Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167

Volume 10, Number 2, 2024, 45–55

APPROXIMATION OF A NONLINEAR STOCHASTIC FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATION WITH MEASURES OF QUALITY AND CERTAINTY

AZAM AHADI, REZA SAADATI, AND DONAL O'REGAN*

Abstract. In this paper, we introduce novel control functions incorporating measures of quality and certainty, which lay the foundation for analyzing the stability of stochastic fractional Volterra integro-differential equations. We utilize the Hyers-Ulam and Hyers-Ulam-Rassias stability methodologies to investigate the stability properties of such equations, providing theoretical insights and potential applications in various fields of mathematics and engineering.

1. Introduction

Fractional calculus extends the concepts of differentiation and integration to non-integer orders, allowing for a richer mathematical framework that can model complex phenomena more accurately than classical methods. The application of fractional derivatives and integrals has gained traction across diverse disciplines, including engineering, physics, biology, mathematics, and medicine. For instance, in engineering, fractional differential equations can describe systems with memory and hereditary properties, while in biology, they can model processes like diffusion and population dynamics.

Stochastic fractional differential equations (SFDEs) introduce uncertainty into these models, incorporating randomness that is often observed in real-world systems. Such equations are particularly relevant in fields like finance, where asset prices exhibit stochastic behavior, and in ecology, where species populations are influenced by random environmental factors.

Research into the stability of solutions for implicit SFDEs is crucial for understanding their behavior under various conditions. The stability of these solutions often determines the reliability of the models in predicting real-world phenomena. Key studies in this area, such as those referenced in [10, 6, 4, 9], provide valuable insights into the theoretical underpinnings of SFDEs and contribute to their practical applicability in modeling complex systems.

Overall, the intersection of fractional calculus, stochastic processes, and differential equations represents a vibrant area of research that continues to evolve, presenting both challenges and opportunities for mathematicians and scientists alike.

2020 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 47H10, 20C20.

Key words and phrases. Stability, generalized Z-number, stochastic fractional Volterra integral, Ψ-Hilfer stochastic fractional derivative, integro-differential equations, fixed point.

In the paper, we explore the stability of solutions for a stochastic fractional nonlinear Volterra integro-differential equation (VIDE) defined as follows:

$$\begin{cases} {}^{H}\mathbb{D}_{0+}^{\iota,\kappa;\Psi}\mu(\varrho,\varsigma) = \boldsymbol{f}(\varrho,\varsigma,\mu(\varrho,\varsigma)) + \int_0^{\varsigma} \boldsymbol{k}(\varrho,\varsigma,\vartheta,\mu(\varrho,\varsigma))d\vartheta, \\ \mathcal{I}_{0+}^{1-\gamma}\mu(\varrho,0) = \sigma, \end{cases} \tag{1.1}$$

with $\varsigma \in [0,T]$. The main components of this equation include operators $\boldsymbol{f}$ and $\boldsymbol{k}$. Both are defined as continuous random operators (CRO). Specifically, $\boldsymbol{f}(\varrho,\varsigma,\mu)$ and $\boldsymbol{k}(\varrho,\varsigma,\vartheta,\mu)$ depend on the variables ϱ, ς, ϑ, and the unknown function μ. The CROs imply some level of randomness, and they are continuous in all their arguments, defined over appropriate spaces such as $\Upsilon \times [0,T] \times \mathbb{R}$ and $\Upsilon \times [0,T] \times \mathbb{R} \times \mathbb{R}$, respectively. Also the equation includes the Stochastic Fractional Derivative ${}^{H}\mathbb{D}_{0+}^{\iota,\kappa;\Psi}$. This denotes the fractional derivative of the function μ, parameterized by ι, κ, and possibly influenced by a weight function Ψ. The fractional order of differentiation introduces memory effects into the equation. Finally the equation includes the Riemann-Liouville Stochastic Fractional Integral $\mathcal{I}_{0+}^{1-\gamma}$. This operator provides a fractional integral of the function μ, and σ is an initial condition specified at $t = 0$. The parameter γ is within the range $0 \leq \gamma < 1$, aligning with the characteristic behavior of fractional integrals.

2. Preliminaries

Here, we define $\Xi_1 = [0,T]$, where $T > 0$, $\Xi_2 = (0,\infty)$, $\Xi_3 = (0,1]$, $\Xi_4 = [0,\infty]$ and $\Xi_5 = [0,1]$ (note that $\Xi_5^\circ = (0,1)$ refers to the interior of Ξ_5).
Now we let

$$\boldsymbol{\Xi} = diag\ \Xi_5 = \left\{ \begin{bmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_{\mathrm{n}} \end{bmatrix} = diag[\xi_1,\ldots,\xi_{\mathrm{n}}],\ \xi_1,\ldots,\xi_{\mathrm{n}} \in \Xi_5 \right\}.$$

We define $diag[\xi_1,\ldots,\xi_{\mathrm{n}}]$ as less than or equal to $diag[\mathrm{f}_1,\ldots,\mathrm{f}_{\mathrm{n}}]$ denoted $diag[\xi_1,\ldots,\xi_{\mathrm{n}}] \preceq diag[\mathrm{f}_1,\ldots,\mathrm{f}_{\mathrm{n}}]$ if and only if $\xi_{\mathrm{i}} \leq \mathrm{f}_{\mathrm{i}}$ for all indices $\mathrm{i} = 1,...,\mathrm{n}$.

Definition 2.1 ([1])**.** A function $\circledast : \boldsymbol{\Xi} \times \boldsymbol{\Xi} \to \boldsymbol{\Xi}$ is referred to as a generalized continuous t-norm (GCTN) if it satisfies the following conditions: $\xi \circledast \mathbf{1} = \xi, \xi \circledast \mathrm{f} = \mathrm{f} \circledast \xi, \xi \circledast (\mathrm{f} \circledast \mathrm{e}) = (\xi \circledast \mathrm{f}) \circledast \mathrm{e}$ for all $\mathrm{e}, \mathrm{f}, \xi \in \boldsymbol{\Xi}$ with $\mathbf{1} = diag[1,...,1]$. Additionaly, if $\mathrm{e} \preceq \mathrm{f}$ and $\xi \preceq \mathrm{h}$ then $\mathrm{e} \circledast \xi \preceq \mathrm{f} \circledast \mathrm{h}$. Furthermore, for sequences $\{\mathrm{f}_{\mathrm{n}}\}$ and $\{\xi_{\mathrm{n}}\}$ in $\boldsymbol{\Xi}$ converging to points f and ξ in $\boldsymbol{\Xi}$, we have $\lim_{\mathrm{n}}(\mathrm{f}_{\mathrm{n}} \circledast \xi_{\mathrm{n}}) = \mathrm{f} \circledast \xi$.

The minimum t-norm, denoted by $\circledast_M : \boldsymbol{\Xi} \times \boldsymbol{\Xi} \to \boldsymbol{\Xi}$, is defined as follows:

$$\varpi \circledast_M \rho = diag[\varpi_1,...,\varpi n] \circledast_M diag[\rho_1,...,\rho_n] = diag[min\{\varpi_1,\rho_1\},...,min\{\varpi_n,\rho_n\}].$$

In this paper, we define the generalized Z-number space. For more information, we refer the reader to [1].

Definition 2.2 ([1])**.** Consider a linear space S and two triples: $(S, \aleph, \circledast)$, a fuzzy normed space, and $(S, \mu_\tau, \circledast)$, a random normed space, and we define a matrix valued function $\widetilde{Z} : S \times \Xi_2 \longrightarrow \mathbf{\Xi}$ as $\widetilde{Z}(\tau, \xi) = diag[\aleph(\tau, \xi), \mu_\tau(\xi), \aleph(\tau, \xi) \circledast \mu_\tau(\xi)]$, which is called a generalized $Z-$ number (G$Z-$N), provided that the following conditions hold for all $\tau, \theta \in S$, $\xi, \zeta > 0$ and $\alpha \neq 0$:

Z1: $\widetilde{Z}(\tau, \xi) = diag[1, 1, 1]$ if and only if $\tau = 0$;

Z2: $\widetilde{Z}(\alpha\tau, \xi) = \widetilde{Z}(\tau, \dfrac{\xi}{|\alpha|})$;

Z3: $\widetilde{Z}(\tau + \theta, \xi + \zeta) \succeq \widetilde{Z}(\tau, \xi) \circledast_M \widetilde{Z}(\theta, \zeta)$.

Let $(S, \widetilde{Z})$ denote a G$Z-$N normed space. A sequence $\{\xi_n\} \subset S$ is said to converge to $\xi \in S$ in the G$Z-$N normed space $(S, \widetilde{Z})$, if for any $\epsilon \in \Xi_5^\circ$ and $\tau \in \Xi_2$, there exists a positive integer $N_{\epsilon,\tau} \in \Xi_2$ such that $\widetilde{Z}(\xi_n - \xi, \tau) \succ diag(1 - \epsilon, 1 - \epsilon, 1 - \epsilon)$ for all $n \geq N_{\epsilon,\tau}$. Similarly, $\{\xi_n\}$ is G$Z-$N Cauchy in $(S, \widetilde{Z})$ if, for any $\epsilon \in \Xi_5^\circ$ and $\tau \in \Xi_2$, there exists a positive integer $N_{\epsilon,\tau} \in \Xi_2$ such that $\widetilde{Z}(\xi_n - \xi_m, \tau) > diag(1 - \epsilon, 1 - \epsilon, 1 - \epsilon)$ whenever $n, m \geq N_{\epsilon,\tau}$. A G$Z-$N normed space in which every Cauchy sequence converges is called a generalized Z-number Banach space (denoted by G$Z-$NB space).

An example of a generalized Z-number norm is

$$\widetilde{Z}\Big(\xi, \tau\Big) = diag\Big(\exp(-\frac{\|\xi\|}{\tau}), \frac{\tau}{\tau + \|\xi\|}, \exp(-\frac{\|\xi\|}{\tau}) \circledast_M \frac{\tau}{\tau + \|\xi\|}\Big),$$

for all $\tau \in \Xi_2$ and ξ is a member of a normed linear space $(W, \|.\|)$.

Consider the probability space (Υ, Ξ_2, ξ) and let $(U, \boldsymbol{B}_U)$ and $(S, \boldsymbol{B}_S)$ be Borel measurable spaces, where U and S are G$Z-$NB spaces. If for every ξ in U and $B \in \boldsymbol{B}_S$, the set $\{\varrho : \mathcal{F}(\varrho, \xi) \in B\} \in \Xi_2$, we call $\mathcal{F} : \Upsilon \times U \to S$ a random operator. A random operator $\mathcal{F} : \Upsilon \times U \to S$ is considered *linear* if for each for each $\xi_1, \xi_2 \in U$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}$, we have $\mathcal{F}(\varrho, \mathbf{a}\xi_1 + \mathbf{b}\xi_2) = \mathbf{a}\mathcal{F}(\varrho, \xi_1) + \mathbf{b}\mathcal{F}(\varrho, \xi_2)$ almost everywhere. It is called bounded if there exists a real-valued random variable $M(\varrho) \in \Xi_2$ such that

$$\widetilde{Z}\Big(\mathcal{F}(\varrho, \xi_1) - \mathcal{F}(\varrho, \xi_2), M(\varrho)\tau\Big) \succcurlyeq \widetilde{Z}\Big(\xi_1 - \xi_2, \tau\Big),$$

almost everywhere for each ξ_1, ξ_2 in U, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$.

Theorem 2.3 ([2, 3])**.** *Let (U, ρ) be a complete metric space with values in the set Ξ_4 and let $\Lambda : U \to U$ be a strictly contractive function with Lipschitz constant $\iota < 1$. Then for a given element $\xi \in U$, one of the following two scenarios must occur:*

1. *The distance $\rho\left(\Lambda^n\xi, \Lambda^{n+1}\xi\right)$ diverges to infinity for all $n \in \mathbb{N}$.*
2. *There exists an integer $n_0 \in \mathbb{N}$ such that:*
   (i) *The distance $\rho(\Lambda^n\xi, \Lambda^{n+1}\xi)$, remains finite for all $n \geq n_0$;*
   (ii) *The sequence $\{\Lambda^n\xi\}$ converges to the fixed point ζ^* of Λ;*

(iii) *In the set* $V = \{\zeta \in U \mid \rho(\Lambda^{n_0}\xi, \zeta) < \infty\}$*, the fixed point* ζ^* *is the unique fixed point of* Λ*;*

(iv) *For every* $\zeta \in V$*, the inequality* $(1-\iota)\rho(\zeta, \zeta^*) \leq \rho(\zeta, \Lambda\zeta)$ *holds.*

Definition 2.4 ([7])**.** In this paper, we consider the gamma function

$$\Gamma(z) = \int_0^{\infty} e^{-\varsigma}\varsigma^{z-1} d\varsigma,$$

where $z \in \mathbb{C}$, $Re(z) > 0$.

Consider $\iota \in \mathring{\Xi}_5$, the integrable random operator $\boldsymbol{f}$ on Ξ_5 and the nondecreasing random operator $\Psi \in C^1(\Upsilon \times \Xi_5)$ where $\Psi'(\varrho, \varsigma) \neq 0$, for every $\varsigma \in \Xi_5$. The right-sided Ψ-Hilfer stochastic fractional derivative, is given by [8, 5]

$$ {}^{H}\mathbb{D}_{0+}^{\iota,\kappa;\Psi} \boldsymbol{f}(\varrho, \varsigma) = \mathcal{I}_{0+}^{\kappa(1-\iota);\Psi} \left(\frac{1}{\Psi'(\varrho, \varsigma)} \frac{d}{d\varsigma} \right) \mathcal{I}_{0+}^{(1-\kappa)(1-\iota);\Psi} \boldsymbol{f}(\varrho, \varsigma). \tag{2.1}$$

Definition 2.5. Consider the continuously differentiable random operator $\mu(\varrho, \varsigma)$ and let $\varphi(\varsigma, \tau)$ represent a matrix generalized Z-number set that satisfies the following condition:

$$\widetilde{Z}\left({}^{H}\mathbb{D}_{0+}^{\iota,\kappa;\Psi} \mu(\varrho, \varsigma) - \boldsymbol{f}(\varrho, \varsigma, \mu(\varrho, \varsigma)) - \int_0^{\varsigma} \boldsymbol{k}(\varrho, \varsigma, \vartheta, \mu(\varrho, \varsigma)) d\vartheta, \tau \right) \succcurlyeq \varphi\Big(\varsigma, \tau\Big),$$

for each $\varsigma \in \Xi_5$, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$. If there exists a solution $\mu_0(\varrho, \varsigma)$ of the VIDE (1.1), along with a fixed constant $\mathbf{C} > 0$, such that

$$\widetilde{Z}\Big(\mu(\varrho, \varsigma) - \mu_0(\varrho, \varsigma), \tau \Big) \succcurlyeq \varphi\left(\varsigma, \frac{\tau}{\mathbf{C}}\right),$$

for all $\varsigma \in \Xi_5$, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$, where $\mathbf{C}$ is independent of $\mu(\varrho, \varsigma)$ and $\mu_0(\varrho, \varsigma)$, then we say that the system described by (1.1) exhibits Hyers-Ulam-Rassias stability.

3. Main results

In this study, we adopt the following definition for $\int_0^{\varsigma} f(\xi).diag\big(g(\xi, \tau), h(\xi, \tau), k(\xi, \tau)\big) d\xi$:

$$\int_0^{\varsigma} f(\xi).diag\Big(g(\xi, \tau), h(\xi, \tau), k(\xi, \tau) \Big) d\xi$$
$$= diag\Big(\int_0^{\varsigma} f(\xi).g(\xi, \tau) d\xi, \int_0^{\varsigma} f(\xi).h(\xi, \tau) d\xi, \int_0^{\varsigma} f(\xi).k(\xi, \tau) d\xi \Big).$$

Consider the following assumption:

(H0). Let $M, L_{\boldsymbol{f}}, L_{\boldsymbol{k}} > 0$ be fixed constants such that $M(L_{\boldsymbol{f}} + L_{\boldsymbol{k}}) \in \mathring{\Xi}_5$. Furthermore, let the CROs $\boldsymbol{f} : \Upsilon \times \Xi_5 \times \mathbb{R} \to \mathbb{R}$ and $\boldsymbol{k} : \Upsilon \times \Xi_5 \times \Xi_5 \times \mathbb{R} \to (\mathbb{R}, \widetilde{Z}, \circledast)$ satisfy the following condition: For all $\varsigma \in \Xi_5$, $\mu_1, \mu_2 \in \mathbb{R}$, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$, we have:

$$\widetilde{Z}\Big(\boldsymbol{f}(\varrho, \varsigma, \mu_1) - \boldsymbol{f}(\varrho, \varsigma, \mu_2), \tau \Big) \succcurlyeq \widetilde{Z}\left(\mu_1 - \mu_2, \frac{\tau}{L_{\boldsymbol{f}}} \right), \tag{3.1}$$

for all $\varsigma, \vartheta \in \Xi_5$, $\mu_1, \mu_2 \in \mathbb{R}$, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$, and we also have:

$$\widetilde{Z}\Big(\boldsymbol{k}(\varrho,\varsigma,\vartheta,\mu_1) - \boldsymbol{k}(\varrho,\varsigma,\vartheta,\mu_2), \tau\Big) \succcurlyeq \widetilde{Z}\left(\mu_1 - \mu_2, \frac{\tau}{L_{\boldsymbol{k}}}\right), \tag{3.2}$$

Theorem 3.1. *Assume that hypothesis* $(H0)$ *holds, and consider the nondecreasing random operator* $\Psi \in C(\Upsilon \times \Xi_5)$, *where* $\Psi'(\varrho,\varsigma) \neq 0$. *Also, let* μ *be a continuously differentiable random operator mapping from* $\Upsilon \times \Xi_5$ *to* $\mathbb{R}$, *satisfying:*

$$\widetilde{Z}\left({}^{H}\mathbb{D}_{0+}^{\iota,\kappa;\Psi}\mu(\varrho,\varsigma) - \boldsymbol{f}(\varrho,\varsigma,\mu(\varrho,\varsigma)) - \int_0^{\varsigma} \boldsymbol{k}(\varrho,\varsigma,\vartheta,\mu(\varrho,\vartheta))d\vartheta, \tau\right) \succcurlyeq \varphi\Big(\varsigma,\tau\Big), \tag{3.3}$$

for all $\varsigma, \vartheta \in \Xi_5$, $\mu \in \mathbb{R}$, $\tau \in \Xi_2$ *and* $\varrho \in \Upsilon$, *where the matrix* $\varphi\big(\varsigma,\tau\big)$ *is defined as*

$$\varphi\Big(\varsigma,\tau\Big) = diag\Big(\phi(\varsigma,\tau), \psi(\varsigma,\tau), \phi(\varsigma,\tau) \circledast \psi(\varsigma,\tau)\Big)$$

representing a generalized Z*-number matrix and*

$$\widetilde{Z}\left(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma} \Psi'(\varrho,\xi)(\Psi(\varrho,\varsigma)-\Psi(\varrho,\xi))^{\iota-1}\varphi(\xi,\tau)d\xi, \tau\right) \succcurlyeq \varphi\left(\varsigma, \frac{\tau}{M}\right). \tag{3.4}$$

Then a a unique CRO $\mu_0 : \Upsilon \times \Xi_5 \to \mathbb{R}$ *exists, such that*

$$\begin{aligned} \mu_0(\varrho,\varsigma) = {} & \frac{(\Psi(\varrho,\varsigma)-\Psi(\varrho,0))^{\gamma-1}}{\Gamma(\gamma)}\sigma \\ & + \mathcal{I}_{0+}^{\iota;\Psi}\boldsymbol{f}(\varrho,\varsigma,\mu_0(\varrho,\varsigma)) \\ & + \mathcal{I}_{0+}^{\iota;\Psi}\left[\int_0^{\xi}\boldsymbol{k}(\varrho,\varsigma,\vartheta,\mu_0(\varrho,\vartheta))d\vartheta\right], \end{aligned} \tag{3.5}$$

with the condition $\mathcal{I}_{0+}^{1-\gamma;\Psi}\mu(\varrho,0) = \sigma$, $0 < \iota < 1$, $0 \leq \kappa \leq 1$ *and*

$$\widetilde{Z}\Big(\mu(\varrho,\varsigma) - \mu_0(\varrho,\varsigma), \tau\Big) \succcurlyeq \varphi\left(\varsigma, \frac{M\tau}{1 - M\left(L_{\boldsymbol{f}} + L_{\boldsymbol{k}}\right)}\right), \tag{3.6}$$

for all $\varsigma \in \Xi_5$, $\tau \in \Xi_2$, $\varrho \in \Upsilon$.

Proof. For $\alpha, \beta \in U$, let

$$\rho(\alpha,\beta) = \inf\left\{\mathbf{C} \in \Xi_4 : \widetilde{Z}\Big(\alpha(\varrho,\varsigma) - \beta(\varrho,\varsigma), \tau\Big) \succcurlyeq \varphi\left(\varsigma, \frac{\tau}{\mathbf{C}}\right)\right\}, \tag{3.7}$$

for each $\varsigma \in \Xi_5$, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$, where

$$U = \{\alpha : \Upsilon \times \Xi_5 \to \mathbb{R} \text{ is CRO}\}.$$

Let $\Lambda : U \to U$ be defined by

$$\begin{aligned} \Lambda\alpha(\varrho,\varsigma) = {} & \frac{(\Psi(\varrho,\varsigma)-\Psi(\varrho,0))^{\gamma-1}}{\Gamma(\gamma)}\sigma \\ & + \mathcal{I}_{0+}^{\iota;\Psi}\boldsymbol{f}(\varrho,\varsigma,\alpha(\varrho,\varsigma)) \\ & + \mathcal{I}_{0+}^{\iota;\Psi}\left[\int_0^{\xi}\boldsymbol{k}(\varrho,\varsigma,\vartheta,\alpha(\varrho,\vartheta))d\vartheta\right], \end{aligned} \tag{3.8}$$

for all $\alpha \in \Xi_5$, $\varsigma \in \Xi_5$ and $\varrho \in \Upsilon$.

We begin by demonstrating that Λ is strictly contractive on U. Let $\mathbf{C}_{\alpha\beta} \in \Xi_4$ represent a fixed constant such that $\rho(\alpha,\beta) \leq \mathbf{C}_{\alpha\beta}$ for all $\alpha,\beta \in U$. From (3.7), it follows that

$$\widetilde{Z}\Big(\alpha(\varrho,\varsigma)-\beta(\varrho,\varsigma),\tau\Big) \succcurlyeq \varphi\left(\varsigma,\frac{\tau}{\mathbf{C}_{\alpha\beta}}\right), \tag{3.9}$$

for each $\varsigma \in \Xi_5$, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$. Also from (3.1), (3.2), (3.4), (3.8) and (3.9), we obtain

(3.10)

$$\begin{aligned}
&\widetilde{Z}\left(\Lambda\alpha(\varrho,\varsigma)-\Lambda\beta(\varrho,\varsigma),\tau\right)\\
&=\widetilde{Z}\Bigg(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\xi)(\Psi(\varrho,\varsigma)-\Psi(\varrho,\xi))^{\iota-1}\widetilde{Z}\Bigg(\boldsymbol{f}(\varrho,\xi,\alpha(\varrho,\xi))-\boldsymbol{f}(\varrho,\xi,\beta(\varrho,\xi))\\
&\quad+\int_0^{\xi}\boldsymbol{k}(\varrho,\varsigma,\vartheta,\alpha(\varrho,\vartheta))-\boldsymbol{k}(\varrho,\varsigma,\vartheta,\beta(\varrho,\vartheta))d\vartheta,\tau\Bigg)d\xi,\tau\Bigg)\\
&\succcurlyeq\widetilde{Z}\Bigg(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\xi)(\Psi(\varrho,\varsigma)-\Psi(\varrho,\xi))^{\iota-1}\Big[\widetilde{Z}\left(\boldsymbol{f}(\varrho,\xi,\alpha(\varrho,\xi))-\boldsymbol{f}(\varrho,\xi,\beta(\varrho,\xi)),\tau\right)\\
&\quad\circledast_M\widetilde{Z}\left(\int_0^{\xi}\boldsymbol{k}(\varrho,\varsigma,\vartheta,\alpha(\varrho,\vartheta))-\boldsymbol{k}(\varrho,\varsigma,\vartheta,\beta(\varrho,\vartheta))d\vartheta,\tau\right)\Big]d\xi,\tau\Bigg)\\
&\succcurlyeq\widetilde{Z}\Bigg(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\xi)(\Psi(\varrho,\varsigma)-\Psi(\varrho,\xi))^{\iota-1}\widetilde{Z}\left(\alpha(\varrho,\xi)-\beta(\varrho,\xi),\frac{\tau}{L_{\boldsymbol{f}}}\right)\\
&\quad\circledast_M\widetilde{Z}\left(\alpha(\varrho,\xi)-\beta(\varrho,\xi),\frac{\tau}{L_{\boldsymbol{k}}}\right)\Big]d\xi,\tau\Bigg)\\
&\succcurlyeq\widetilde{Z}\left(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\xi)(\Psi(\varrho,\varsigma)-\Psi(\varrho,\xi))^{\iota-1}\widetilde{Z}\left(\alpha(\varrho,\xi)-\beta(\varrho,\xi),\frac{\tau}{L_{\boldsymbol{f}}+L_{\boldsymbol{k}}}\right)d\xi,\tau\right)\\
&\succcurlyeq\widetilde{Z}\left(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\xi)(\Psi(\varrho,\varsigma)-\Psi(\varrho,\xi))^{\iota-1}\varphi\left(\xi,\frac{\tau}{\mathbf{C}_{\alpha\beta}(L_{\boldsymbol{f}}+L_{\boldsymbol{k}})}\right)d\xi,\tau\right)\\
&\succcurlyeq\varphi\left(\varsigma,\frac{\tau}{M\mathbf{C}_{\alpha\beta}(L_{\boldsymbol{f}}+L_{\boldsymbol{k}})}\right),
\end{aligned}$$

and we conclude

$$\rho(\Lambda\alpha,\Lambda\beta)\leqslant\frac{\tau}{M\mathbf{C}_{\alpha\beta}(L_{\boldsymbol{f}}+L_{\boldsymbol{k}})},$$

for all $\varsigma \in \Xi_5$ and $\tau \in \Xi_2$. Hence, we deduce:

$$\rho(\Lambda\alpha,\Lambda\beta)\leq[M\,(L_{\boldsymbol{f}}+L_{\boldsymbol{k}})]\rho(\alpha,\beta)$$

for any $\alpha,\beta \in U$, where $0 < M\,(L_{\boldsymbol{f}}+L_{\boldsymbol{k}}) < 1$.

From (3.8), we can identify a constant $\mathbf{C} \in \Xi_2$, such that

$$\widetilde{Z}\Big(\Lambda\beta_0(\varrho,\varsigma)-\beta_0(\varrho,\varsigma),\tau\Big)$$

$$= \widetilde{Z}\bigg(\frac{(\Psi(\varrho,\varsigma)-\Psi(\varrho,0))^{\gamma-1}}{\Gamma(\gamma)}\sigma + \mathcal{I}_{0+}^{\iota;\Psi}\boldsymbol{f}(\varrho,\varsigma,\beta_0(\varrho,\varsigma))$$
$$+\mathcal{I}_{0+}^{\iota;\Psi}\bigg[\int_0^{\xi}\boldsymbol{f}(\varrho,\varsigma,\vartheta,\beta_0(\varrho,\vartheta))d\vartheta\bigg]-\beta_0(\varrho,\varsigma),\tau\bigg)$$
$$\succcurlyeq \varphi\bigg(\varsigma,\frac{\tau}{\mathbf{C}}\bigg),$$

for any $\beta_0 \in U$, for all $\varsigma \in \Xi_5$, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$. The boundedness of

$$\boldsymbol{f}(\varrho,\xi,\beta_0(\varrho,\xi)),\ \ \boldsymbol{k}(\varrho,\varsigma,\vartheta,\beta_0(\varrho,\vartheta))\ \text{ and }\ \beta_0(\varrho,\varsigma),$$

together with (3.7), guarantees that $\rho(\Lambda\beta_0,\beta_0) < \infty$. From Theorem 2.3, we can find a CRO $\mu_0 : \Upsilon \times \Xi_5 \to \mathbb{R}$ such that $\Lambda^n\mu_0 \to \mu_0$ in (U,ρ) and $\Lambda\mu_0 = \mu_0$.

Since β and μ_0 are bounded on Ξ_5 for each $\beta \in U$ and $\min_{\varsigma\in\Xi_5}\varphi(\varsigma,\tau) > 0$, we obtain a fixed constant $\mathbf{C}_{\alpha\beta} \in \Xi_4$ such that:

$$\widetilde{Z}\bigg(\beta_0(\varrho,\varsigma)-\beta(\varrho,\varsigma),\tau\bigg) \succcurlyeq \varphi\left(\varsigma,\frac{\tau}{\mathbf{C}_\beta}\right),$$

for any $\varsigma \in \Xi_5$, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$. Thus $\rho(\beta_0,\beta) < \infty$ for all $\beta \in U$.

Therefore, we have $U = \{\beta \in U : \rho(\beta_0,\beta) < \infty\}$. Additionally, Theorem 2.3 and (3.5), imply the uniqueness of μ_0.

Using (3.3) and [8, Theorem 5], we obtain

$$\widetilde{Z}\bigg(\mu(\varrho,\varsigma)-\frac{(\Psi(\varrho,\varsigma)-\Psi(\varrho,0))^{\gamma-1}}{\Gamma(\gamma)}\sigma$$
$$-\mathcal{I}_{0+}^{\iota;\Psi}\boldsymbol{f}(\varrho,\varsigma,\mu(\varrho,\varsigma))-\mathcal{I}_{0+}^{\iota;\Psi}\bigg[\int_0^{\xi}\boldsymbol{k}(\varrho,\varsigma,\vartheta,\mu(\varrho,\vartheta))d\vartheta\bigg],\tau\bigg)$$
$$\succcurlyeq \frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\xi)(\Psi(\varrho,\varsigma)-\Psi(\varrho,\xi))^{\iota-1}\varphi(\xi,\tau)d\xi.$$

Then, from (3.4) and (3.8), we obtain

$$\widetilde{Z}\bigg(\mu(\varrho,\varsigma)-\Lambda\mu(\varrho,\varsigma),\tau\bigg)$$
$$\succcurlyeq \frac{1}{\Gamma(\iota)}\int_0^{\varsigma}\Psi'(\varrho,\xi)(\Psi(\varrho,\varsigma)-\Psi(\varrho,\xi))^{\iota-1}\varphi(\xi,\tau)d\xi$$
$$\succcurlyeq \varphi\left(\varsigma,\frac{\tau}{M}\right),$$

for any $\varsigma \in \Xi_5$, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$, and it follows that

$$\rho(\mu,\Lambda\mu) \le M. \tag{3.11}$$

where, by applying Theorem 2.3 and (3.11), we infer that

$$\rho(\mu,\mu_0) \le \frac{1}{1-M(L_{\boldsymbol{f}}+L_{\boldsymbol{k}})}\rho(\Lambda\mu,\mu) \le \frac{M}{1-M(L_{\boldsymbol{f}}+L_{\boldsymbol{k}})},$$

which leads to the conclusion in (3.6). □

Theorem 3.2. *Let $\iota, \kappa \in \mathring{\Xi}_5$. Consider the nondecreasing random operator $\varPsi \in C^1(\Upsilon \times \Xi_5)$ with $\varPsi'(\varrho, \varsigma) \neq 0$ for all $\varsigma \in \Xi_5$. Let $L_{\boldsymbol{f}}, L_{\boldsymbol{k}} \in \Xi_2$ be fixed constants such that $\frac{(L_{\boldsymbol{f}}+L_{\boldsymbol{k}})}{\Gamma(\iota+1)} \in \mathring{\Xi}_5$. Let the CROs $\boldsymbol{f} : \Upsilon \times \Xi_5 \times \mathbb{R} \to \mathbb{R}$ and $\boldsymbol{k} : \Upsilon \times \Xi_5 \times \Xi_5 \times \mathbb{R} \to \mathbb{R}$ satisfy the conditions in (3.1) and (3.2), respectively. Let $\varepsilon \in \mathring{\Xi}_5$, and consider the continuously differentiable random operator $\mu : \Upsilon \times \Xi_5 \to \mathbb{R}$ such that*

$$\widetilde{Z}\left({}^{H}\mathbb{D}_{0+}^{\iota,\kappa;\varPsi}\mu(\varrho,\varsigma) - \boldsymbol{f}(\varrho,\varsigma,\mu(\varrho,\varsigma)) - \int_0^{\varsigma} \boldsymbol{k}(\varrho,\varsigma,\vartheta,\mu(\varrho,\vartheta))d\vartheta, \tau\right) \succcurlyeq diag\Big(\varepsilon,\varepsilon,\varepsilon\Big),$$

and

$$\widetilde{Z}\Big((\varPsi(\varrho,\varsigma) - \varPsi(\varrho,0))^{\iota}, \tau\Big) \succcurlyeq \widetilde{Z}\Big(\varsigma,\tau\Big),$$

for all $\varsigma, \vartheta \in \Xi_5$, $\mu \in \mathbb{R}$, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$. Under these conditions, we can find a unique CRO $\mu_0 : \Upsilon \times \Xi_5 \to \mathbb{R}$ satisfying (3.5) and

$$\begin{aligned}\widetilde{Z}\Big(\mu(\varrho,\varsigma) - \mu_0(\varrho,\varsigma), \tau\Big) \\ \succcurlyeq diag\Bigg(&\frac{(\varPsi(\varrho,T)-\varPsi(\varrho,0))^{\iota}\varepsilon}{\Gamma(\iota+1) - (\varPsi(\varrho,T)-\varPsi(\varrho,0))^{\iota}[T\,(L_{\boldsymbol{f}}+L_{\boldsymbol{k}})]}, \\ &\frac{(\varPsi(\varrho,T)-\varPsi(\varrho,0))^{\iota}\varepsilon}{\Gamma(\iota+1) - (\varPsi(\varrho,T)-\varPsi(\varrho,0))^{\iota}[T\,(L_{\boldsymbol{f}}+L_{\boldsymbol{k}})]}, \\ &\frac{(\varPsi(\varrho,T)-\varPsi(\varrho,0))^{\iota}\varepsilon}{\Gamma(\iota+1) - (\varPsi(\varrho,T)-\varPsi(\varrho,0))^{\iota}[T\,(L_{\boldsymbol{f}}+L_{\boldsymbol{k}})]}\Bigg),\end{aligned} \tag{3.12}$$

for all $\varsigma \in \Xi_5$, $\mu \in \mathbb{R}$, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$.

Proof. Let $U = \{\alpha : \Upsilon \times \Xi_5 \to \mathbb{R} \text{ is CRO}\}$. Consider the Ξ_4-valued metric on U defined by

$$\rho(\alpha,\beta) = \inf\left\{\mathbf{C} \in \Xi_4 : \widetilde{Z}\Big(\alpha(\varrho,\varsigma) - \beta(\varrho,\varsigma), \tau\Big) \succcurlyeq diag\Big(\frac{\tau}{\tau+\mathbf{C}}, \frac{\tau}{\tau+\mathbf{C}}, \frac{\tau}{\tau+\mathbf{C}}\Big)\right\}, \tag{3.13}$$

for each $\varsigma \in \Xi_5$, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$. In [5] the authors established the completeness of the metric space (U, ρ).

Let $\Lambda : U \to U$ be defined by

$$\begin{aligned}\Lambda\alpha(\varrho,\varsigma) = &\frac{(\varPsi(\varrho,\varsigma)-\varPsi(\varrho,0))^{\gamma-1}}{\Gamma(\gamma)}\sigma \\ &+ \mathcal{I}_{0+}^{\iota;\varPsi}\boldsymbol{f}(\varrho,\varsigma,\alpha(\varrho,\varsigma)) \\ &+ \mathcal{I}_{0+}^{\iota;\varPsi}\left[\int_0^{\xi}\boldsymbol{k}(\varrho,\varsigma,\vartheta,\alpha(\varrho,\vartheta))d\vartheta\right],\end{aligned} \tag{3.14}$$

for all $\varsigma \in \Xi_5$ and $\varrho \in \Upsilon$.

Let $\alpha, \beta \in U$ and assume a fixed constant $\mathbf{C}_{\alpha\beta} \in \Xi_4$ such that $\rho(\alpha,\beta) \leq \mathbf{C}_{\alpha\beta}$, and

$$\widetilde{Z}\Big(\alpha(\varrho,\varsigma) - \beta(\varrho,\varsigma), \tau\Big) \succcurlyeq diag\left(\frac{\tau}{\tau+\mathbf{C}_{\alpha\beta}}, \frac{\tau}{\tau+\mathbf{C}_{\alpha\beta}}, \frac{\tau}{\tau+\mathbf{C}_{\alpha\beta}}\right), \tag{3.15}$$

for every $\varsigma \in \Xi_5$, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$. Now we will show

$$\rho(\Lambda\alpha, \Lambda\beta) \leq \left(\frac{\tau\left(L_{\boldsymbol{f}} + L_{\boldsymbol{k}}\right)}{\Gamma(\iota+1)\left(\tau + \mathbf{C}_{\alpha\beta}\right)}\right)\rho(\alpha, \beta).$$

To see this note from (3.1), (3.2), (3.14) and (3.15) that

$$\begin{aligned}
&\widetilde{Z}\bigg(\Lambda\alpha(\varrho,\varsigma) - \Lambda\beta(\varrho,\varsigma), \tau\bigg) \\
&= \widetilde{Z}\bigg(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma} \Psi'(\varrho,\xi)(\Psi(\varrho,\varsigma)-\Psi(\varrho,\xi))^{\iota-1}\widetilde{Z}\bigg(\boldsymbol{f}(\varrho,\xi,\alpha(\varrho,\xi)) - \boldsymbol{f}(\varrho,\xi,\beta(\varrho,\xi)) \\
&\quad + \int_0^{\xi} \boldsymbol{k}(\varrho,\varsigma,\vartheta,\alpha(\varrho,\vartheta)) - \boldsymbol{k}(\varrho,\varsigma,\vartheta,\beta(\varrho,\vartheta))d\vartheta, \tau\bigg)d\xi, \tau\bigg) \\
&\succcurlyeq \widetilde{Z}\bigg(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma} \Psi'(\varrho,\xi)(\Psi(\varrho,\varsigma)-\Psi(\varrho,\xi))^{\iota-1}\bigg[\widetilde{Z}\left(\boldsymbol{f}(\varrho,\xi,\alpha(\varrho,\xi)) - \boldsymbol{f}(\varrho,\xi,\beta(\varrho,\xi)), \tau\right) \\
&\quad \circledast_M \widetilde{Z}\left(\int_0^{\xi} \boldsymbol{k}(\varrho,\varsigma,\vartheta,\alpha(\varrho,\vartheta)) - \boldsymbol{k}(\varrho,\varsigma,\vartheta,\beta(\varrho,\vartheta))d\vartheta, \tau\right)\bigg]d\xi, \tau\bigg) \\
&\succcurlyeq \widetilde{Z}\bigg(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma} \Psi'(\varrho,\xi)(\Psi(\varrho,\varsigma)-\Psi(\varrho,\xi))^{\iota-1}\bigg[\widetilde{Z}\left(\alpha(\varrho,\xi) - \beta(\varrho,\xi), \frac{\tau}{L_{\boldsymbol{f}}}\right) \\
&\quad \circledast_M \widetilde{Z}\left(\alpha(\varrho,\xi) - \beta(\varrho,\xi), \frac{\tau}{L_{\boldsymbol{k}}}\right)\bigg]d\xi, \tau\bigg) \\
&\succcurlyeq \widetilde{Z}\left(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma} \Psi'(\varrho,\xi)(\Psi(\varrho,\varsigma)-\Psi(\varrho,\xi))^{\iota-1}\widetilde{Z}\left(\alpha(\varrho,\xi) - \beta(\varrho,\xi), \frac{\tau}{L_{\boldsymbol{f}} + L_{\boldsymbol{k}}}\right)d\xi, \tau\right) \\
&\succcurlyeq \widetilde{Z}\left(\frac{1}{\Gamma(\iota)}\int_0^{\varsigma} \Psi'(\varrho,\xi)(\Psi(\varrho,\varsigma)-\Psi(\varrho,\xi))^{\iota-1}\left(\frac{\tau\left(L_{\boldsymbol{f}} + L_{\boldsymbol{k}}\right)}{\tau + \mathbf{C}_{\alpha\beta}}\right)d\xi, \tau\right) \\
&\succcurlyeq \widetilde{Z}\left((\Psi(\varrho,\varsigma)-\Psi(\varrho,0))^{\iota}\left(\frac{\tau\Gamma(\iota+1)\left(L_{\boldsymbol{f}} + L_{\boldsymbol{k}}\right)}{\Gamma(\iota+1)\left(\tau + \mathbf{C}_{\alpha\beta}\right)}\right), \tau\right) \\
&\succcurlyeq \widetilde{Z}\left((\Psi(\varrho,\varsigma)-\Psi(\varrho,0))^{\iota}, \frac{\tau}{\left(\frac{\tau\left(L_{\boldsymbol{f}}+L_{\boldsymbol{k}}\right)}{\Gamma(\iota+1)\left(\tau+\mathbf{C}_{\alpha\beta}\right)}\right)}\right) \\
&\succcurlyeq \widetilde{Z}\left(\varsigma, \frac{\tau}{\left(\frac{\tau\left(L_{\boldsymbol{f}}+L_{\boldsymbol{k}}\right)}{\Gamma(\iota+1)\left(\tau+\mathbf{C}_{\alpha\beta}\right)}\right)}\right),
\end{aligned}$$

for each $\varsigma \in \Xi_5$, $\tau \in \Xi_2$ and $\varrho \in \Upsilon$. Thus, we conclude:

$$\rho(\Lambda\alpha, \Lambda\beta) \leq \left(\frac{\tau\left(L_{\boldsymbol{f}} + L_{\boldsymbol{k}}\right)}{\Gamma(\iota+1)\left(\tau + \mathbf{C}_{\alpha\beta}\right)}\right)\rho(\alpha, \beta),$$

for each $\alpha, \beta \in U$ and $\varrho \in \Upsilon$. Let $\beta_0 \in U$. We can identify a constant $\mathbf{C} \in \Xi_2$ such that the following holds:

$$\widetilde{Z}\bigg(\Lambda\beta_0(\varrho,\varsigma) - \beta_0(\varrho,\varsigma), \tau\bigg)$$

$$
\begin{aligned}
&= \widetilde{Z}\bigg(\frac{(\Psi(\varrho,\varsigma)-\Psi(\varrho,0))^{\gamma-1}}{\Gamma(\gamma)}\sigma + \mathcal{I}_{0+}^{\iota;\Psi}\boldsymbol{f}(\varrho,\varsigma,\beta_0(\varrho,\varsigma)) \\
&+ \mathcal{I}_{0+}^{\iota;\Psi}\bigg[\int_0^{\xi}\boldsymbol{k}(\varrho,\varsigma,\vartheta,\beta_0(\varrho,\vartheta))d\vartheta\bigg] - \beta_0(\varrho,\varsigma),\tau\bigg) \\
&\succcurlyeq diag\bigg(\frac{\tau}{\tau+\mathbf{C}},\frac{\tau}{\tau+\mathbf{C}},\frac{\tau}{\tau+\mathbf{C}}\bigg),
\end{aligned}
$$

for all $\varsigma \in \Xi_5$ and $\varrho \in \Upsilon$.

The boundedness of the functions

$$\boldsymbol{f}(\varrho,\xi,\beta_0(\varrho,\xi)),\ \boldsymbol{k}(\varrho,\varsigma,\vartheta,\beta_0(\varrho,\vartheta)),\ \text{ and }\ \beta_0(\varrho,\varsigma),$$

along with (3.13), ensures that $\rho(\Lambda\beta_0,\beta_0) < \infty$.

By applying Theorem 2.3, we can construct a CRO $\mu_0 : \Upsilon \times \Xi_5 \to \mathbb{R}$, where $\Lambda^n\beta_0 \to \mu_0$ in (U,ρ) and $\Lambda\mu_0 = \mu_0$, ensuring that μ_0 satisfies (3.5). Using a method analogous to that in Theorem 3.1, we conclude that the set $\{\beta \in U : \rho(\beta_0,\beta) < \infty\} = U$. Additionally Theorem 2.3 and (3.5) guarantee the uniqueness of μ_0.

Next, applying (3.3) along with [8, Theorem 5], we obtain

$$
\begin{aligned}
&\widetilde{Z}\bigg(\mu(\varrho,\varsigma) - \frac{(\Psi(\varrho,\varsigma)-\Psi(\varrho,0))^{\gamma-1}}{\Gamma(\gamma)}\sigma - \mathcal{I}_{0+}^{\iota;\Psi}\boldsymbol{f}(\varrho,\varsigma,\mu_0(\varrho,\varsigma)) \\
&- \mathcal{I}_{0+}^{\iota;\Psi}\bigg[\int_0^{\xi}\boldsymbol{k}(\varrho,\varsigma,\vartheta,\mu_0(\varrho,\vartheta))d\vartheta\bigg], \frac{\tau\Gamma(\iota+1)}{(\Psi(\varrho,T)-\Psi(\varrho,0))^{\iota}}\bigg) \\
&\succcurlyeq diag\bigg(\varepsilon,\varepsilon,\varepsilon\bigg),
\end{aligned}
$$

for all $\varsigma \in \Xi_5$ and $\varrho \in \Upsilon$, which implies that

$$\rho(\mu,\Lambda\mu) \le \varepsilon\frac{(\Psi(\varrho,T)-\Psi(\varrho,0))^{\iota}}{\Gamma(\iota+1)}.$$

Using Theorem 2.3 and equation (3.7), we deduce:

$$
\begin{aligned}
&\widetilde{Z}\left(\mu(\varrho,\varsigma)-\mu_0(\varrho,\varsigma), \frac{\tau\left(\Gamma(\iota+1)-(\Psi(\varrho,T)-\Psi(\varrho,0))^{\iota}[L_{\boldsymbol{f}}+\frac{T}{2}L_{\boldsymbol{k}}]\right)}{(\Psi(\varrho,T)-\Psi(\varrho,0))^{\iota}}\right) \\
&\qquad\qquad \succcurlyeq diag\bigg(\varepsilon,\varepsilon,\varepsilon\bigg),
\end{aligned}
$$

which gives (3.12) for all $\varsigma \in \Xi_5$. □

References

[1] A. Ahadi and R. Saadati, *Generalized Z-number approximation for the fractional two-point iterative equation with a boundary condition*, Information Sciences **649** (2023): 119673.

[2] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4** (2003): Article 4, 7 pp.

[3] J. B. Diaz and B. Margolis *A fixed point theorem of the alternative, for contractions on a generalized complete metric space*, Bull. In Amer. Math. Soc. **74**(1968), 305–309.

[4] P. Muniyappan and S. Rajan, *Stability of a class of fractional integro-differential equation with nonlocal initial conditions*, Acta Math. Univ. Comenian. (N.S.) **87** (2018), 85–95.
[5] S. Sevgin and H. Sevli, *Stability of a nonlinear Volterra integro-differential equation via a fixed point approach*, J. Nonlinear Sci. Appl. **9** (2016), 200–207.
[6] J. V. da C. Sousa, K. D. Kucche and E. C. de Oliveira, *On the Ulam-Hyers stabilities of the solutions of Ψ-Hilfer fractional differential equation with abstract Volterra operator*, Math. Methods Appl. Sci. **42** (2019), 3021–3032.
[7] J. V. da C. Sousa and E. C. de Oliveira, *On a new operator in fractional calculus and applications*, J. Fixed Point Theory Appl. **20** (2018): Paper No. 96, 21 pp.
[8] J. V. da C. Sousa and E. C. de Oliveira, *On the Ψ-Hilfer fractional derivative*, Commun. Nonlinear Sci. Numer. Simul. **60** (2018), 72–91.
[9] J. R. Wang, M. Feckan and Y. Zhou, *A survey on impulsive fractional differential equations*, Fract. Calc. Appl. Anal. **19** (2016), 806–831.
[10] J. R. Wang and Y. Zhou, *Mittag-Leffler-Ulam stabilities of fractional evolution equations*, Appl. Math. Lett. **25** (2012), 723–728.

Manuscript received December 18 2024

Azam Ahadi
School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran
E-mail address: `ahadi_azam@mathdep.iust.ac.ir`

Reza Saadati
School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran
E-mail address: `rsaadati@eml.cc`

Donal O'Regan
School of Mathematical and Statistical Sciences, University of Galway, University Road, Galway, Ireland
E-mail address: `donal.oregan@nuigalway.ie`