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CONVERGENCE THEOREMS FOR SOLVING SPLIT EQUALITY VARIATIONAL INEQUALITY PROBLEMS INVOLVING QUASIMONOTONE MAPPINGS IN BANACH SPACES

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Abstract. In this work, we propose and study projection-based algorithms for solving split equality variational inequality problems involving quasimonotone mappings. We provide weak and strong convergence theorems for the algorithms produced in real Banach spaces that are reflexive. In addition, we give some specific applications of the main results and finally provide a numerical example to demonstrate the workability of our method.

1. Introduction

Let $\mathbb{E}$ to be a Banach space with norm $\|.\|$, and let $\mathbb{E}^*$ be its dual. The mapping $\mathcal{T} : C \to \mathbb{E}^*$ is called

(1) $\alpha-$strongly monotone on C if there exists $\alpha > 0$ such that
$$\langle \mathcal{T}(x) - \mathcal{T}(y), x - y\rangle \geq \alpha||x - y||^2, \text{ for all } x, y \in C;$$
(2) monotone on C if
$$\langle \mathcal{T}(x) - \mathcal{T}(y), x - y\rangle \geq 0, \text{ for all } x, y \in C;$$
(3) pseudomonotone on C if for all $x, y \in C$, we have
$$\langle T(y), x - y\rangle \geq 0 \Rightarrow \langle T(x), x - y\rangle \geq 0;$$
(4) quasimonotone on C if for all $x, y \in C$, we have
$$\langle T(y), x - y\rangle > 0 \Rightarrow \langle T(x), x - y\rangle \geq 0;$$
(5) Lipschitz continuous on C if there exists a constant $L > 0$, called the Lipschitz constant, such that
$$||\mathcal{T}(x) - \mathcal{T}(y)|| \leq L||x - y||, \text{ for all } x, y \in C.$$
If $L < 1$, then $\mathcal{T}$ is called a *contraction* and if $L = 1$, then $\mathcal{T}$ is said to be *nonexpansive*.

Remark 1.1. We note that all α-strongly monotone mappings are monotone, all monotone mappings are pseudo-monotone, and all pseudo-monotone mappings are quasi-monotone. However, the reverse is not true (see, e.g., [31]).

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Let C be a non-empty, convex and closed subset of a Hilbert space H. Suppose $\mathcal{T} : H \to H$ is a mapping. The Variational Inequality Problem (VIP) can be defined as follows:

$$\text{find a vector } x^* \in C \text{ such that } \langle \mathcal{T}(x^*), y - x^* \rangle \geq 0, \quad \text{for all } y \in C. \tag{1.1}$$

$VI(C, \mathcal{T})$ is the solution set of the VIP (1.1). The formulation of the Dual Variational Inequality Problem (DVIP), often called the Minty Variational Inequality Problem (MVIP), is as follows:

$$\text{find a vector } x^* \in C \text{ such that } \langle \mathcal{T}(y), y - x^* \rangle \geq 0, \text{for all } y \in C. \tag{1.2}$$

The DVIP solution set (1.2) is indicated by $DVI(C, \mathcal{T})$. Clearly, $DVI(C, \mathcal{T})$ is closed and convex. If $\mathcal{T}$ is continuous and C is convex, then $DVI(C, \mathcal{T}) \subseteq VI(C, \mathcal{T})$ (see to [32]). However, when $\mathcal{T}$ is a continuous and quasi-monotone mapping, the inclusion $VI(C, \mathcal{T}) \subseteq DVI(C, \mathcal{T})$ may not always hold (refer to [31]). In fact, if $\mathcal{T}$ is continuous and pseudo-monotone, then $VI(C, \mathcal{T}) = DVI(C, \mathcal{T})$ (see, [6].

Since its inception in 1964 by Stampacchia [25] and Fichera [9], the theory of variational inequalities has received substantial attention and remains a hot topic. One of the main causes of this is the wide range of problems that can be expressed as variational inequalities, including those emerging in the domains of optimization and control, engineering science, mechanics, game theory, elasticity, physics, economics, transportation equilibrium, etc.

According to [11], the Dual Variational Inequality problem has real-world applications. A dynamical system with $\mathcal{T}$ has been shown to have solutions of a DVIP associated with a single-valued continuous mapping $\mathcal{T}$ defined on an open convex domain, which can be viewed as the subset of stable equilibria within the set of all equilibria (represented by the Stampacchia variational inequality solutions).

Many numerical iterative approaches have been developed to tackle variational inequalities and optimization problems (e.g., [4, 22]). The authors investigated a variety of projection-type techniques to tackle the variational inequality problem. The gradient method (GM) is the first projection technique for solving the VIP and is provided by

$$x_1 \in \mathbb{H}, \; x_{n+1} = P_C(x_n - \lambda \mathcal{T} x_n), \tag{1.3}$$

where P_C is the projection operator onto the convex and closed subset C of a Hilbert space $\mathbb{H}$ and $\lambda > 0$ is a suitable stepsize. With only one projection onto the feasible set, the gradient method (GM) is the most straightforward approach for solving the VIP. The approach, however, only converges when the cost operator $\mathcal{T}$ is α-strongly monotone and L-Lipschitz continuous with $\lambda \in (0, \frac{2\alpha}{L^2})$. These stringent conditions greatly limit the scope of applications of the GM (1.3).

To avoid the notion of strong monotonicity, Korpelevich [12] introduced the extragradient method (EGM), described below, which is an algorithm for resolving variational inequalities in a Hilbert space $\mathbb{H}$:

$$x_1 \in \mathbb{H},\ y_n = P_C\left(x_n - \lambda \mathcal{T} x_n\right);\ x_{n+1} = P_C\left(x_n - \lambda \mathcal{T} y_n\right), \tag{1.4}$$

where $\mathcal{T} : C \to \mathbb{H}$ is monotone and L-Lipschitz continuous and $\lambda \in \left(0, \frac{1}{L}\right)$ and $C \subseteq \mathbb{H}$ is closed and convex. He established the weak convergence of (1.4) to a solution of the VIP (1.1) in Hilbert spaces if $\mathcal{T}$ is monotone and Lipschitz.

The algorithm (1.4) has significant drawbacks, as shown by the above technique. It is necessary that the mapping have the properties of Lipschitz continuity, monotonicity, and a known Lipschitz constant. This algorithm's complexity and low efficiency stem from the need to compute projections twice for the feasible set C during each iteration. To ensure algorithm convergence, research on variational inequality problems focuses on weakening the mapping and speeding the convergence rate.

Scholars have made significant improvements to the EGM, as seen in [7, 18, 23] and references. One of the method's primary areas for improvement is to reduce the number of projections onto the feasible set C every iteration. Censor *et al.* [4] made the first attempt in this way. They altered the EGM and substituted a projection onto a half-space for the second projection. This strategy, known as the subgradient extragradient technique (SEGM), only requires one projection onto the feasible set C. The SEGM is presented as follows:

$$x_1 \in \mathbb{H},\ y_n = P_C(x_n - \lambda \mathcal{T} x_n);\ x_{n+1} = P_{\mathcal{T}_n}(x_n - \lambda T y_n), \tag{1.5}$$

where $\mathcal{T}_n = \{z \in H : \langle x_n - \lambda \mathcal{T} x_n - y_n, z - y_n \rangle \leq 0\}$ and $\lambda \in (0, \frac{1}{L})$. As seen, the projection for x_{n+1} in the scheme (1.5) is computed on a half-space T_n which is inherently explicit. Using the same assumptions as the EGM (1.4), the Censor *et al.* [4] obtained a weak convergence for the SEGM (1.5). The SEGM is an advance over the EGM since it allows for explicit computation of projection into a half-space. We should remark, however, that each iteration of the SEGM still requires computing two projections onto the closed convex sets C and $\mathcal{T}_n$. This may be a substantial hurdle to the SEGM's implementation.

The second attempt is the following Tseng extragradient method: choose $x_0 \in \mathbb{H}$, for each $n \geq 0$, compute:

$$y_n = P_C(x_n - \lambda \mathcal{T} x_n);\ x_{n+1} = y_n - \lambda(\mathcal{T} y_n - \mathcal{T} x_n), \tag{1.6}$$

where $\lambda \in (0, \frac{1}{L})$. The scheme (1.6) is based on Tseng's modified forward-backward splitting technique, which was first introduced in [29]. He showed that the sequence $\{x_n\}$ converges weakly to a VIP solution (1.1). Tseng's approach offers an advantage over Korpelevich's in that it just requires one projection every iteration. It still requires two evaluations of the mapping T per iteration. Several researchers have studied changes to Tseng's algorithm (see, for example, [18, 22, 23, 34]).

It is worth noting that the aforementioned approaches use norm distance and metric projections. Applying this theory to Banach spaces creates complications, as many beneficial qualities of nonexpansive operators in Hilbert space, such as the metric projection P_K onto a nonempty, closed, and convex subset K of $\mathbb{H}$, are no longer nonexpansive in Banach spaces. There are several strategies for solving these obstacles. One of these is to use the Bregman distance, which requires no symmetry or triangle inequality properties. Instead of employing metric projections, which are less flexible and broad, researchers in this situation use the Bregman distance and projection. As a result, the Bregman distance and projection methods for approximating VIP solutions are useful for analyzing a wide range of problems.

In 1999, Solodov and Svaiter [24] studied the following double projection method with the use of Bregman distance and Bregman projection for solving variational inequalities in Euclidean spaces.

Algorithm: Choose $x_0 \in \mathbb{H}$, and $\gamma, \sigma \in (0,1)$. Take $k = 0$. Calculate $\{x_k\}$ as follows:

Step 1: Compute $z_k = (\nabla g)^{-1}\left[\nabla g(x_k) - \mathcal{T}(x_k)\right];$
If $x_k = \Pi_C(z_k)$, then stop; else go to Step 2.
Step 2: Compute

$$m_k = \min\left\{m \in \mathbb{N} : \left\langle \mathcal{T}(x_k) - \mathcal{T}(y_m), x_k - \Pi_C^g(z_k)\right\rangle \leq \sigma D_g\left(\Pi_C^g(z_k), x_k\right)\right\};$$

where $y_m = \gamma^m \Pi_C^g(z_k) + (1-\gamma^m)x_k$; Let $y_k = \gamma^{m_k}\Pi_C^g(z_k) + (1-\gamma^{m_k})x_k$;
and Π_C^g is the Bregman projection(see Section 2 for the definition).
Step 3: Compute $x_{k+1} = \Pi^g_{C\bigcap H_k}(x_k)$; where $H_k = \{v \in \mathbb{E} : \langle \mathcal{T}(y_k), v - y_k\rangle \leq 0\};$
Step 4 : Let $k = k+1$ and return to **Step 1**.

In 2018, Zheng [4] extended the results from Euclidian to Banach spaces. The suggested technique strongly converges to the solution of variational inequalities in reflexive real Banach spaces, assuming $VI(C,\mathcal{T}) \neq \emptyset$ and $\mathcal{T}$ is uniformly continuous and quasimonotone mapping. We believe there is a gap in the proof of the strong convergence theorem, therefore the convergence of the double method to the solution of variational inequalities in the same scenario as Zheng [4] is still open.

Consider C and D as convex and closed subsets of real Banach spaces $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively. Let $\mathcal{T} : \mathbb{E}_1 \to \mathbb{E}_1^*$ and $\mathcal{S} : \mathbb{E}_2 \to \mathbb{E}_2^*$ be non-linear mappings, and $A : \mathbb{E}_1 \to \mathbb{E}_3$, $B : \mathbb{E}_2 \to \mathbb{E}_3$ be bounded linear mappings. The Split Equality

Variational Inequality Problem (SEVIP) involves locating two points:

$$x^* \in VI(C,\mathcal{T}) \text{ and } y^* \in VI(D,\mathcal{S}) \text{ such that } Ax^* = By^*. \tag{1.7}$$

On the other hand, the Split Equality Duality Variational Inequality Problem (SED-VIP) consists of locating two points.

$$x^* \in DVI(C,\mathcal{T})\text{and } y^* \in DVI(D,\mathcal{S}) \text{ such that } Ax^* = By^*. \tag{1.8}$$

Special examples of the inclusion problem (1.8) include variational inequality, convex programming, split feasibility, and minimization.
Based on the aforementioned results, we raise the following important questions:

1. *Can we extend the double projection algorithm proposed by Solodov and Svaiter using the Bregman distance to that of split equality variational inequality problems in real Banach spaces*?
2. *Can we prove weak/strong convergence results using the Bregman distance to the solution of the variational inequality problems in real Banach spaces*?

Motivated and inspired by the efforts of Tseng [29], Solodov and Svaiter [24], and Zheng [4], in this paper we introduce and investigate a projection-based algorithms for solving split equality variational inequality problems as well as split equality dual variational inequality problems in reflexive real Banach spaces with uniformly continuous quasimonotone mappings. A Halpern-type projection-based algorithm is utilized to obtain the strong convergence. We also present some concrete applications of the key ideas, as well as a numerical example, to validate our theoretical findings. Our findings complement Zheng's work [4] and broaden the existing literature.

The following is the paper's outline: We provide some fundamental concepts and background information in Sect. 2 that will be helpful in our analysis. In Sect. 3, we present some necessary hypotheses and our algorithms. Subsequently, we establish weak convergence analysis of the first method to the solution of split equality variational inequality problems, and derive strong convergence analysis for the second method, which addresses the solution of split equality dual variational inequality problems. In sect. 4, we present some specific applications of our main results. Some numerical experiments are presented in Sect. 5 illustrating the applicability of the algorithm. Finally, we present some concluding remarks on our work in Sec. 6

2. Preliminaries

Let $\mathbb{E}$ be a reflexive real Banach space with $\mathbb{E}^*$ as its dual and the function $g : \mathbb{E} \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex with the effective domain *dom* $g = \{x \in \mathbb{E} : g(x) < +\infty\}$. The real valued function g^* is the *Fenchel conjugate* of g: $\mathbb{E} \to (-\infty, \infty]$ which is proper, lower semi-continuous and convex, is the real valued function g^*: $\mathbb{E}^* \to (-\infty, \infty]$ defined by $g^*(x^*) = \sup\{\langle x^*, x\rangle - g(x) : x \in \mathbb{E}\}$

for any $x^* \in \mathbb{E}^*$.

We denote by $g'(x,y)$ the right-hand derivative of g at $x \in$ *int(dom g)* in the direction $y \in \mathbb{E}$, that is,

$$g'(x,y) = \lim_{t \downarrow 0+} \frac{g(x+ty) - g(x)}{t}. \tag{2.1}$$

Recall that the function g is called *Gâteaux differentiable* at x if the limit as $t \to 0$ in (2.1) exits at any $y \in \mathbb{E}$. In this case $g'(x,y)$ coincides with $(\nabla g)(x)$, the value of the gradient ∇g of g at x. In the event that the limit in (2.1) is attained uniformly for any $y \in \mathbb{E}$ with $\|y\| = 1$, then g is considered to be *Fréchet differentiable* at x, and if the limit (2.1) is attained uniformly for $x \in C$ and $||y|| = 1$, then g is said to be *uniformly Fréchet differentiable* at x. It is known that if $g : \mathbb{E} \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of $\mathbb{E}$, then ∇g is uniformly continuous on bounded subsets of $\mathbb{E}$ (see, e.g., [21]) and it is also uniformly smooth (see, e.g., [33]).

If a function $g : \mathbb{E} \to \mathbb{R}$ meets both of the following two requirements, it is referred to be a *Legendre function*.

(L1) g is Gâteaux differentiable, *int(dom g)* $\neq \emptyset$ and *dom* $\nabla g =$*int(dom g)*;

(L2) g^* is Gâteaux differentiable, *int(dom* g^**)* $\neq \emptyset$ and *dom* $\nabla g^* =$*int(dom* g^**)*.

We say that the space $\mathbb{E}$ is *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$, exists for $x, y \in B$ and $\mathbb{E}$ is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in B$ with $x \neq y$, where $B = \{x \in \mathbb{E} : \|x\| = 1\}$.

If $\mathbb{E}$ is a Banach space that is smooth and absolutely convex, then $g(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$ is a lower semi-continuous, Legendre function with the Fenchel conjugate $g^*(x^*) = \frac{1}{q}\|x^*\|^q$ $(1 < q < \infty)$ which is characterized by $\frac{1}{p} + \frac{1}{q} = 1$ (see, e.g., [1]). In this case, we have $\nabla g = J_p$, where $J_p : \mathbb{E} \to 2^{\mathbb{E}^*}$ is the multi-valued function defined by

$$J_p(x) = \left\{x^* \in \mathbb{E}^* : \langle x^*, x\rangle = \|x\|^p \text{ and } \|x^*\| = \|x\|^{p-1}\right\},$$

for all $x \in \mathbb{E}$. The special case when $p = 2$ gives that $J_p = J$, where J is the normalized duality mapping. If $\mathbb{E} = \mathbb{H}$, where $\mathbb{H}$ is a real Hilbert space, then $J = I$, where I is the identity mapping on $\mathbb{H}$. We will refer to the normalized duality mapping J on $\mathbb{E}$ as $J_{\mathbb{E}}$ throughout the remainder of the paper.

Moreover, if $\mathbb{E} = \mathbb{H}$, where $\mathbb{H}$ is a real Hilbert space, then $J = I$, where I is the identity mapping on $\mathbb{H}$. Throughout the rest of the paper, we will denote the normalized duality mapping J on $\mathbb{E}$ by $J_{\mathbb{E}}$.

If g: $\mathbb{E} \to (-\infty, +\infty]$ is a Legendre function and $\mathbb{E}$ is a reflexive Banach space, then $\nabla g^* = (\nabla f)^{-1}$(see, [2]). Moreover, we note that g is a Legendre function if and only if g^* is a Legendre function (see, [1]).

Definition 2.1. Let g: $\mathbb{E} \to (-\infty, \infty]$ be a convex and Gâteaux differentiable function.

(1) The nonnegative real-valued function D_g: $dom\ g \times int(dom\ g) \to [0, \infty)$ defined by

$$D_g(y, x) = g(y) - g(x) - \langle \nabla g(x), y - x \rangle,\ x \in int(dom\ g), \text{ and } y \in dom\ g, \tag{2.2}$$

is called the *Bregman distance* with respect to g (see, Censor and Lent [5]).

(2) The modulus of total convexity of g at the point $x \in \mathbb{E}$ is the function $v_g : \mathbb{E} \times [0, \infty) \to [0, \infty)$ defined as

$$v_g(x, t) = \inf \{D_g(x, y) : y \in \mathbb{E}, ||y - x|| = t\}. \tag{2.3}$$

(3) A function g is said to be totally convex if $v_g(x, t) > 0$ for all $t > 0$ and $x \in \mathbb{E}$.

(4) g is said to be a β-strongly convex function if there exists $\beta > 0$ such that

$$g(x) - g(y) - \langle \nabla g(y), x - y \rangle \geq \beta ||x - y||^2, \forall x, y \in \mathbb{E}. \tag{2.4}$$

note that ∇g^* is $\frac{1}{\beta}-$ Lipschitz continuous if a function g is $\beta-$strongly convex (see, [35]).

A function g: $\mathbb{E} \to (-\infty, \infty]$ is said to be strongly coercive if $\lim_{\|x\| \to \infty} \left(\frac{g(x)}{\|x\|} \right) = \infty$. If $\mathbb{E}$ is a smooth and 2-uniformly convex Banach space, then the function $f(x) = \frac{1}{2}\|x\|^2$ is strongly coercive, lower semi-continuous, bounded, uniformly Fréchet differentiable and strongly convex with strong convexity constant $\beta \in (0, 1]$ and conjugate $f^*(x^*) = \frac{1}{2}\|x^*\|^2$.

Lemma 2.2 ([33])**.** *If g is convex and bounded on bounded subsets of $\mathbb{E}$. Then the following are equivalent:*

i. *g is strongly coercive and uniformly convex on bounded subsets of $\mathbb{E}$.*
ii. *$dom\ g^* = \mathbb{E}^*$, g^* is bounded on bounded subsets and uniformly smooth on bounded subsets of $\mathbb{E}^*$.*
iii. *$dom\ g^* = \mathbb{E}^*$, g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of $\mathbb{E}^*$.*

Lemma 2.3 ([16])**.** *Let g: $\mathbb{E} \to (-\infty, \infty]$ be a Gâteaux differentiable function which is totally convex on bounded subsets of $\mathbb{E}$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be bounded in $\mathbb{E}$. Then, $\lim_{n\to\infty} D_g(x_n, y_n) = 0$ if and only if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Definition 2.4. Let g: $\mathbb{E} \to (-\infty, \infty]$ be a convex and Gâteaux differentiable function and let $C \subseteq int(dom\ g)$ be a nonempty, closed and convex subset of $\mathbb{E}$.

Then, the Bregman projection of $x \in int(dom\ g)$ onto C is the unique vector $\Pi_C^g(x)$ of C with the property

$$D_g(\Pi_C^g(x), x) = \inf \{D_g(y, x) : y \in C\}.$$

The Bregman projection also satisfies the following properties:

$$z = \Pi_C^g(x) \ if \ and \ only \ if \ \langle \nabla gx - \nabla gz, y - z\rangle \leq 0, \text{ for all } y \in C, \ and \tag{2.5}$$

$$D_g(y, \Pi_C^g(x)) + D_g(\Pi_C^g(x), x) \leq D_g(y, x), \text{ for all } x \in E, y \in C. \tag{2.6}$$

Lemma 2.5 ([20]). *Let $g : \mathbb{E} \to (-\infty, +\infty]$ be a convex Gâteaux differentiable function. The Bregman distance $D_g(.,.)$ has the following property called three point identity*

$$D_g(y, z) + D_g(z, x) - D_g(y, x) = \langle \nabla g(z) - \nabla g(x), z - y\rangle \tag{2.7}$$

for any $y \in domf$ and $x, z \in int(domf)$.

Lemma 2.6 ([15]). *Let C be a closed, non-empty, convex subset of $\mathbb{E}$, and let $g : \mathbb{E} \to \mathbb{R}$ be a totally convex, Frechet-differentiable function. If the level sets $D_g(x, .)$ are bounded for all $x \in E$ and ∇g is uniformly continuous on bounded subsets of E, then $\Pi_C^g : \mathbb{E}_1 \to C$ maps bounded subset of $\mathbb{E}_1$ into bounded subset of C.*

Lemma 2.7 ([16]). *Let $\mathbb{E}$ be a real Banach space. The non-negative real valued $V_g : \mathbb{E} \times \mathbb{E}^* \to (-\infty, \infty]$ associated with a Gâteaux differentiable Legendre function $g : \mathbb{E}_1 \to \mathbb{R}$ defined by*

$$V_g(x, x^*) = g(x) + \nabla g'(y^*) - \langle x^*, x\rangle, \forall x \in \mathbb{E}, x^* \in \mathbb{E}^* \tag{2.8}$$

satisfies the following two properties

$$V_g(x, x^*) = D_g(x, (\nabla g)^{-1}(x^*)) \tag{2.9}$$

and

$$V_g(x, x^*) + \left\langle y^*, (\nabla g)^{-1}(x^*) - x\right\rangle \leq V_g(x, x^* + y^*), \forall x \in \mathbb{E}, x^*, y^* \in \mathbb{E}^* \tag{2.10}$$

Lemma 2.8 ([14]). *Let $g : \mathbb{E} \to \mathbb{R}$ be a totally convex function. If the sequence $\{D_g(x_k, x_0)\}$ is bounded for any $x_0 \in \mathbb{E}$, then $\{x_k\}$ is bounded.*

Lemma 2.9 ([24]). *Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be bounded spaces. Let U be a bounded subset of $\mathbb{E}_1$. If $\mathcal{T} : \mathbb{E}_1 \to \mathbb{E}_2$ is uniformly continuous on bounded subsets of $\mathbb{E}_1$ then $\mathcal{T}$ is bounded on U.*

Lemma 2.10 ([10]). *Assume $y \in C$ and $\mathcal{T}$ to be a quasimonotone, continuous operator on $C \subset \mathbb{E}$. If for some $x_0 \in C$ we have $\langle \mathcal{T}(y), x_0 - y\rangle \geq 0$, then at least one of the following must hold: $\langle \mathcal{T}(y), x_0 - y\rangle \geq 0$ or $\langle \mathcal{T}(y), x - y\rangle \geq 0$, $\forall x \in C$.*

Lemma 2.11 ([32]). *Let C be a non-empty, closed and convex subset of $\mathbb{E}$ and let g be β-strongly convex and continuously differentiable function such that the level sets $D_g(x,.)$ are bounded for all $x \in E$. Define $h : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ by $h(x,v) = \langle \mathcal{T}(v), x - v \rangle$ for any given $v \in \mathbb{E}$ and the mapping $\mathcal{T} : \mathbb{E} \to \mathbb{E}^*$ and take $K(v) = \{x \in C : h(x,v) \leq 0\}$. If $K(v) \neq \emptyset$ and $h(x,v)$ is Lipschitz continuous with respect to x on C with modulus $L > 0$, then*

$$D_g(x,v) \geq \frac{\beta}{L^2} h^2(x,v), \forall x \in C \setminus K(v), y \in K(v), v \in \mathbb{E}. \tag{2.11}$$

Lemma 2.12 ([26]). *Consider a smooth and strictly convex real Banach space, $\mathbb{E}$. The normalized duality mapping on $\mathbb{E}_1$, represented by $J_{\mathbb{E}_1}$, possesses the subsequent attribute:*

$$||x+y||^2 \leq ||x||^2 + 2\left\langle J_{\mathbb{E}_1}(x+y), y \right\rangle, \forall x, y \in \mathbb{E}_1. \tag{2.12}$$

Lemma 2.13 ([19]). *If $g : \mathbb{E} \to (-\infty, \infty]$ is a proper, lower semi-continuous, convex and Gâteaux differentiable function, then $g^* : \mathbb{E}^* \to (-\infty, \infty]$ is a proper weak* lower semi-continuous and convex function. Thus, for all $x \in \mathbb{E}_1$, we have*

$$D_f\left(x, (\nabla g)^{-1}\left(\sum_{i=1}^{N} \beta_i \nabla g(x_i)\right)\right) \leq \sum_{i=1}^{N} \beta_i D_g(x, x_i), \tag{2.13}$$

where $\{x_i\} \subseteq \mathbb{E}, \{\beta_i\} \subseteq (0,1)$ and $N \in \mathbb{N}$ such that $\sum_{i=1}^{N} \beta_i = 1$.

Lemma 2.14 ([13]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers. If $\{a_{n_i}\}$ is a subsequence of $\{a_n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$, then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad and \quad a_k \leq a_{m_k+1}.$$

In fact, m_k=max $\{n \leq k : a_n < a_{n+1}\}$.

Lemma 2.15 ([30]). *If $\{a_k\}$ is a sequence of nonnegative real numbers such that*

$$a_{k+1} \leq (1 - \alpha_k)\, a_k + \alpha_k d_k,$$

where $\{\alpha_k\} \subset (0,1)$ such that $\sum_{k=1}^{\infty} \alpha_k = \infty$ and $\{d_k\}$ is a sequence of real numbers with $\limsup_{k\to\infty} d_k \leq 0$, then $\lim_{k\to\infty} a_k = 0$.

3. Main results

In this section, we describe our algorithms and their convergence outcomes under the following circumstances:

(H_1) Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be reflexive real Banach spaces with duals $\mathbb{E}_1^*$ and $\mathbb{E}_2^*$, respectively, and let C and D be non-empty, closed and convex subsets of $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively.

(H_2) Let $T : \mathbb{E}_1 \to \mathbb{E}_1^*$ and $S : \mathbb{E}_2 \to \mathbb{E}_2^*$ be quasi-monotone mappings which are uniformly continuous on bounded subsets of C and D, respectively, satisfying

$$||T(x)|| \leq \liminf_{k\to\infty} ||T(x_k)|| \text{ and } ||S(y)|| \leq \liminf_{k\to\infty} ||S(y_k)||, \tag{3.1}$$

whenever $\{x_k\}$ and $\{y_k\}$ are sequences in $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively, such that $x_k \rightharpoonup x$ and $y_k \rightharpoonup y$.

(H_3) Let $A : \mathbb{E}_1 \to \mathbb{E}_3$ and $B : \mathbb{E}_2 \to \mathbb{E}_3$, where $\mathbb{E}_3$ is another real Banach space, be bounded linear mappings with adjoints A^* and B^*, respectively.

(H_4) Let the set $\Omega = \{(x^*, y^*) \in DVI(C,T) \times DVI(D,S) : Ax^* = By^*\}$ be non-empty.

(H_5) Let the proper lower semi-continuous functions $g : \mathbb{E}_1 \to \mathbb{R}$ and $f : \mathbb{E}_2 \to \mathbb{R}$ be strongly coercive Legendre functions which are bounded, uniformly Fréchet differentiable and β-strongly convex on bounded subsets of $\mathbb{E}_1$ and $\mathbb{E}_2$ respectively.

(H_6) Let the sequence $\{\alpha_k\} \subset (0,1]$ be such that $\sum_{k=1}^{\infty} \alpha_k = \infty$ and $\lim_{k\to\infty} \alpha_k = 0$.

Algorithm 3.1 Choose $x_0 \in \mathbb{E}_1$, $y_0 \in \mathbb{E}_2$ and $\gamma, \sigma \in (0,1)$. Take $k = 1$. Calculate $\{x_k\}$ and $\{y_k\}$ as follows:

Step 1: Compute

$$a_k = \Pi_C^g (\nabla g)^{-1} \left[\nabla g(x_k) - \gamma_k A^* \left(J_{\mathbb{E}_3}(Ax_k - By_k)\right)\right]; \tag{3.2}$$

$$b_k = \Pi_D^f (\nabla f)^{-1} \left[\nabla f(y_k) - \gamma_k B^* \left(J_{\mathbb{E}_3}(By_k - Ax_k)\right)\right],$$

where $0 < \rho \leq \gamma_k \leq \rho_k$ with

$$\rho_k = \min\left\{\rho + 1, \frac{\beta\|Ax_k - By_k\|^2}{2[\|A^* J_{\mathbb{E}_3}(Ax_k - By_k)\|^2 + \|B^* J_{\mathbb{E}_3}(By_k - Ax_k)\|^2]}\right\}, \tag{3.3}$$

for $k \in \Upsilon = \{m \in \mathbb{N} : Ax_m - By_m \neq 0\}$, otherwise $\gamma_k = \rho$.

Step 2: Compute

$$z_k = (\nabla g)^{-1} \left[\nabla g(a_k) - \beta_k T(a_k)\right]; \; u_k = (\nabla f)^{-1} \left[\nabla f(b_k) - \beta_k S(b_k)\right], \tag{3.4}$$

where $\{\beta_k\} \subset [a, 1] \subset (0,1]$, for all $k \geq 1$;

Step 3: Compute

$$m_k = \min\left\{m \in \mathbb{N} : \left\langle T(a_k) - T(d_m), a_k - \Pi_C^g(z_k)\right\rangle \leq \sigma D_g\left(\Pi_C^g(z_k), a_k\right)\right\}; \tag{3.5}$$

$$l_k = \min\left\{l \in \mathbb{N} : \left\langle S(b_k) - S(e_l), b_k - \Pi_D^f(u_k)\right\rangle \leq \sigma D_f\left(\Pi_D^f(u_k), b_k\right)\right\}, \tag{3.6}$$

where

$$d_m = \gamma^m \Pi_C^g(z_k) + (1-\gamma^m)a_k;\; e_l = \gamma^l \Pi_D^f(u_k) + (1-\gamma^l)b_k. \tag{3.7}$$

Let

$$d_k = \gamma^{m_k} \Pi_C^g(z_k) + (1-\gamma^{m_k})a_k;\; e_k = \gamma^{l_k} \Pi_D^f(u_k) + (1-\gamma^{l_k})b_k. \tag{3.8}$$

Step 4: Compute

$$x_{k+1} = \Pi^g_{C\bigcap H_k}(a_k);\; y_{k+1} = \Pi^f_{D\bigcap R_k}(b_k); \tag{3.9}$$

where

$$H_k = \{x \in \mathbb{E}_1 : \langle T(d_k), x - d_k\rangle \le 0\}\,;\; R_k = \{y \in \mathbb{E}_2 : \langle S(e_k), y - e_k\rangle \le 0\}\,. \tag{3.10}$$

Step 5 : Let $k = k+1$ and return to **Step 1**.

Remark 3.1. The method we introduced is a newly proposed projection-based method for solving the more general problem called split equality variational inequality problems associated with the class of uniformly continuous quasimonotone mappings in Banach spaces. As far as our knowledge is concerned, our method is the first method of its kind proposed in the setting of reflexive Banach spaces. It does not require prior knowledge of the Lipschitz constants of the underlying mappings. It extends all the results in the literature from Hilbert spaces to the more general reflexive Banach spaces under mild conditions.

Lemma 3.2. *Let conditions $(H_1)-(H_5)$ be satisfied. Then the line search rules (3.5) and (3.6) are well defined.*

Proof. Suppose that $a_k = \Pi_C^g(z_k)$, then (3.5) is satisfied for $m = 0$. Let $a_k \neq \Pi_C^g(z_k)$ and assume on the contrary that

$$\left\langle T(a_k) - T(d_m), a_k - \Pi_C^g(z_k)\right\rangle > \sigma D_g\left(\Pi_C^g(z_k), a_k\right)\; \forall m \ge 1. \tag{3.11}$$

Given that $\gamma \in (0,1)$, we have $\lim_{m\to\infty} d_m = \lim_{m\to\infty} \gamma^m\left(\Pi_C^g(z_k)\right) + (1-\gamma^m)\,a_k = a_k$.
Letting $m \to \infty$ in (3.11) and by the continuity of T, we get
$0 = \lim_{m\to\infty}\left\langle T(a_k) - T(d_m), a_k - \Pi_C^g(z_k)\right\rangle \ge \sigma D_g\left(\Pi_C^g(z_k), a_k\right)$, and the fact that $\sigma > 0$, we get

$$D_g\left(\Pi_C^g(z_k), a_k\right) \le 0. \tag{3.12}$$

But as g is strongly convex and $a_k \neq \Pi_C^g(z_k)$, we have

$$D_g\left(\Pi_C^g(z_k), a_k\right) > 0, \tag{3.13}$$

which is a contradiction. Thus, (3.5) is well defined. Similarly, the same can be shown for (3.6). ☐

Lemma 3.3. *Assume the conditions* $(H_1), (H_2)$ *and* (H_5) *hold. Let* T *and* S *be continuous mappings on* C *and* D*, respectively. For all* $x \in C$ *and* $y \in D$*, we have*

$$\left\langle T(x), x - \Pi_C^g \left[(\nabla g)^{-1} (\nabla g(x) - \beta_k T(x)) \right] \right\rangle \geq D_g \left(\Pi_C^g \left[(\nabla g)^{-1} (\nabla g(x) - \beta_k T(x)) \right], x \right), \tag{3.14}$$

and

$$\left\langle S(y), y - \Pi_D^f \left[(\nabla f)^{-1} (\nabla f(y) - \beta_k S(y)) \right] \right\rangle \geq D_f \left(\Pi_D^f \left[(\nabla f)^{-1} (\nabla f(y) - \beta_k S(y)) \right], y \right). \tag{3.15}$$

Proof. Lemma 2.5 of [28]. □

Remark 3.4. Note from Algorithm 3.1 and Lemma 3.3 the following hold.

$$\begin{aligned} \left\langle T(d_k), a_k - \Pi_C^g(z_k) \right\rangle &\geq (1-\sigma) D_g \left(\Pi_C^g(z_k), a_k \right), \\ \left\langle S(e_k), b_k - \Pi_D^f(u_k) \right\rangle &\geq (1-\sigma) D_f \left(\Pi_D^f(u_k), b_k \right). \end{aligned} \tag{3.16}$$

Proof. From Step 3 of the algorithm we have

$$\left\langle T(d_m), a_k - \Pi_C^g(z_k) \right\rangle \geq \left\langle T(a_k), a_k - \Pi_C^g(z_k) \right\rangle - \sigma D_g \left(\Pi_C^g(z_k), a_k \right). \tag{3.17}$$

From Lemma 3.3, we get

$$\left\langle T(a_k), a_k - \Pi_C^g(z_k) \right\rangle \geq D_g \left(\Pi_C^g(z_k), a_k \right). \tag{3.18}$$

Hence,

$$\left\langle T(d_m), a_k - \Pi_C^g(z_k) \right\rangle \geq (1-\sigma) D_g \left(\Pi_C^g(z_k), a_k \right). \tag{3.19}$$

Therefore, for $m = m_k$, we obtain

$$\left\langle T(d_k), a_k - \Pi_C^g(z_k) \right\rangle \geq (1-\sigma) D_g \left(\Pi_C^g(z_k), a_k \right). \tag{3.20}$$

Similarly, the relation (3.16) holds. □

Lemma 3.5. *Assume the conditions* $(H_1) - (H_5)$ *hold. If* $\{a_k\}$ *and* $\{b_k\}$ *are sequences generated by Algorithm 3.1, we get*

(i) $DVI(C,T) \subseteq C \cap H_k$ *and* $DVI(D,S) \subseteq D \cap R_k$*, for all* $k \geq 1$*;*

(ii) $h_k(a_k) \geq c_k (1-\sigma) D_g \left(\Pi_C^g(z_k), a_k \right)$ *and* $w_k(b_k) \geq t_k (1-\sigma) D_g \left(\Pi_D^g(u_k), b_k \right)$ *for* $c_k = \gamma^{m_k}$*,* $t_k = \gamma^{l_k}$*,* $h_k(v) = \langle T d_k, v - d_k \rangle$ *for* $v \in C$ *and* $w_k(u) = \langle S(e_k), u - e_k \rangle$*, for* $u \in D$*, for all* $k \geq 1$*.*

Proof. Let $(x^*, y^*) \in \Omega$. Then $\langle T(d_k), x^* - d_k \rangle \leq 0$ from which it follows that $x^* \in C \cap H_k$ and so $DVI(C,T) \subset C \cap H_k$, for all $k \geq 1$. Similarly, we get $DVI(D,S) \subset D \cap R_k$, for all $k \geq 1$.

(ii) From Step 4 and Remark 3.4, we obtain that

$$\begin{aligned} h_k(a_k) = \langle T(d_k), a_k - d_k \rangle &= c_k \left\langle T(d_k), a_k - \Pi_C^g(z_k) \right\rangle \\ &\geq c_k (1-\sigma) D_g \left(\Pi_C^g(z_k), a_k \right), \end{aligned}$$

which yields that the first part of (ii) holds. Similarly, we obtain that the second part of (ii) holds. □

Lemma 3.6. *Assume the conditions* $(H_1)-(H_5)$ *hold. Let* $(x^*,y^*)\in\Omega$. *Then, the sequences* $\{x_k\}$, $\{a_k\}$, $\{y_k\}$ *and* $\{b_k\}$ *generated by Algorithm 3.1 satisfy the following relations.*

(1) $D_g(x_{k+1},a_k)\le D_g(x^*,x_k)-D_g(x^*,x_{k+1})-\gamma_k\langle A^*J_{\mathbb{E}_3}(Ax_k-By_k),p_k-x^*\rangle$;
$D_f(y_{k+1},b_k)\le D_f(y^*,y_k)-D_f(y^*,y_{k+1})-\gamma_k\langle B^*J_{\mathbb{E}_3}(By_k-Ax_k),t_k-y^*\rangle.$
where

$$p_k=(\nabla g)^{-1}\left[\nabla g(x_k)-\gamma_kA^*(J_{\mathbb{E}_3}(Ax_k-By_k))\right]$$

and

$$t_k=(\nabla f)^{-1}\left[\nabla f(y_k)-\gamma_kB^*(J_{\mathbb{E}_3}(By_k-Ax_k))\right].$$

(2) $\{x_k\}$ and $\{y_k\}$ are bounded.

Proof. (1) From the three point identity, taking $z=x_{k+1}$, $x=a_k$, Lemma 2.5 and (2.5) we get that:

$$D_g(y,x_{k+1})+D_g(x_{k+1},a_k)-D_g(y,a_k)=\langle\nabla g(x_{k+1})-\nabla g(a_k),x_{k+1}-y\rangle\le 0,\quad\forall y\in C\cap H_k. \tag{3.21}$$

Let $y=x^*$ in (3.21), then

$$D_g(x_{k+1},a_k)\le D_g(x^*,a_k)-D_g(x^*,x_{k+1}). \tag{3.22}$$

Let $p_k=(\nabla g)^{-1}\left[\nabla g(x_k)-\gamma_kA^*(J_{\mathbb{E}_3}(Ax_k-By_k)\right]$ and $t_k=(\nabla f)^{-1}[\nabla f(f_k)-\gamma_kB^*(J_{\mathbb{E}_3}(Ay_k-Bx_k))]$. From (2.6), (2.9), (2.10) and (3.2) we have

$$\begin{aligned}D_g(x^*,a_k)\le D_g(x^*,p_k)&=D_g\left(x^*,(\nabla g)^{-1}\left[\nabla g(x_k)-\gamma_kA^*(J_{\mathbb{E}_3}(Ax_k-By_k)\right]\right)\\&=V_g\left(x^*,\nabla g(x_k)-\gamma_kA^*\left(J_{\mathbb{E}_3}(Ax_k-By_k)\right)\right)\\&\le V_g\left(x^*,\nabla g(x_k)\right)-\gamma_k\langle A^*(J_{\mathbb{E}_3}(Ax_k-By_k)),p_k-x^*\rangle\\&=D_g(x^*,x_k)-\gamma_k\langle A^*J_{\mathbb{E}_3}(Ax_k-By_k),p_k-x^*\rangle.\end{aligned} \tag{3.23}$$

Combining (3.22) and (3.23), we get

$$D_g(x_{k+1},a_k)\le D_g(x^*,x_k)-D_g(x^*,x_{k+1})-\gamma_k\langle A^*J_{\mathbb{E}_3}(Ax_k-By_k),p_k-x^*\rangle. \tag{3.24}$$

Following the same approach, we arrive at

$$D_f(y_{k+1},b_k)\le D_f(y^*,y_k)-D_f(y^*,y_{k+1})-\gamma_k\langle B^*J_{\mathbb{E}_3}(By_k-Ax_k),t_k-y^*\rangle. \tag{3.25}$$

(2) From the fact that $D_g(x_{k+1}, a_k) \geq 0$, $D_f(y_{k+1}, b_k) \geq 0$, (3.24), (3.25) and $Ax^* = By^*$, we have

$$\begin{aligned}
&D_g(x_{k+1}, a_k) + D_f(y_{k+1}, b_k) \\
&\leq D_g(x^*, x_k) - D_g(x^*, x_{k+1}) - \gamma_k \langle A^* J_{\mathbb{E}_3}(Ax_k - By_k), p_k - x^* \rangle \\
&\quad + D_f(y^*, y_k) - D_f(y^*, y_{k+1}) - \gamma_k \langle B^* J_{\mathbb{E}_3}(By_k - Ax_k), t_k - y^* \rangle \\
&= D_g(x^*, x_k) - D_g(x^*, x_{k+1}) + D_f(y^*, y_k) - D_f(y^*, y_{k+1}) \\
&\quad - \gamma_k \langle J_{\mathbb{E}_3}(Ax_k - By_k), Ap_k - Bt_k \rangle .
\end{aligned} \tag{3.26}$$

Furthermore, we have

$$\begin{aligned}
&- \langle J_{\mathbb{E}_3}(Ax_k - By_k), Ap_k - Bt_k \rangle \\
&= - \langle J_{\mathbb{E}_3}(Ax_k - By_k), Ax_k - By_k \rangle - \langle J_{\mathbb{E}_3}(Ax_k - By_k), Ap_k - Ax_k \rangle \\
&\quad - \langle J_{\mathbb{E}_3}(Ax_k - By_k), By_k - Bt_k \rangle \\
&= -||Ax_k - By_k||^2 - \langle A^* J_{\mathbb{E}_3}(Ax_k - By_k), p_k - x_k \rangle \\
&\quad - \langle B^* J_{\mathbb{E}_3}(Ax_k - By_k), y_k - t_k \rangle \\
&\leq -||Ax_k - By_k||^2 + ||p_k - x_k|| ||A^* J_{\mathbb{E}_3}(Ax_k - By_k)|| \\
&\quad + ||y_k - t_k|| ||B^* J_{\mathbb{E}_3}(Ax_k - By_k)||.
\end{aligned} \tag{3.27}$$

Using the Lipschitz continuity of $(\nabla g)^{-1}$, we get

$$\begin{aligned}
||p_k - x_k|| &= ||(\nabla g)^{-1} [\nabla g(x_k) - \gamma_k A^* J_{\mathbb{E}_3}(Ax_k - By_k)] - (\nabla g)^{-1} (\nabla g(x_k))|| \\
&\leq \frac{\gamma_k}{\beta} ||A^* J_{\mathbb{E}_3}(Ax_k - By_k)||.
\end{aligned} \tag{3.28}$$

Similarly, the Lipschitz continuity of $(\nabla f)^{-1}$ yields

$$||t_k - y_k|| \leq \frac{\gamma_k}{\beta} ||B^* J_{\mathbb{E}_3}(By_k - Ax_k)||. \tag{3.29}$$

Combining (3.27), (3.28), (3.29) and making use of (3.3), we obtain

$$\begin{aligned}
&- \gamma_k \langle J_{\mathbb{E}_3}(Ax_k - By_k), Ap_k - Bt_k \rangle \\
&\leq -\gamma_k ||Ax_k - By_k||^2 + \frac{\gamma_k^2}{\beta} ||A^* J_{\mathbb{E}_3}(Ax_k - By_k)||^2 + \frac{\gamma_k^2}{\beta} ||B^* J_{\mathbb{E}_3}(By_k - Ax_k)||^2 \\
&\leq -\frac{\rho}{2} ||Ax_k - By_k||^2 - \frac{\gamma_k}{2} ||Ax_k - By_k||^2 + \frac{\gamma_k}{2} \left(\frac{2\gamma_k}{\beta} \left[||A^* J_{\mathbb{E}_3}(Ax_k - By_k)||^2 \right] \right) \\
&\quad + \frac{\gamma_k}{2} \left(\frac{2\gamma_k}{\beta} \left[||B^* J_{\mathbb{E}_3}(By_k - Ax_k)||^2 \right] \right) \leq -\frac{\rho}{2} ||Ax_k - By_k||^2.
\end{aligned} \tag{3.30}$$

As such, from (3.26) and (3.30), we obtain

$$\begin{aligned}
D_g(x_{k+1}, a_k) + D_f(y_{k+1}, b_k) \leq\ & D_g(x^*, x_k) - D_g(x^*, x_{k+1}) - D_f(y^*, y_{k+1}) \\
&+ D_f(y^*, y_k) - \frac{\rho}{2} ||Ax_k - By_k||^2.
\end{aligned} \tag{3.31}$$

Hence,

$$D_g(x^*, x_{k+1}) + D_f(y^*, y_{k+1}) \leq D_g(x^*, x_k) + D_f(y^*, y_k) - \frac{\rho}{2}||Ax_k - By_k||^2. \tag{3.32}$$

We get that $\{D_g(x^*, x_k) + D_f(y^*, y_k)\}$ is decreasing and bounded below. Thus, $\{D_g(x^*, x_k)\}$ and $\{D_f(y^*, y_k)\}$ are bounded which implies that $\{x_k\}$ and $\{y_k\}$ are bounded. □

Lemma 3.7. *Assume that conditions $(H_1) - (H_5)$ hold. Then, the sequences $\{x_k\}$, $\{a_k\}$, $\{y_k\}$ and $\{b_k\}$ generated by Algorithm 3.1 satisfy the following relations.*

(1) $\lim_{k\to\infty} D_g(x_{k+1}, a_k) = 0$ *and* $\lim_{k\to\infty} D_f(y_{k+1}, b_k) = 0$.

(2) $D_g(x_{k+1}, a_k) \geq \frac{\alpha c_k^2}{L_1^2}(1-\sigma)^2 D_g\left(\Pi_C^g(z_k), a_k\right)$ *and*
$D_f(y_{k+1}, b_k) \geq \frac{\alpha t_k^2}{L_2^2}(1-\sigma)^2 D_f\left(\Pi_D^f(u_k), b_k\right)$.

Proof. (1) From (3.31), we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty} D_g(x_{k+1}, a_k)\\
&\leq \sum_{k=0}^{\infty} D_g(x_{k+1}, a_k) + D_f(y_{k+1}, b_k)\\
&\leq \sum_{k=0}^{\infty} \left(D_g(x^*, x_k) - D_g(x^*, x_{k+1}) - D_f(y^*, y_{k+1}) + D_f(y^*, y_k)\right)\\
&\leq D_g(x^*, x_0) + D_f(y^*, y_0),
\end{aligned} \tag{3.33}$$

which implies that

$$\lim_{k\to\infty} D_g(x_{k+1}, a_k) = 0. \tag{3.34}$$

Again with a similar approach, we have

$$\lim_{k\to\infty} D_f(y_{k+1}, b_k) = 0.$$

2) From Lemma 3.6, (3.23) and (3.34), we have that $\{a_k\}$ is bounded. Similarly, $\{b_k\}$ is bounded. Using (H_5) and the fact that T is uniformly continuous on bounded subsets of $\mathbb{E}_1$, we get that $\{z_k\}$ is bounded. Also by Lemma 2.6, we get that $\left\{\Pi_C^g(z_k)\right\}$ is bounded. Thus, we get that $\{d_k\}$ is bounded and hence $\{T(d_k)\}$ is bounded. Therefore, there exists $L_1 > 0$ such that

$$||T(d_k)|| \leq L_1, \forall k \geq 1.$$

Since $h_k(v) = \langle T(d_k), v - d_k\rangle$, we get

$$\begin{aligned}
|h_k(a_k) - h_k(x_{k+1})| &= |\langle T(d_k), a_k - x_{k+1}\rangle|\\
&\leq ||T(d_k)||||a_k - x_{k+1}||\\
&\leq L_1||a_k - x_{k+1}||,
\end{aligned}$$

which implies that h_k is Lipschitz continuous on C. Using Lemmas 3.5 and 2.11, we obtain

$$D_g(x_{k+1}, a_k) \geq \frac{\alpha c_k^2}{L_1^2}(1-\sigma)^2 D_g^2\left(\Pi_C^g(z_k), a_k\right).$$

Similarly, we get

$$D_f(y_{k+1}, b_k) \geq \frac{\alpha t_k^2}{L_2^2}(1-\sigma)^2 D_f^2\left(\Pi_D^f(u_k), b_k\right). \tag{3.35}$$

□

Lemma 3.8. *Assume that conditions* $(H_1) - (H_5)$ *hold. If* $\{a_k\}$ *and* $\{b_k\}$ *are sequences generated by Algorithm 3.1, then* $VI(C,T)$ *and* $VI(D,S)$ *contain weak accumulation points of* $\{a_k\}$ *and* $\{b_k\}$*, respectively.*

Proof. From Lemma 3.7 (2), we have

$$\lim_{k\to\infty} D_g(x_{k+1}, a_k) \geq \lim_{k\to\infty} \frac{\alpha c_k^2}{L_1^2}(1-\sigma)^2 D_g^2\left(\Pi_C^g(z_k), a_k\right). \tag{3.36}$$

Since $\alpha > 0$, inequality (3.36) and equation (3.34) imply that

$$\lim_{k\to\infty} c_k D_g\left(\Pi_C^g(z_k), a_k\right) = 0. \tag{3.37}$$

Let $\{a_{k_i}\}$ be a sub-sequence of $\{a_k\}$ which converges weakly to a^*. If there exists $N > 0$ such that $a_{k_i} = \Pi_C^g z_{k_i}$ for all $i \geq N$, then $Ta_{k_i} = 0$, for all $i \geq N$ and hence by (H_2), we get that $0 \leq ||Ta^*|| \leq \liminf_{i\to\infty} ||Ta_{k_i}|| = 0$, which implies that $a^* \in VI(C,T)$. Otherwise, we can take a subsequence of $\{a_{k_i}\}$, with no loss of generality, which could still be denoted $\{a_{k_i}\}$ such that $a_{k_i} \neq \Pi_C^g z_{k_i}$ for all $i \geq 1$. In this case, the equality in (3.37) yields

$$\lim_{i\to\infty} c_{k_i} D_g\left(\Pi_C^g(z_{k_i}), a_{k_i}\right) = 0. \tag{3.38}$$

Now, we show that $a^* \in VI(T,C)$ under two cases.
Case 1. If $\limsup_{i\to\infty} c_{k_i} > 0$, then there exists a sub-sequence, with no loss of generality, still denoted by $\{a_{k_i}\}$, and constant $\theta > 0$, such that $c_{k_i} > \theta$, for $i \geq N$ and for some $N > 0$. From (3.38), we get

$$\lim_{i\to\infty} D_g\left(\Pi_C^g(z_{k_i}), a_{k_i}\right) = 0. \tag{3.39}$$

So by Lemma 2.3, we have

$$\lim_{i\to\infty} ||\Pi_C^g(z_{k_i}) - a_{k_i}|| = 0. \tag{3.40}$$

On the other hand, (2.5) implies that

$$\left\langle \nabla g(z_{k_i}) - \nabla g\left(\Pi_C^g(z_{k_i})\right), y - \Pi_C^g(z_{k_i})\right\rangle \leq 0, \forall y \in C. \tag{3.41}$$

Now, from the definition of z_{k_i} and (3.41), we obtain

$$\left\langle \nabla g(a_{k_i}) - \nabla g\left(\Pi_C^g(z_{k_i})\right), y - \Pi_C^g(z_{k_i})\right\rangle \leq \beta_k \left\langle T(a_{k_i}), y - \Pi_C^g(z_{k_i})\right\rangle. \tag{3.42}$$

Consequently, we get

$$\left\langle \nabla g\left(a_{k_i}\right) - \nabla g\left(\Pi_C^g\left(z_{k_i}\right)\right), y - \Pi_C^g\left(z_{k_i}\right)\right\rangle - \beta_k\left\langle T\left(a_{k_i}\right), a_{k_i} - \Pi_C^g\left(z_{k_i}\right)\right\rangle \leq \beta_k\left\langle T\left(a_{k_i}\right), y - a_{k_i}\right\rangle. \tag{3.43}$$

From the uniform continuity of ∇g, the fact that $\beta_k \geq a > 0$ for all $k \geq 1$, (3.40), boundedness of $\{a_{k_i}\}$ and $\left\{\Pi_C^g\left(z_{k_i}\right)\right\}$, (see Lemma 3.7 (2)), and letting $i \to \infty$ in (3.43), we get that

$$\liminf_{i\to\infty}\left\langle T\left(a_{k_i}\right), y - a_{k_i}\right\rangle \geq 0. \tag{3.44}$$

Hence, for any $\epsilon > 0$, $\exists N > 0$ such that for $i \geq N$,we have

$$\left\langle T\left(a_{k_i}\right), y - a_{k_i}\right\rangle + \epsilon \geq 0. \tag{3.45}$$

Since $a_{k_i} \neq \Pi_C^g(z_{k_i})$, we have $T\left(a_{k_i}\right) \neq 0$. Take $v_{k_i} \in \mathbb{E}_1$ such that $\left\langle T\left(a_{k_i}\right), v_{k_i}\right\rangle = 1$, then

$$\left\langle T\left(a_{k_i}\right), y + \epsilon v_{k_i} - a_{k_i}\right\rangle \geq 0,\ \forall i \geq N. \tag{3.46}$$

So, by Lemma 2.10, either

$$\left\langle T\left(y + \epsilon v_{k_i}\right), y + \epsilon v_{k_i} - a_{k_i}\right\rangle \geq 0,\ \forall i \geq N \tag{3.47}$$

or

$$\left\langle T\left(a_{k_i}\right), z - a_{k_i}\right\rangle \geq 0,\quad \forall z \in C,\quad \forall i \geq N. \tag{3.48}$$

However, the inequality in (3.48) suggests that $a_{k_i} = \Pi_C^g\left(z_{k_i}\right)$, which contradicts the assumption that $a_{k_i} \neq \Pi_C^g(z_{k_i})$. So, we get (3.47) that can be expressed as

$$\left\langle T(y), y - a_{k_i}\right\rangle \geq \left\langle T(y) - T\left(y + \epsilon v_{k_i}\right), y + \epsilon v_{k_i} - a_{k_i}\right\rangle - \epsilon\left\langle T(y), v_{k_i}\right\rangle, \forall i \geq N. \tag{3.49}$$

Taking $\epsilon \to 0$ and using the continuity of T and boundedness of $\{a_{k_i}\}$, we get

$$\left\langle T(y), y - a_{k_i}\right\rangle \geq 0, \forall y \in C. \tag{3.50}$$

Hence, letting $i \to \infty$, we get

$$\left\langle T(y), y - a^*\right\rangle \geq 0, \forall y \in C. \tag{3.51}$$

Therefore, $a^* \in DVI(C,T)$ and hence $a^* \in VI(C,T)$.

Case 2. If $\limsup_{i\to\infty} c_{k_i} = 0$, then we get $\lim_{i\to\infty} c_{k_i} = 0$. Now, we show that $\lim_{i\to\infty} D_g\left(\Pi_C^g\left(z_{k_i}\right), a_{k_i}\right) = 0$. Take $\overline{d}_{k_i} = \frac{c_{k_i}}{\gamma}\Pi_C^g\left(z_{k_i}\right) + \left(1 - \frac{c_{k_i}}{\gamma}\right)a_{k_i}$ which implies $\overline{d}_{k_i} - a_{k_i} = \frac{c_{k_i}}{\gamma}\left(\Pi_C^g\left(z_{k_i}\right) - a_{k_i}\right)$. From the boundedness of $\left\{\Pi_C^g\left(z_{k_i}\right) - a_{k_i}\right\}$ and the fact that $\lim_{i\to\infty} c_{k_i} = 0$, we get

$$\lim_{i\to\infty} ||\overline{d}_{k_i} - a_{k_i}|| = 0. \tag{3.52}$$

Hence, from the definition of c_{k_i} and Step 3 of the Algorithm 3.1, we obtain

$$\left\langle T\left(a_{k_i}\right) - T\left(\overline{d}_{k_i}\right), a_{k_i} - \Pi_C^g\left(z_{k_i}\right)\right\rangle > \sigma D_g\left(\Pi_C^g\left(z_{k_i}\right), a_{k_i}\right). \tag{3.53}$$

Applying the fact that T is uniformly continuous on bounded subsets of $\mathbb{E}_1$, $\sigma > 0$ and boundedness of $\left\{\Pi_C^g\left(z_{k_i}\right)\right\}$ and $\{a_{k_i}\}$, we have

$$\lim_{i\to\infty} D_g\left(\Pi_C^g\left(z_{k_i}\right), a_{k_i}\right) = 0. \tag{3.54}$$

Then following the argument in Case 1, we arrive at the desired conclusion. □

Remark 3.9. Note that, if in Lemma 3.8, we assume that $T(x) \neq 0$ for all $x \in C$ and $S(y) \neq 0$ for all $y \in D$, then the method of proof of Lemma 3.8 provides that $DVI(C,T)$ and $DVI(D,S)$ contain the weak accumulation points of $\{a_k\}$ and $\{b_k\}$, respectively.

In the sequel, we shall use the following lemma.

Lemma 3.10. *Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be real Banach spaces and $g : \mathbb{E}_1 \to (-\infty,\infty], f : \mathbb{E}_2 \to (-\infty,\infty]$ be proper and strictly convex Gâteaux differentiable functions such that ∇f and ∇g are weakly sequentially continuous mappings. Suppose the sequence $\{(x_k, y_k)\}$ in $\mathbb{E}_1 \times \mathbb{E}_2$ converges weakly to $(\overline{x}, \overline{y})$. Then*

$$\limsup_{k\to\infty}[D_g(\overline{x}, x_k) + D_f(\overline{y}, y_k)] < \limsup_{k\to\infty}[D_g(\hat{x}, x_k) + D_f(\hat{y}, y_k)] \tag{3.55}$$

for any point $(\hat{x}, \hat{y}) \in \mathbb{E}_1 \times \mathbb{E}_2$ such that $(\hat{x}, \hat{y}) \neq (\overline{x}, \overline{y})$.

Proof. Using the Bregman distance definition, we get that

$$\begin{aligned}
&D_g(\overline{x}, x_k) + D_f(\overline{y}, y_k) - [D_g(\hat{x}, x_k) + D_f(\hat{y}, y_k)]\\
&= g(\overline{x}) - g(x_k) - \langle \overline{x} - x_k, \nabla g(x_k)\rangle + f(\overline{y}) - f(y_k) - \langle \overline{y} - y_k, \nabla f(y_k)\rangle\\
&\quad - [g(\hat{x}) - g(x_k) - \langle \hat{x} - x_k, \nabla g(x_k)\rangle + f(\hat{y}) - f(y_k) - \langle \hat{y} - y_k, \nabla f(y_k)\rangle]\\
&= g(\overline{x}) - g(\hat{x}) - \langle \overline{x} - \hat{x}, \nabla g(x_k)\rangle + f(\overline{y}) - f(\hat{y}) - \langle \overline{y} - \hat{y}, \nabla f(y_k)\rangle\\
&= -[g(\hat{x}) - g(\overline{x}) - \langle \hat{x} - \overline{x}, \nabla g(x_k)\rangle + f(\hat{y}) - f(\overline{y}) - \langle \hat{y} - \overline{y}, \nabla f(y_k)\rangle]\\
&= -[g(\hat{x}) - g(\overline{x}) - \langle \hat{x} - \overline{x}, \nabla g(x_k) + \nabla g(\overline{x}) - \nabla g(\overline{x})\rangle + f(\hat{y}) - f(\overline{y})\\
&\quad - \langle \hat{y} - \overline{y}, \nabla f(y_k) - \nabla f(\overline{y}) + \nabla f(\overline{y})\rangle]\\
&= -[g(\hat{x}) - g(\overline{x}) - \langle \hat{x} - \overline{x}, \nabla g(\overline{x})\rangle - \langle \hat{x} - \overline{x}, \nabla g(x_k) - \nabla g(\overline{x})\rangle\\
&\quad + f(\hat{y}) - f(\overline{y}) - \langle \hat{y} - \overline{y}, \nabla f(\overline{y})\rangle\\
&\quad\; - \langle \hat{y} - \overline{y}, \nabla f(y_k) - \nabla f(\overline{y})\rangle]\\
&= -D_g(\hat{x}, \overline{x}) + \langle \hat{x} - \overline{x}, \nabla g(x_k) - \nabla g(\overline{x})\rangle - D_f(\hat{y}, \overline{y})\\
&\quad + \langle \hat{y} - \overline{y}, \nabla f(y_k) - \nabla f(\overline{y})\rangle .
\end{aligned} \tag{3.56}$$

Now, using the fact that ∇f and ∇g are weakly sequentially continuous we get that $\limsup_{k\to\infty}[D_g(\overline{x}, x_k)+D_f(\overline{y}, y_k)-[D_g(\hat{x}, x_k)+D_f(\hat{y}, y_k)]] = -D_g(\hat{x}, \overline{x})-D_f(\hat{y}, \overline{y})$. Consequently,

$$\begin{aligned}
&\limsup_{k\to\infty}[D_g(\overline{x}, x_k) + D_f(\overline{y}, y_k)]\\
&\leq \limsup_{k\to\infty}[D_g(\overline{x}, x_k) + D_f(\overline{y}, y_k) - D_g(\hat{x}, x_k) - D_f(\hat{y}, y_k)]\\
&\quad + \limsup_{k\to\infty}[D_g(\hat{x}, x_k) + D_f(\hat{y}, y_k)]\\
&= -D_g(\hat{x}, \overline{x}) - D_f(\hat{y}, \overline{y}) + \limsup_{k\to\infty}[D_g(\hat{x}, x_k) + D_f(\hat{y}, y_k)]
\end{aligned}$$

$$< \limsup_{k\to\infty}[D_g(\hat{x}, x_k) + D_f(\hat{y}, y_k)]. \tag{3.57}$$

Therefore, the conclusion of the lemma holds. □

Theorem 3.11. *Assume that conditions $(H_1)-(H_5)$ hold. In addition, let ∇g and ∇f be weakly sequentially continuous mappings. The sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.1 converges weakly to a point in Ω^*, where $\Omega^* = \{(x^*, y^*) \in VI(C,T) \times VI(D,S) : Ax^* = By^*\}$.*

Proof. Let $(x^*, y^*) \in \Omega^*$. From Lemma 3.6, we obtain that

$$\Omega^*_{k+1} \leq \Omega^*_k - \frac{\rho}{2}||Ax_k - By_k||^2, \tag{3.58}$$

where $\Omega^*_k = D_g(x^*, x_k) + D_f(y^*, y_k)$. Hence, $\{\Omega^*_k\}$ is a decreasing and hence convergent sequence. Thus (3.58) implies $||Ax_k - By_k|| \to 0$ as $k \to \infty$. Since $\{(x_k, y_k)\}$ is bounded by Lemma 3.6 , a sub-sequence $\{(x_{k_i}, y_{k_i})\}$ of $\{(x_k, y_k)\}$ such that $(x_{k_i}, y_{k_i}) \rightharpoonup (\hat{x}, \hat{y})$ exists. This implies that $x_{k_i} \rightharpoonup \hat{x}$ and $y_{k_i} \rightharpoonup \hat{y}$. By Lemmas 3.7 and 2.8, we get $a_{k_i-1} \rightharpoonup \hat{x}$ and $b_{k_i-1} \rightharpoonup \hat{y}$ as $i \to \infty$. Thus, by Lemma 3.8, we get that $\hat{x} \in VI(C,T)$ and $\hat{y} \in VI(D,S)$. Moreover, from (3.58) and the fact that A and B are bounded linear mappings and $\{\Omega_k\}$ is convergent, we have $A\hat{x} = B\hat{y}$. Hence, $(\hat{x}, \hat{y}) \in \Omega^*$.

Now we prove that $(x_k, y_k) \rightharpoonup (\hat{x}.\hat{y})$ as $k \to \infty$. Suppose to the contrary that this is not the case. Then there exists a sub-sequence $\{(x_{k_j}, y_{k_j})\}$ of $\{(x_k, y_k)\}$ such that $(x_{k_j}, y_{k_j}) \rightharpoonup (\overline{x}, \overline{y})$ as $j \to \infty$, where $(\bar{x}, \bar{y}) \neq (\hat{x}, \hat{y})$. This implies that $x_{k_j} \rightharpoonup \overline{x}$ and $y_{k_j} \rightharpoonup \overline{y}$ as $j \to \infty$. Note that one may also show that $(\overline{x}, \overline{y}) \in \Omega^*$. Now, using the fact that $\{\Omega^*_k\}$ is convergent and the property in Lemma 3.10, we get

$$\begin{aligned}
&\lim_{k\to\infty} [D_g(\hat{x}, x_{k_i}) + D_f(\hat{y}, y_{k_i})] \\
&= \limsup_{i\to\infty} [D_g(\hat{x}, x_{k_i}) + D_f(\hat{y}, y_{k_i})] \\
&< \limsup_{i\to\infty} [D_g(\overline{x}, x_{k_i}) + D_f(\overline{y}, y_{k_i})] \\
&= \lim_{k\to\infty} [D_g(\overline{x}, x_k) + D_f(\overline{y}, y_k)] = \limsup_{j\to\infty} \left[D_g(\overline{x}, x_{k_j}) + D_f(\overline{y}, y_{k_j})\right] \\
&< \limsup_{j\to\infty} \left[D_g(\hat{x}, x_{k_j}) + D_f(\hat{y}, y_{k_j})\right] = \lim_{k\to\infty} [D_g(\hat{x}, x_{k_i}) + D_f(\hat{y}, y_{k_i})],
\end{aligned}$$

which is a contradiction. This yields that $(\hat{x}, \hat{y}) = (\overline{x}, \overline{y})$. Therefore, the whole sequence $\{(x_k, y_k)\}$ converges weakly to $(\hat{x}, \hat{y}) \in \Omega^*$ as $k \to \infty$ as desired. □

Next, we propose Halpern-type projection-based algorithm for approximating a solution of variational problems in Banach spaces.

Algorithm 3.2 Choose $x_0 \in \mathbb{E}_1$, $y_0 \in \mathbb{E}_2$ and two parameters: $\gamma, \sigma \in (0, 1)$. Take $k = 0$. For arbitrary points $u \in C$ and $v \in D$, calculate $\{x_k\}$ and $\{y_k\}$ as follows:

Step 1: Compute

$$a_k = \Pi_C^g (\nabla g)^{-1} \left[\nabla g(x_k) - \gamma_k A^* \left(J_{\mathbb{E}_3}(Ax_k - By_k)\right)\right], \tag{3.59}$$

$$b_k = \Pi_D^f (\nabla f)^{-1} \left[\nabla f(y_k) - \gamma_k B^* \left(J_{\mathbb{E}_3}(By_k - Ax_k)\right)\right], \tag{3.60}$$

where $0 < \rho \leq \gamma_k \leq \rho_k$ with

$$\rho_k = \min\left\{\rho + 1, \frac{\beta \|Ax_k - By_k\|^2}{2[\|A^* J_{\mathbb{E}_3}(Ax_k - By_k)\|^2 + \|B^* J_{\mathbb{E}_3}(By_k - Ax_k)\|^2]}\right\},$$

for $k \in \Upsilon = \{m \in \mathbb{N} : Ax_m - By_m \neq 0\}$, otherwise $\gamma_k = \rho$.
Step 2: Compute

$$z_k = (\nabla g)^{-1} [\nabla g(a_k) - \beta_k T(a_k)], \;\; u_k = (\nabla f)^{-1} [\nabla f(b_k) - \beta_k S(b_k)],$$

where $\{\beta_k\} \subset [a, 1] \subset (0, 1]$ for all $k \geq 1$.
Step 3 : Compute

$$m_k = \min\left\{m \in \mathbb{N} : \left\langle T(a_k) - T(d_m), a_k - \Pi_C^g(z_k)\right\rangle \leq \sigma D_g \left(\Pi_C^g\left(z_k\right), a_k\right)\right\},$$

$$l_k = \min\left\{l \in \mathbb{N} : \left\langle S(b_k) - S(e_l), b_k - \Pi_D^f(u_k)\right\rangle \leq \sigma D_f \left(\Pi_D^f\left(u_k\right), b_k\right)\right\},$$

where

$$d_m = \gamma^m \Pi_C^g(z_k) + (1 - \gamma^m) a_k, \;\; e_l = \gamma^l \Pi_D^f(u_k) + (1 - \gamma^l) b_k.$$

Let

$$d_k = \gamma^{m_k} \Pi_C^g(z_k) + (1 - \gamma^{m_k}) a_k, \;\; e_k = \gamma^{l_k} \Pi_D^f(u_k) + (1 - \gamma^{l_k}) b_k.$$

Step 4 : Compute

$$x_{k+1} = \alpha_k u + (1 - \alpha_k) \Pi^g_{C \bigcap H_k}(a_k), \;\; y_{k+1} = \alpha_k v + (1 - \alpha_k) \Pi^f_{D \bigcap R_k}(b_k),$$

where

$$H_k = \{x \in \mathbb{E}_1 : \langle T(d_k), x - d_k\rangle \leq 0\} \text{ and } R_k = \{y \in \mathbb{E}_2 : \langle S(e_k), y - e_k\rangle \leq 0\}.$$

Step 5 : Let $k = k + 1$ and return to **Step 1**.

Remark 3.12. The novelty of the proposed Halpern-type projection-based method is that it provides strong convergence to the solution of the more general problem know as split equality dual variational inequality problems associated with the class of uniformly continuous quasimonotone mappings in Banach spaces. It does not require prior knowledge of the Lipschitz constants of the underlying mappings. It extends all the results in the literature from Hilbert spaces to the more general reflexive Banach spaces under mild conditions.

Lemma 3.13. *Assume that conditions $(H_1) - (H_6)$ hold. Then, the sequences $\{x_k\}$, $\{a_k\}$, $\{y_k\}$ and $\{b_k\}$ generated by Algorithm 3.2 are bounded.*

Proof. Let $(\overline{x},\overline{y})\in\Omega$. Then from (3.23), (2.11) and Lemma 2.13, we obtain that

$$\begin{aligned}
D_g(\overline{x},x_{k+1}) &= D_g(\overline{x},(\alpha_k u+(1-\alpha_k)\Pi^g_{C\bigcap H_k}(a_k))\\
&\le \alpha_k D_g(\overline{x},u)+(1-\alpha_k)D_g(\overline{x},\Pi^g_{C\bigcap H_k}(a_k))\\
&\le \alpha_k D_g(\overline{x},u)+(1-\alpha_k)D_g(\overline{x},a_k)\\
&\le \alpha_k D_g(\overline{x},u)+(1-\alpha_k)\left(D_g\left(\overline{x},x_k\right)-\gamma_k\left\langle A^*J_{\mathbb{E}_3}(Ax_k-By_k),p_k-\overline{x}\right\rangle\right). \qquad (3.61)
\end{aligned}$$

Similarly, we get

(3.62)
$$D_f(\overline{y},y_{k+1})\le \alpha_k D_f(\overline{y},v)+(1-\alpha_k)\left(D_f\left(\overline{y},y_k\right)-\gamma_k\left\langle B^*J_{\mathbb{E}_3}(By_k-Ax_k),t_k-\overline{y}\right\rangle\right).$$

Denote: $\Omega_k = D_g(\overline{x},x_k)+D_f(\overline{y},y_k)$. Then, from (3.61), (3.62) and (3.30) we obtain that

$$\begin{aligned}
\Omega_{k+1} &\le (1-\alpha_k)\Omega_k+\alpha_k\left[D_g(\overline{x},u)+D_f(\overline{y},v)\right]-\frac{\rho}{2}||Ax_k-By_k||^2\\
&\le (1-\alpha_k)\Omega_k+\alpha_k\left[D_g(\overline{x},u)+D_f(\overline{y},v)\right]\\
&\le \max\left\{\Omega_k, D_g(\overline{x},u)+D_f(\overline{y},v)\right\}\\
&\quad\vdots\\
&\le \max\left\{\Omega_0, D_g(\overline{x},u)+D_f(\overline{y},v)\right\},\forall k\ge 0. \qquad (3.63)
\end{aligned}$$

Thus, the sequence $\{\Omega_k\}$ is bounded and hence the sequences $\{D_g(\overline{x},x_k)\}$ and $\{D_f(\overline{y},y_k)\}$ are bounded. Therefore, by Lemma 2.8, we have $\{x_k\}$ and $\{y_k\}$ are bounded.

□

Theorem 3.14. *Assume that conditions $(H_1)-(H_6)$ hold. Let $T(x)\neq 0$, for all $x\in C$ and $S(y)\neq 0$, for all $y\in D$. Then, the sequence $\{(x_k,y_k)\}$ generated by Algorithm 3.2 converges strongly to $(x^*,y^*)=\Pi^h_\Omega(u,v)$, where $h(x,y)=(g(x),f(y))$.*

Proof. Let $(x^*,y^*)=\Pi^h_\Omega(u,v)$. Then, from (2.5) we get that

(3.64) $$\langle(\nabla g(u),\nabla f(v))-(\nabla g(x^*),\nabla f(y^*)),(s,t)-(x^*,y^*)\rangle\le 0\ \forall(s,t)\in\Omega.$$

From (2.9), (2.10), Lemma 2.13 and (3.24), we have

$$\begin{aligned}
D_g(x^*,x_{k+1}) &= D_g(x^*,(\nabla g)^{-1}\big(\alpha_k\nabla g(u)+(1-\alpha_k)\nabla g(\Pi^g_{C\bigcap H_k}(a_k))\big)\\
&= V_g(x^*,(\alpha_k\nabla g(u)+(1-\alpha_k)\nabla g(\Pi^g_{C\bigcap H_k}(a_k)))\\
&\le V_g(x^*,(1-\alpha_k)\nabla g(\Pi^g_{C\bigcap H_k}(a_k))+\alpha_k\nabla g(x^*))\\
&\quad+\alpha_k\left\langle\nabla g(u)-\nabla g(x^*),x_{k+1}-x^*\right\rangle\\
&= D_g(x^*,(\nabla g)^{-1}((1-\alpha_k)\nabla g(\Pi^g_{C\bigcap H_k}(a_k))+\alpha_k\nabla g(x^*))+\alpha_k\nabla g(x^*)\\
&\quad+\alpha_k\left\langle\nabla g(u)-\nabla g(x^*),x_{k+1}-x^*\right\rangle\\
&\le (1-\alpha_k)D_g(x^*,\Pi^g_{C\bigcap H_k}(a_k))+\alpha_k\left\langle\nabla g(u)-\nabla g(x^*),x_{k+1}-x^*\right\rangle \qquad (3.65)\\
&\le (1-\alpha_k)D_g(x^*,a_k)+\alpha_k\left\langle\nabla g(u)-\nabla g(x^*),x_{k+1}-x^*\right\rangle \qquad (3.66)\\
&\le (1-\alpha_k)D_g\left(x^*,x_k\right)-(1-\alpha_k)\gamma_k\left\langle A^*J_{\mathbb{E}_3}(Ax_k-By_k),p_k-x^*\right\rangle
\end{aligned}$$

$$+ \alpha_k \langle \nabla g(u) - \nabla g(x^*), x_{k+1} - x^* \rangle . \tag{3.67}$$

Similarly,

$$\begin{aligned} D_f(y^*, y_{k+1}) &\le (1-\alpha_k) D_f(y^*, y_k) - (1-\alpha_k)\gamma_k \langle B^* J_{\mathbb{E}_3}(By_k - Ax_k), t_k - y^* \rangle \\ &\quad + \alpha_k \langle \nabla f(v) - \nabla f(y^*), y_{k+1} - y^* \rangle . \end{aligned} \tag{3.68}$$

Let $\Omega_k = D_g(x^*, x_k) + D_f(y^*, y_k)$, then from (3.67), (3.68) and (3.30), we get

$$\begin{aligned} \Omega_{k+1} &\le (1-\alpha_k)\Omega_k - (1-\alpha_k)\frac{\rho}{2}||Ax_k - By_k||^2 \\ &\quad + \alpha_k \left[\langle \nabla g(u) - \nabla g(x^*), x_{k+1} - x^* \rangle + \langle \nabla f(v) - \nabla f(y^*), y_{k+1} - y^* \rangle \right] \qquad (3.69) \\ &\le (1-\alpha_k)\Omega_k \\ &\quad + \alpha_k \left[\langle \nabla g(u) - \nabla g(x^*), x_k - x^* \rangle + \langle \nabla f(v) - \nabla f(y^*), y_k - y^* \rangle \right] \\ &\quad + \alpha_k \left[\langle \nabla g(u) - \nabla g(x^*), x_{k+1} - x_k \rangle + \langle \nabla f(v) - \nabla f(y^*), y_{k+1} - y_k \rangle \right] \\ &\le (1-\alpha_k)\Omega_k \\ &\quad + \alpha_k \left[\langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_k, y_k) - (x^*, y^*) \rangle \right] \\ &\quad + \alpha_k \left[\langle \nabla g(u) - \nabla g(x^*), x_{k+1} - x_k \rangle + \langle \nabla f(v) - \nabla f(y^*), y_{k+1} - y_k \rangle \right] . \qquad (3.70) \end{aligned}$$

We now consider two cases on the sequence $\{\Omega_k\}$.

Case I. Suppose there exists a natural number N such that $\Omega_{k+1} \le \Omega_k$ for all $k \ge N$, then we have by the Monotone Convergence Theorem that the sequence $\{\Omega_k\}$ converges. Taking the limit as $k \to \infty$ in (3.69), we obtain

$$\lim_{k\to\infty} \|Ax_k - By_k\| = 0. \tag{3.71}$$

Moreover, since $\{(x_k, y_k)\}$ is bounded in $\mathbb{E}_1 \times \mathbb{E}_2$ which is reflexive, then there exists a subsequence $\{(x_{k_i}, y_{k_i})\}$ of $\{(x_k, y_k)\}$ such that $(x_{k_i}, y_{k_i}) \rightharpoonup (\overline{x}, \overline{y}) \in \mathbb{E}_1 \times \mathbb{E}_2$ and

$$\begin{aligned} &\limsup_{k\to\infty} \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_k, y_k) - (x^*, y^*) \rangle \\ &= \lim_{i\to\infty} \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_{k_i}, y_{k_i}) - (x^*, y^*) \rangle \\ &= \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (\overline{x}, \overline{y}) - (x^*, y^*) \rangle . \end{aligned} \tag{3.72}$$

Consequently, we have $x_{k_i} \rightharpoonup \overline{x}$ and $y_{k_i} \rightharpoonup \overline{y}$. Now, we show that $(\overline{x}, \overline{y}) \in \Omega$.

From the definition of a_k, (2.11), (2.9), (2.10), the property of the Bregman projection and the Cauchy Schwarz inequality, we have

$$\begin{aligned} D_g(x_k, a_k) &\le D_g\left(x_k, (\nabla g)^{-1}\left[\nabla g(x_k) - \gamma_k A^*(J_{\mathbb{E}_3}(Ax_k - By_k)\right]\right) \\ &= V_g\left(x_k, \nabla g(x_k) - \gamma_k A^*\left(J_{\mathbb{E}_3}(Ax_k - By_k)\right)\right) \\ &\le V_g\left(x_k, \nabla g(x_k)\right) - \gamma_k \langle A^*(J_{\mathbb{E}_3}(Ax_k - By_k), p_k - x_k \rangle \\ &= D_g(x_k, x_k) - \gamma_k \langle A^* J_{\mathbb{E}_3}(Ax_k - By_k), p_k - x_k \rangle \\ &\le \gamma_k ||A^* J_{\mathbb{E}_3}(Ax_k - By_k)||.||p_k - x_k||. \end{aligned} \tag{3.73}$$

Substituting (3.28) into (3.73), taking the limit on both sides and making use of the inequality in (3.71), we obtain

$$\begin{aligned} 0 &\le \lim_{k\to\infty} D_g(x_k, a_k) \\ &\le \lim_{k\to\infty} \big(\frac{\gamma_k^2}{\beta}\|A\|^2\|J_{\mathbb{E}_3}(Ax_k - By_k)\|^2\big) \\ &\le \lim_{k\to\infty} \big(\frac{\gamma_k^2}{\beta}\|A\|^2\|Ax_k - By_k\|^2\big) \\ &= 0. \end{aligned} \tag{3.74}$$

This implies that $\lim_{k\to\infty} D_g(x_k, a_k) = 0$ and hence by Lemma 2.3, we get

$$\lim_{k\to\infty} \|x_k - a_k\| = 0. \tag{3.75}$$

Similarly, we have that

$$\lim_{k\to\infty} \|y_k - b_k\| = 0. \tag{3.76}$$

Hence, we obtain that $a_{k_i} \rightharpoonup \overline{x}$ and $b_{k_i} \rightharpoonup \overline{y}$. Therefore, by Remark 3.9, we get $\overline{x} \in DVI(C,T)$ and $\overline{y} \in DVI(D,S)$. Moreover, since A and B are bounded linear maps, we obtain that $Ax_{k_i} \to A\overline{x}$ and $By_{k_i} \to B\overline{y}$ as $i \to \infty$ and this with (3.71) imply that $A\overline{x} = B\overline{y}$. Therefore, we have $(\overline{x}, \overline{y}) \in \Omega$. Thus, from (3.64) and (3.72), we obtain that

$$\begin{aligned} &\limsup_{k\to\infty} \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_k, y_k) - (x^*, y^*)\rangle \\ &= \lim_{i\to\infty} \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_{k_i}, y_{k_i}) - (x^*, y^*)\rangle \\ &= \langle (\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (\overline{x}, \overline{y}) - (x^*, y^*)\rangle \le 0. \end{aligned} \tag{3.77}$$

Let $v_k = \Pi^g_{C\bigcap H_k}(a_k)$ and $u_k = \Pi^f_{D\bigcap R_k}(b_k)$. Then, from the convergence of Ω_k, (3.65),(3.80), (3.67) and (3.68) we get

$$\begin{aligned} \lim_{k\to\infty} \Big[D_g(x^*, a_k) + D_f(y^*, b_k)\Big] &= \lim_{k\to\infty} \Big[D_g(x^*, v_k) + D_f(y^*, u_k)\Big] \\ &= \lim_{k\to\infty} \Big[D_g(x^*, x_{k+1}) + D_f(y^*, y_{k+1})\Big]. \end{aligned} \tag{3.78}$$

Moreover, using (2.6) and (3.78) we get

$$\begin{aligned} &\lim_{k\to\infty} \Big[D_g(v_k, a_k) + D_g(u_k, b_k)\Big] \\ &= \lim_{k\to\infty} \Big[D_g(x^*, a_k) - D_g(x^*, v_k) + D_f(y^*, b_k) - D_f(y^*, u_k)\Big] \\ &= \lim_{k\to\infty} \Big[D_g(x^*, a_k) + D_f(y^*, b_k) - (D_g(x^*, x_{k+1}) + D_g(y^*, y_{k+1})\Big] \\ &\quad + \lim_{k\to\infty} \Big[D_g(x^*, x_{k+1}) + D_g(y^*, y_{k+1}) - (D_g(x^*, v_k) + D_g(y^*, u_k))\Big] \\ &\to 0 \text{ as } k \to \infty, \end{aligned} \tag{3.79}$$

and hence $D_g(v_k, a_k) \to 0$ and $D_g(u_k, b_k) \to 0$ as $k \to \infty$. This, along with Lemma 2.3, provides that $v_k - a_k \to 0$ and $u_k - b_k \to 0$ as $k \to \infty$. Thus, from this and Step 4 of Algorithm 3.2, we obtain

$$x_{k+1} - x_k = \alpha_k(u - x_k) + (1 - \alpha_k)(v_k - x_k) \to 0, \text{ as } k \to \infty. \tag{3.80}$$

Similarly,

$$y_{k+1} - y_k \to 0, \text{ as } k \to \infty. \tag{3.81}$$

Now, from (3.70), (3.77), (3.80), (3.81) and Lemma 2.15, we obtain that $\Omega_k \to 0$ as $k \to \infty$, and hence $D_g(x^*, x_k) \to 0$ and $D_f(y^*, y_k) \to 0$ as $k \to \infty$. Therefore, by Lemma 2.3, we obtain that $x_k \to x^*$ and $y_k \to y^*$ as $k \to \infty$.

Case II. Suppose there exists a subsequence $\{\Omega_{k_i}\}$ of $\{\Omega_k\}$ with $\Omega_{k_i} < \Omega_{k_i+1}$ for all $i \geq 0$. Then, by Lemma 2.14, there exists a non-decreasing sequence $\{m_l\}$ of positive integers such that $\lim_{l\to\infty} m_l = \infty$ and

$$\Omega_{m_l} \leq \Omega_{m_l+1} \text{ and } \Omega_k \leq \Omega_{m_l+1}, \tag{3.82}$$

for all positive integers l. We have from (3.70) that

$$\begin{aligned} \frac{\rho}{2}||Ax_{m_l} - By_{m_l}||^2 \leq (1 - \alpha_l)\Omega_{m_l} &- \Omega_{m_l+1} \\ &+ \alpha_{m_l}\big[\langle(\nabla g(u) - \nabla g(x^*)), x_{m_l+1} - x^*\rangle \\ &+ \langle(\nabla f(v) - \nabla f(y^*)), y_{m_l+1} - y^*\rangle\big]. \end{aligned} \tag{3.83}$$

Taking the limit as $l \to \infty$ on both sides of (3.83) and taking (3.82) and the property of α_{m_l} into account, we obtain that $\lim\limits_{l\to\infty} \|Ax_{m_l} - By_{m_l}\| = 0$. Moreover, following similar methods used in Case I, we obtain

$$\limsup_{l\to\infty} \langle(\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_{m_l}, y_{m_l}) - (x^*, y^*)\rangle \leq 0. \tag{3.84}$$

Thus, the inequalities in (3.70) and (3.82) imply that

$$\begin{aligned} \alpha_{m_l}\Omega_{m_l} &\leq \Omega_{m_l} - \Omega_{m_l+1} \\ &+ \alpha_{m_l}\langle(\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_{m_l}, y_{m_l}) - (x^*, y^*)\rangle \\ &+ \alpha_{m_l}\left[\langle(\nabla g(u) - \nabla g(x^*)), x_{m_l+1} - x_{m_l}\rangle \right. \\ &+ \left.\langle\nabla f(v) - \nabla f(y^*), y_{m_l+1} - y_{m_l}\rangle\right]. \end{aligned} \tag{3.85}$$

which yields

$$\begin{aligned} \Omega_{m_l} &\leq \langle(\nabla g(u), \nabla f(v)) - (\nabla g(x^*), \nabla f(y^*)), (x_{m_l}, y_{m_l}) - (x^*, y^*)\rangle \\ &+ \left[\langle(\nabla g(u) - \nabla g(x^*)), x_{m_l+1} - x_{m_l}\rangle + \langle\nabla f(v) - \nabla f(y^*), y_{m_l+1} - y_{m_l}\rangle\right]. \end{aligned} \tag{3.86}$$

Taking the limsup as $l \to \infty$ on both sides of (3.86) and using (3.84), (3.80) and (3.81), we obtain that $\lim_{l\to\infty} \Omega_{m_l} = 0$. This together with (3.69) imply that $\lim_{k\to\infty} \Omega_{m_l+1} = 0$, which implies by (3.82) that $\lim_{k\to\infty} \Omega_k = 0$. As a consequence, we get $\lim_{k\to\infty} D_f(x^*, x_k) = \lim_{k\to\infty} D_g(y^*, y_k) = 0$ and hence, we obtain by Lemma 2.3 that $\lim_{k\to\infty} x_k = x^*$ and $\lim_{k\to\infty} y_k = y^*$. Therefore, we conclude from Cases I and II that the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.2 converges strongly to (x^*, y^*), where $(x^*, y^*) = P^h_\Omega(\hat{x}, \hat{y})$, and hence the proof is complete. □

We notice that if the mappings T and S in Theorem 3.11 and Theorem 3.14 are assumed to be Lipschitz continuous and quasimonotone, then we get the following corollaries.

Corollary 3.15. *Assume that (H_1) and $(H_3)-(H_5)$ hold. In addition, let ∇f and ∇g be weakly sequentially continuous mappings. If $T : \mathbb{E}_1 \to \mathbb{E}_1^*$ and $S : \mathbb{E}_2 \to \mathbb{E}_2^*$ are quasimonotone mappings that are Lipschitz continuous on C and D, respectively, satisfying (3.1), then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.1 converges weakly to $(x^*, y^*) \in \Omega^*$, where $\Omega^* = \{(x^*, y^*) \in VI(C, T) \times VI(D, S) : Ax^* = By^*\}$.*

Corollary 3.16. *Assume that (H_1) and $(H_3)-(H_6)$ hold. Let $T : \mathbb{E}_1 \to \mathbb{E}_1^*$ and $S : \mathbb{E}_2 \to \mathbb{E}_2^*$ be quasimonotone mappings that are Lipschitz continuous on C and D, respectively, satisfying (3.1). Let $T(x) \neq 0$, for all $x \in C$ and $S(y) \neq 0$, for all $y \in D$. The sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.2 converges strongly to (x^*, y^*), where $(x^*, y^*) = \Pi^h_\Omega(u, v)$.*

If we assume that $\mathbb{E}_1$ and $\mathbb{E}_2$ are smooth and 2-uniformly convex Banach spaces and take $g(\cdot) = \frac{1}{2}||\cdot||^2$ and $f(\cdot) = \frac{1}{2}||\cdot||^2$, then $\nabla g = J_{\mathbb{E}_1}, \nabla f = J_{\mathbb{E}_2}$, $(\nabla g)^{-1} = J^{-1}_{\mathbb{E}_1^*}$ and $(\nabla f)^{-1} = J^{-1}_{\mathbb{E}_2^*}$. Thus, we get the following corollaries.

Corollary 3.17. *Let C and D be nonempty, closed and convex subsets of the smooth and 2-uniformly convex Banach spaces $\mathbb{E}_1$ and $\mathbb{E}_2$, and assume that conditions $(H_2)-(H_4)$ are satisfied. In addition, let $\nabla g = J_{\mathbb{E}_1}, \nabla f = J_{\mathbb{E}_2}, (\nabla g)^{-1} = J^{-1}_{\mathbb{E}_1^*}$ and $(\nabla f)^{-1} = J^{-1}_{\mathbb{E}_2^*}$ be weakly sequentially continuous mappings. Then, the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.1 converges weakly to $(x^*, y^*) \in \Omega^*$, where $\Omega^* = \{(x^*, y^*) \in VI(C, T) \times VI(D, S) : Ax^* = By^*\}$.*

Corollary 3.18. *Let C and D be nonempty, closed and convex subsets of the smooth and 2-uniformly convex Banach spaces $\mathbb{E}_1$ and $\mathbb{E}_2$, and assume that the conditions $(H_2)-(H_4)$ and (H_6) are satisfied. Let $T(x) \neq 0$, for all $x \in C$ and $S(y) \neq 0$, for all $y \in D$. Then, the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.2 with $\nabla g = J_{\mathbb{E}_1}, \nabla f = J_{\mathbb{E}_2}, (\nabla g)^{-1} = J^{-1}_{\mathbb{E}_1^*}$ and $(\nabla f)^{-1} = J^{-1}_{\mathbb{E}_2^*}$, converges strongly to (x^*, y^*), where $(x^*, y^*) = \Pi^h_\Omega(u, v)$.*

If we assume, in Theorem 3.11 and Theorem 3.14 that $\mathbb{E}_1$, $\mathbb{E}_2$ and $\mathbb{E}_3$ are real Hilbert spaces, $g(\cdot) = \frac{1}{2}||\cdot||^2$ and $f(\cdot) = \frac{1}{2}||\cdot||^2$, then $\nabla g = I_{\mathbb{E}_1}, \nabla f = I_{\mathbb{E}_2}$, $(\nabla g)^{-1} = I^{-1}_{\mathbb{E}_1^*}$ and $(\nabla f)^{-1} = I^{-1}_{\mathbb{E}_2^*}$ and hence we obtain the following corollaries.

Corollary 3.19. *Let $\mathbb{E}_1, \mathbb{E}_2$ and $\mathbb{E}_3$ be real Hilbert spaces and let C and D be nonempty, closed and convex subsets of $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively. Assume that conditions $(H_2)-(H_4)$ are satisfied. Then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.1 with $\nabla g = I_{\mathbb{E}_1}, \nabla f = I_{\mathbb{E}_2}, (\nabla g)^{-1} = I^{-1}_{\mathbb{E}_1^*}$ and $(\nabla f)^{-1} = I^{-1}_{\mathbb{E}_2^*}$, converges weakly to $(x^*, y^*) \in \Omega^*$, where $\Omega^* = \{(x^*, y^*) \in VI(C, T) \times VI(D, S) : Ax^* = By^*\}$.*

Corollary 3.20. *Let $\mathbb{E}_1, \mathbb{E}_2$ and $\mathbb{E}_3$ be real Hilbert spaces and let C and D be nonempty, closed and convex subsets of $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively. Assume that conditions $(H_2) - (H_4)$ and (H_6) are satisfied. Let $T(x) \neq 0$, for all $x \in C$ and $S(y) \neq 0$, for all $y \in D$. Then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.2 with $\nabla g = I_{\mathbb{E}_1}, \nabla f = I_{\mathbb{E}_2}, (\nabla g)^{-1} = I^{-1}_{\mathbb{E}_1^*}$ and $(\nabla f)^{-1} = I^{-1}_{\mathbb{E}_2^*}$, converges strongly to $(x^*, y^*) = P_\Omega(u, v)$.*

4. Applications

In this section, we apply our main result to solve some specific problems.

4.1. **Split Variational Inequality Problems (SVIP).** Let C and D be nonempty, closed and convex subsets of real Banach spaces $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively. Let $T : \mathbb{E}_1 \to \mathbb{E}_1^*$ and $S : \mathbb{E}_2 \to \mathbb{E}_2^*$ be two non-linear mappings. Let $A : \mathbb{E}_1 \to \mathbb{E}_2$ be a bounded linear mapping with A^* as its adjoint. The SVIP (see, e.g., Censor et al. [3]) is to:

$$\text{find } x^* \in VI(C, T),\ y^* \in VI(D, S) \text{ such that } y^* = Ax^*, \tag{4.1}$$

where as SDVIP is to:

$$\text{find } x^* \in DVI(C, T),\ y^* \in DVI(D, S) \text{ such that } y^* = Ax^*. \tag{4.2}$$

Censor et al. [3] introduced the SVIP in 2012. It is widely known to have a number applications such as in phase retrieval, image and signal processing, data compression, among others (see, for example [27] and the references therein). Denote $\Gamma = \{(x^*, y^*) \in DVI(C, T) \times DVI(D, S) : Ax^* = y^*\}$ and $\Gamma^* = \{(x^*, y^*) \in VI(C, T) \times VI(D, S) : Ax^* = y^*\}$. Then, we have the following corollaries.

Corollary 4.1. *Assume that conditions $(H_1) - (H_3)$ and (H_5), with $\mathbb{E}_2 = \mathbb{E}_3$ and $B = I_{\mathbb{E}_2}$ hold. Let ∇f and ∇g be weakly sequentially continuous mappings. If $\Gamma \neq \emptyset$, then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.1 converges weakly to $(x^*, y^*) \in \Gamma^*$.*

Corollary 4.2. *Assume that conditions $(H_1) - (H_3)$ and $(H_5) - (H_6)$ with $\mathbb{E}_2 = \mathbb{E}_3$ and $B = I_{\mathbb{E}_2}$ hold. If $\Gamma \neq \emptyset$, then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.2, with $B = I_{\mathbb{E}_2}$, converges strongly to (x^*, y^*), where $(x^*, y^*) = \Pi^h_\Gamma(u, v)$.*

4.2. **Common Solutions of Variational Inequality Problems.** Let C and D be nonempty, closed and convex subsets of a real Banach space $\mathbb{E}$ with it dual $\mathbb{E}^*$. Let $T, S : \mathbb{E} \to \mathbb{E}^*$ be two nonlinear mappings. The Common Solutions of Variational Inequality Problem is to:

$$\text{find a point } x^* \in C \text{ such that } x^* \in VI(C, T) \cap VI(C, S), \tag{4.3}$$

where as the Common Solutions of Dual Variational Inequality Problem is to:

$$\text{find a point } x^* \in C \text{ such that } x^* \in DVI(C, T) \cap DVI(C, S), \tag{4.4}$$

Denote $\Psi = \{(x^*, y^*) \in DVI(C, T) \times DVI(D, S) : x^* = y^*\}$ and $\Psi^* = \{(x^*, y^*) \in VI(C, T) \times VI(D, S) : x^* = y^*\}$. Now, we have the following corollaries.

Corollary 4.3. *Assume that conditions* $(H_1)-(H_3)$ *and* (H_5)*, with* $\mathbb{E}_1=\mathbb{E}_2=\mathbb{E}_3=\mathbb{E}$ *hold. Let* ∇f *and* ∇g *be weakly sequentially continuous mappings. If* $\Psi\neq\emptyset$*, then the sequence* $\{(x_k,y_k)\}$ *generated by Algorithm 3.1, with* $A=B=I_{\mathbb{E}}$*, converges weakly to* $(x^*,x^*)\in\Psi^*$*.*

Corollary 4.4. *Assume that conditions* $(H_1)-(H_3)$ *and* $(H_5)-(H_6)$ *with* $\mathbb{E}_1=\mathbb{E}_2=\mathbb{E}_3=\mathbb{E}$ *hold. If* $\Psi\neq\emptyset$*, then the sequence* $\{(x_k,y_k)\}$ *generated by Algorithm 3.2, with* $A=B=I_{\mathbb{E}}$*, converges strongly to* (x^*,x^*)*, where* $(x^*,x^*)=\Pi^h_{\Psi}(u,v)$*.*

5. Numerical example

In this section, we give a numerical example with illustrations to demonstrate the applicability of the algorithms.

Example 5.1. Let $\mathbb{E}=\mathbb{E}_1=\mathbb{E}_2=\mathbb{E}_3=\mathbb{R}^2$ with the usual norm. Let

$$C=\left\{(x,y)\in\mathbb{E}:(x-2)^2+y^2\leq 1\right\} \text{ and } D=\left\{(x,y)\in\mathbb{E}:x^2+(y-2)^2\leq 1\right\}.$$

Clearly, C and D are closed and convex subsets of $\mathbb{E}$. Define the mappings T,S: $\mathbb{E}\to\mathbb{E}$ by $T(x,y)=\left(\frac{x^2}{1+x^2},0\right)$ and $S(x,y)=\left(0,\frac{y^2}{1+y^2}\right)$. Clearly, the mappings T and S are uniformly continuous on bounded subsets of C and D, respectively. It remains to show that the mappings T and S are quasimonotone on $\mathbb{E}$. Indeed, suppose that for $(x_1,x_2),(y_1,y_2)\in\mathbb{E}$, we have $\langle T(x_1,x_2),(y_1,y_2)-(x_1,x_2)\rangle>0$, that is,

$$\left\langle\left(\frac{x_1^2}{1+x_1^2},0\right),(y_1,y_2)-(x_1,x_2)\right\rangle=\frac{x_1^2}{1+x_1^2}(y_1-x_1)>0.$$

Then, this implies that $y_1-x_1>0$, and hence

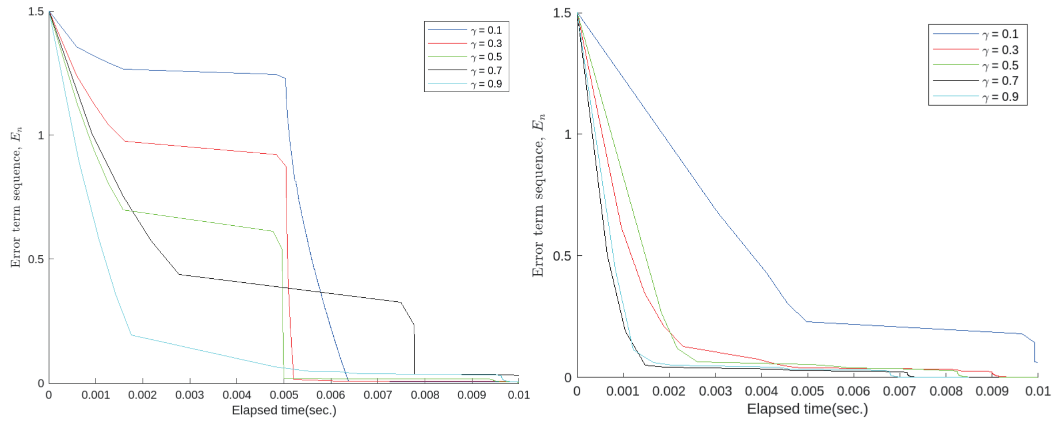
$$\langle T(y_1,y_2),(y_1,y_2)-(x_1,x_2)\rangle=\left\langle\left(\frac{y_1^2}{1+y_1^2},0\right),(y_1,y_2)-(x_1,x_2)\right\rangle=\frac{y_1^2}{1+y_1^2}(y_1-x_1)\geq 0.$$

This yields that T is quasimonotone. However, if we take $(x_1,x_2)=(0,0)$ and $(y_1,y_2)=\left(\frac{-1}{2},0\right)$, then $\langle T(x_1,x_2),(y_1,y_2)-(x_1,x_2)\rangle=0$, but $\langle T(y_1,y_2),(y_1,y_2)-(x_1,x_2)\rangle<0$, which implies that T is not pseudomonotone. Similarly, one can show that S is uniformly continuous on bounded subsets of D and quasimonotone. Moreover, $DVI(C,T)=VI(C,T)=\{(1,0)\}$ and $DVI(D,S)=VI(D,S)=\{(0,1)\}$. Let A: $\mathbb{E}\to\mathbb{E}$ and B: $\mathbb{E}\to\mathbb{E}$ be defined by $A(x_1,x_2)=\left(0,\frac{x_2}{2}\right)$ and $B(x_1,x_2)=(3x_1,0)$. Clearly, A and B are bounded linear mappings with adjoints $A^*(x_1,x_2)=\left(0,\frac{x_2}{2}\right)$ and $B^*(x_1,x_2)=(3x_1,0)$, respectively. In addition, we have $A(1,0)=(0,0)=B(0,1)$. Therefore, we have that

$$(x^*,y^*)=\big((1,0),(0,1)\big)\in\Omega=\{(x,y)\in VI(C,T)\times VI(D,S):Ax=By\}.$$

Define g: $\mathbb{E}\to(-\infty,+\infty]$ and f: $\mathbb{E}\to(-\infty,+\infty]$ by $g(x)=\frac{1}{2}\|x\|^2$ and $f(y)=\frac{1}{2}\|y\|^2$. Then, $\nabla g(x)=(\nabla g)^{-1}x=I_{\mathbb{E}}x=x$ and $\nabla f(y)=(\nabla f)^{-1}y=I_{\mathbb{E}}y=y$. Take $\alpha_n=\frac{1}{n+1}$, $\sigma=0.9$, $\beta_n=0.01+\frac{1}{n+1}$. Then the conditions $(H_1)-(H_6)$ are satisfied. We have conducted the numerical experiments to demonstrate that the error term sequence $E_n=\{||(x_n,y_n)-(x^*,y^*)||\}$, $n\geq 1$, of Algorithm 3.1 and 3.2 converges to zero for different parameters.

In Figure 1(A) and Figure 1(B), the convergence of the error term sequence $E_n=\{||(x_n,y_n)-(x^*,y^*)||\}$ with the aforementioned parameters is indicated for

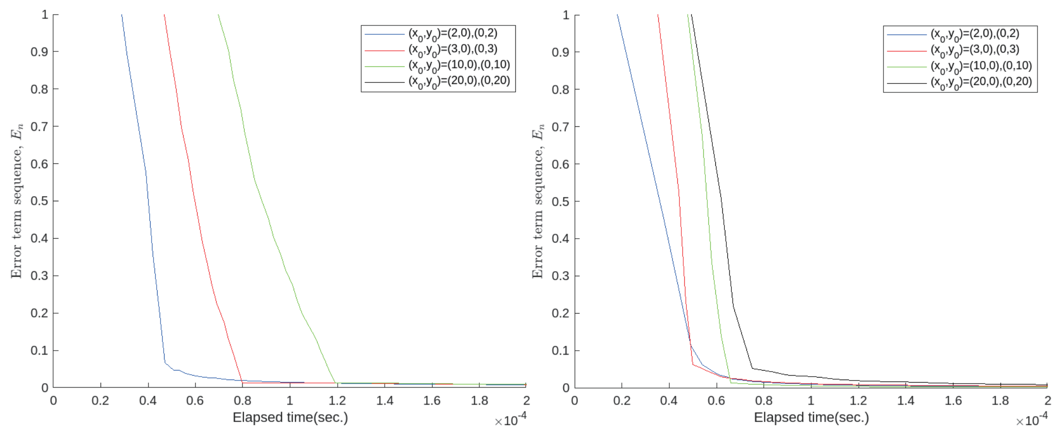


(A) Convergence of **Algorithm 3.1** for different choices of γ.

(B) Convergence of **Algorithm 3.2** for different choices of γ.

FIGURE 1. Convergence rates of Alg. 3.1 and Alg. 3.2 for different values of γ.

various values of γ and initial value $(x_0, y_0) = ((2,0),(0,2))$ for Algorithm 3.1 and 3.2, respectively.



(A) Convergence of **Algorithm 3.1** for $\gamma = 0.9$ and varying initial points.

(B) Convergence of **Algorithm 3.2** for $\gamma = 0.9$ and varying initial points.

FIGURE 2. Convergence rates of Alg. 3.1 and Alg. 3.2 for different initial points.

It is clear that Figure 2(A) and Figure 2(B) illustrate how $E_n = \{||(x_n, y_n) -(x^*, y^*)||\}$ converges for the inertial parameter $\gamma = 0.9$ and different initial values $(x_0, y_0) = ((2,0),(0,2))$, $(x_0, y_0) = ((3,0),(0,3))$, $(x_0, y_0) = ((10,0),(0,10))$ and $(x_0, y_0) = ((20,0),(0,20))$ for Algorithms 3.1 and 3.2, respectively.

Remark 5.2. The numerical experiment's results, which can be seen in Figure 1 and Figure 2, demonstrate that $E_n = \| (x_n, y_n) - (x^*, y^*)\|$ converges strongly to zero; that is, for both algorithms, the sequence $\{(x_n, y_n)\}$ converges strongly to the point $(x^*, y^*) = \big((1, 0), (0, 1)\big)$ of Ω.

We can also observe from Figure 1 and Figure 2 that the rate of convergence for both algorithms eventually becomes the same regardless of the beginning locations we choose.

6. Conclusions

In this paper, we proposed and studied an iterative method for solving split equality variational inequality problems in reflexive real Banach spaces. Weak and strong convergence theorems are established for Algorithm 3.1 and Algorithm 3.2, respectively, under the assumption that the underlying mappings are quasimonotone and uniformly continuous. As a consequence, we obtain weak convergence of Algorithm 3.1 to the solution of the variational inequality problems involving uniformly continuous quasimonotone mappings in reflexive real Banach spaces. Our findings provide positive answers to the questions posed and expand on the existing literature. Our main result has been supplemented with specific applications. In addition, a numerical example has been provided to demonstrate the effectiveness of our method.

This paper's main result generalizes a lot of results found in the literature because it deals with a more general split equality problem in the setting of Banach spaces, a more general than Hilbert spaces. Specifically, it extends the works in [32] from a variational inequality problem to a more general split equality variational inequality problem and from weak convergence to strong convergence results.

References

[1] H. H. Bauschke and J. M. Borwein, *Legendre functions and the method of random Bregman projections*, J. Convex Anal. **4**(1997), 27–67.

[2] J. F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer Science & Business Media, New York, 2013.

[3] Y. Censor, A. Gibali and S. Reich, *Algorithms for the split variational inequality problem*, Numerical Algorithms **59** (2010), 301–323.

[4] Y. Censor, A. Gibali and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert spaces*, J. Optim Theory Appl. **148** (2011), 318–335.

[5] Y. Censor and A. Lent, *An iterative row-action method for interval convex programming*, J. Optim. Theory Appl. **34** (1981), 321–353.

[6] R. W. Cottle and J. C. Yao, *Pseudo-monotone complementarity problems in Hilbert space*, J. Optim. Theory Appl. **75** (1992), 281–295.

[7] V. T. Duong and A. Gibali, *Two strong convergence subgradient extragradient methods for solving variational inequalities in Hilbert spaces*, Japan J. Indust. Appl. Math. **36** (2019), 299–321.

[8] F. Facchinei and J. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems, Volumes I and II*, Comprehensive Study in Mathematics, Springer-Verlag, New York, 2003.

[9] G. Fichera, *Sul problema elastostatico di Signorini con ambigue condizioni al contorno*, Atti Accad Naz Lincci VIII Scr Rend Cl Sci Fis Mat Nat. **34** (1963), 138–142.

[10] N. Hadjisavvas and S. Schaible*Quasimonotone variational inequalities in Banach spaces*, J. Optim. Theory Appl. **90** (1996), 95–111.

[11] S. Komlosi, *On the Stampacchia and minty variational inequalities*, in: Generalized Conv exity and Optimization for Economic and Financial Decisions, G. Giorgi and F. A. Rossi (eds.), Pitagora Editrice, Bologna, 1999.

[12] G. M. Korpelevich, *An extragradient method for finding saddle points and other problems*, Ekon. Mat. Metody **12** (1976), 747–756.

[13] P. E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and non-strictly convex minimization*, Set-Valued Anal. **16** (2008), 899–912.

[14] V. Mart´ın-Marquez, S. Reich and S. Sabach, *Right Bregman nonexpansive operators in Banach spaces*, Nonlinear Anal. ´ **75** (2012), 5448–5465.

[15] J. Mashreghi and M. Nasri, *Forcing strong convergence of Korpelevich's method in Banach spaces with its applications in game theory*, Nonlinear Anal. **72** (2010), 2086–2099.

[16] E. Naraghirad and J. C. Yao, *Bregman weak relatively nonexpansive mappings in banach spaces*, Fixed Point Theory appl. **2013** (2013), 1–43.

[17] H. K. Pathak, *An Introduction to Nonlinear Analysis and Fixed Point Theory*, Springer, Singapore, 2018.

[18] G. N. Ogwo, T. O. Alakoya and O. T. Mewomo, *Iterative algorithm with self-adaptive step size for approximating the common solution of variational inequality and fixed point problems*, Optimization, (2021) DOI:10.1080/02331934.2021.1981897.

[19] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Springer, Berlin, 2009.

[20] D. Reem, S. Reich and A. De Piero, *Re-examination of Bregman functions and new properties of their divergences*, Optimization **68** (2019), 279–348.

[21] S. Reich and S. Sabach, *A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces*, J. Nonlinear Convex Anal. **10** (2009), 471–485.

[22] Y. Shehu and O. S. Iyiola, *On a modified extragradient method for variational inequality problem with application to industrial electricity production*, J. Ind. Manag. Optim. **15** (2019), 319–342.

[23] Y. Shehu, *Single projection algorithm for variational inequalities in Banach space with application to contact problem*, Acta Math. Sci. **40B** (2020), 1045–1063.

[24] M. V. Solodov and B. F. Svaiter, *A new projection method for variational inequality problems*, SIAM J. Control Optim. **37** (1999), 765–776.

[25] G. Stampacchia, *Formes bilinearieres coercitivities sur les ensembles convexes*, CR Acad Sci Paris **258** (1964), 4413–4416.

[26] T. M. M. Sow, *New iterative schemes for solving variational inequality and fixed points problems involving demicontractive and quasi-nonexpansive mappings in Banach spaces*, Appl. Math. Nonlinear Sci. **4** (2019), 559–574.

[27] Y. Tang and A. Gibali, *New self-adaptive step size algorithms for solving split variational inclusion problems and its applications*, Numerical Algorithms **83** (2019), 305–331.

[28] G. J. Tang and X. Wang, *An extragradient-type method for variational inequalities on Banach spaces*, Acta Math. Sinica (Chin. Ser.) **59** (2016), 187–198.

[29] P. Tseng, *Amodified forward-backward splitting method formaximal monotone mappings*, SIAM J. Control Optim. **38** (2000), 431–446.

[30] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. **66** (2002), 240–256.

[31] M. L. Ye and Y. R. He, *A double projection method for solving variational inequalities without monotonicity*, Comput. Optim. Appl. **60** (2015), 141–150.

[32] L. Zheng, *A double projection algorithm for quasimonotone variatioal inequalities in Banach spaces*, (2018), 10.1186/s13660-018-1852-2.

[33] C. Zâlinescu, *Convex Analysis in General Vector Spaces*, World Scientific Publishing, Singapore, 2002.

[34] H. Zegeye and N. Shahzad, *Extragradient method for solutions of variational inequality problems in Banach spaces*, Abstr. Appl. Anal. **2012** (2013): 832548.

[35] X. Zhong, *On the fenchel duality between strong convexity and Lipschitz continuous gradient*, arXiv:1803.06573 [math.OC], (2018)

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