCA, is defined as the right-hand side of (8.3) using Zadeh's extension principle. Proposition 8.3 shows that our series (8.1) of fuzzy sets is equivalent to that in Stojaković and Stojaković [12, 13] when FCA is satisfied.

Remark 8.5. In this remark, we present other definitions of series of sets and fuzzy sets. For $\{S_t\}_{t \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n)$, which is not necessary to satisfy CA, define

$$\sum_{t=1}^{\infty} S_t = \left\{ \sum_{t=1}^{\infty} \boldsymbol{x}_t : \boldsymbol{x}_t \in S_t, t \in \mathbb{N}, \text{ and } \sum_{t=1}^{\infty} \boldsymbol{x}_t \text{ converges} \right\}.$$

For $\{\widetilde{a}_t\}_{t \in \mathbb{N}} \subset \mathcal{F}(\mathbb{R}^n)$, define

$$\left(\sum_{t=1}^{\infty} \widetilde{a}_t \right) (\boldsymbol{x}) = \sup \left\{ \inf_{t \in \mathbb{N}} \widetilde{a}_t(\boldsymbol{x}_t) : \sum_{t=1}^{\infty} \boldsymbol{x}_t = \boldsymbol{x} \right\}$$

for each $\boldsymbol{x} \in \mathbb{R}^n$. It can be proved in a similar way to Proposition 8.3 that

$$\sum_{t=1}^{\infty} \widetilde{a}_t = M \left(\left\{ \sum_{t=1}^{\infty} [\widetilde{a}_t]_\alpha \right\}_{\alpha \in]0,1]} \right).$$

For a series of fuzzy sets, the following theorem presents the conditions for which level sets of a series coincide with a series of level sets.

Theorem 8.6. ([12, Theorem 1]) *Let $\{\widetilde{a}_t\}_{t \in \mathbb{N}} \subset \mathcal{FC}(\mathbb{R}^n)$. If $\{\widetilde{a}_t\}_{t \in \mathbb{N}}$ satisfies FACA, then*

$$\left[\sum_{t=1}^{\infty} \widetilde{a}_t \right]_\alpha = \sum_{t=1}^{\infty} [\widetilde{a}_t]_\alpha$$

for any $\alpha \in]0,1]$.

The following proposition presents the properties of orderings for a series of fuzzy sets.

Proposition 8.7. *Let* $\{\widetilde{a}_t\}_{t\in\mathbb{N}}, \{\widetilde{b}_t\}_{t\in\mathbb{N}} \subset \mathcal{FC}(\mathbb{R}^n)$ *satisfy FACA.*

(i) If $\widetilde{a}_t \leq \widetilde{b}_t$ *for any* $t \in \mathbb{N}$*, then*

$$\sum_{t=1}^{\infty} \widetilde{a}_t \leq \sum_{t=1}^{\infty} \widetilde{b}_t.$$

(ii) If $\widetilde{a}_t \leq \widetilde{b}_t$ *for any* $t \in \mathbb{N}$ *and if there exists* $m \in \mathbb{N}$ *such that* $\widetilde{a}_m < \widetilde{b}_m$*, then*

$$\sum_{t=1}^{\infty} \widetilde{a}_t < \sum_{t=1}^{\infty} \widetilde{b}_t.$$

Proof. (i) Fix any $\alpha \in]0,1]$. Because $[\widetilde{a}_t]_\alpha \leq [\widetilde{b}_t]_\alpha$ ($t \in \mathbb{N}$), it follows that $\sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha \leq \sum_{t=1}^{\infty}[\widetilde{b}_t]_\alpha$ from Proposition 4.3 (iii). From Theorem 8.6, $[\sum_{t=1}^{\infty} \widetilde{a}_t]_\alpha = \sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha \leq \sum_{t=1}^{\infty}[\widetilde{b}_t]_\alpha = \left[\sum_{t=1}^{\infty} \widetilde{b}_t\right]_\alpha$. Therefore, we have $\sum_{t=1}^{\infty} \widetilde{a}_t \leq \sum_{t=1}^{\infty} \widetilde{b}_t$ by the arbitrariness of $\alpha \in]0,1]$.

(ii) Fix any $\alpha \in]0,1]$. Because $[\widetilde{a}_t]_\alpha \leq [\widetilde{b}_t]_\alpha$ ($t \in \mathbb{N}$) and $[\widetilde{a}_m]_\alpha < [\widetilde{b}_m]_\alpha$, it follows that $\sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha < \sum_{t=1}^{\infty}[\widetilde{b}_t]_\alpha$ from Proposition 4.3 (iv). From Theorem 8.6, $[\sum_{t=1}^{\infty} \widetilde{a}_t]_\alpha = \sum_{t=1}^{\infty}[\widetilde{a}_t]_\alpha < \sum_{t=1}^{\infty}[\widetilde{b}_t]_\alpha = \left[\sum_{t=1}^{\infty} \widetilde{b}_t\right]_\alpha$. Therefore, we have $\sum_{t=1}^{\infty} \widetilde{a}_t < \sum_{t=1}^{\infty} \widetilde{b}_t$ by the arbitrariness of $\alpha \in]0,1]$. □

The following proposition shows that our series of fuzzy sets is equivalent to the limit of partial sums of a series under some conditions, where the limit is defined as the limit in Definition 7.1 and is a fuzzified version of Painlevé-Kuratowski convergence.

Proposition 8.8. *Let* $\{\widetilde{a}_t\}_{t\in\mathbb{N}} \subset \mathcal{FC}(\mathbb{R}^n)$*. If* $\{\widetilde{a}_t\}_{t\in\mathbb{N}}$ *satisfies FACA, then*

$$\sum_{t=1}^{\infty} \widetilde{a}_t = \lim_{m\to\infty} \sum_{t=1}^{m} \widetilde{a}_t$$

where the limit on the right-hand side is the limit in Definition 7.1.

Proof. From Proposition 4.9, it follows that

$$\sum_{t=1}^{\infty} [\widetilde{a}_t]_\alpha = \lim_{m\to\infty} \sum_{t=1}^{m} [\widetilde{a}_t]_\alpha = \lim_{m\to\infty} \left[\sum_{t=1}^{m} \widetilde{a}_t\right]_\alpha$$

for any $\alpha \in]0,1]$, where $\lim_{m\to\infty}$ means Painlevé-Kuratowski convergence. Therefore, we have

$$\sum_{t=1}^{\infty} \widetilde{a}_t = \lim_{m\to\infty} \sum_{t=1}^{m} \widetilde{a}_t$$

where the limit on the right-hand side is the limit in Definition 7.1. □

Example 8.9. Consider $\{S_t\}$ as in Example 4.10. If we set $\widetilde{a}_t = c_{S_t}$ for each $t \in \mathbb{N}$, then $\{\widetilde{a}_t\} \subset \mathcal{FN}(\mathbb{R}^n)$ and $\{\widetilde{a}_t\} \not\subset \mathcal{FC}(\mathbb{R}^n)$, and then $\{\widetilde{a}_t\}$ satisfies FACA. In this case,

$$\sum_{t=1}^{\infty} \widetilde{a}_t = c_{\sum_{t=1}^{\infty} S_t} \neq c_{\lim_{m\to\infty} \sum_{t=1}^{m} S_t} = \lim_{m\to\infty} \sum_{t=1}^{m} \widetilde{a}_t$$

where the first limit means Painlevé-Kuratowski convergence and the second limit is the limit in Definition 7.1.

9. Conclusion

In this study, we proposed series of sets and fuzzy sets and investigated their properties.

First, we proposed a series of sets and derived the properties of addition, scalar multiplication, and orderings (Proposition 4.3). For a sequence of sets in a series, it was derived that the sequence approaches the origin (Proposition 4.4). For a series of sets, it was derived that the sum of the latter part of the series approaches the origin (Proposition 4.5). It was derived that a series of compact sets is compact (Proposition 4.8). It was derived that under some conditions, our series of sets is equivalent to the limit of partial sums of a series as Painlevé-Kuratowski convergence (Proposition 4.9).

Next, we proposed a series of fuzzy sets and derived the properties of addition and scalar multiplication (Proposition 8.2). It was derived that our series of fuzzy sets is equivalent to that obtained using Zadeh's extension principle (Proposition 8.3). The properties of orderings for a series of fuzzy sets were derived (Proposition 8.7). It was derived that under some conditions, our series of fuzzy sets is equivalent to the limit of partial sums of a series, where the limit is a fuzzified version of Painlevé-Kuratowski convergence (Proposition 8.8).

It is expected that the properties derived in this study will be useful for an infinite-horizon MDP or dynamic programming as a maximization problem of the discounted total reward, in which the rewards are given as compact sets or compact fuzzy sets and are not necessarily convex.

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