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UNCERTAINTY RELATION FOR Q-COMMUTATOR

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Abstract. We show that an uncertainty relation for Wigner-Yanase-Dyson skew information proved by Yanagi [5] can be extended for q-commutator which is defined by $[A, B]_q = AB - qBA$, where A, B are self-adjoint operators and $q > 0$.

1. Introduction

Wigner-Yanase skew information

$$\begin{aligned} I_\rho(H) &= \frac{1}{2}Tr\left[\left(i\left[\rho^{1/2}, H\right]\right)^2\right] \\ &= Tr[\rho H^2] - Tr[\rho^{1/2}H\rho^{1/2}H] \end{aligned}$$

was defined in [4]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H. Here we denote the commutator by $[X, Y] = XY - YX$. This quantity was generalized by Dyson

$$\begin{aligned} I_{\rho,\alpha}(H) &= \frac{1}{2}Tr[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])] \\ &= Tr[\rho H^2] - Tr[\rho^\alpha H\rho^{1-\alpha}H], \alpha \in [0, 1] \end{aligned}$$

which is known as the Wigner-Yanase-Dyson skew information. It is well known that Heisenberg uncertainty relation is represented by the commutator in the following.

Theorem 1.1. *Let ρ be a density operator and A, B be self-adjoint operators. Then the following uncertainty relation holds.*

$$\frac{1}{4}|Tr[\rho[A, B]]|^2 \le V_\rho(A)V_\rho(B), \tag{1.1}$$

where $V_\rho(A) = Tr[\rho A^2] - (Tr[\rho A])^2$.

Luo [3] gave an extension of (1.1) in the following.

Theorem 1.2. *Let ρ be a density operator and A, B be self-adjoint operators. Then the following uncertainty relation holds.*

$$\frac{1}{4}|Tr[\rho[A, B]]|^2 \le U_\rho(A)U_\rho(B), \tag{1.2}$$

where $A_0 = A - Tr[\rho A]I$, $I_\rho(A) = \frac{1}{2}Tr[(i[\rho^{1/2}, A_0])^2]$, $J_\rho(A) = \frac{1}{2}Tr[\{\rho^{1/2}, A_0\}^2] = \frac{1}{2}Tr[(\rho^{1/2}A_0 + A_0\rho^{1/2})^2]$, *and* $U_\rho(A) = \sqrt{I_\rho(A)J_\rho(A)}$.

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Furthermore Yanagi [5] gave an extension of (1.2) in the following.

Theorem 1.3. *Let ρ be a density operator and A, B be self-adjoiint operators. Then for $0 \leq \alpha \leq 1$ the following uncertaity relation holds.*

$$\alpha(1-\alpha)|Tr[\rho[A,B]]|^2 \leq U_{\rho,\alpha}(A)U_{\rho,\alpha}(B), \tag{1.3}$$

where $A_0 = A - Tr[\rho A]I$, $I_{\rho,\alpha}(A) = \frac{1}{2}Tr[(i[\rho^\alpha, A_0])(i[\rho^{1-\alpha}, A_0])]$, $J_{\rho,\alpha}(A) = \frac{1}{2}Tr[\{\rho^\alpha, A_0\}\{\rho^{1-\alpha}, A_0\}]$ *and* $U_{\rho,\alpha}(A) = \sqrt{I_{\rho,\alpha}(A)J_{\rho,\alpha}(A)}$.

In this paper we have several extensions for generalized Heisenberg uncertainty relation (1.3) by using q-commutator.

2. q-commutator

Let $B(H), B(H)_s$ and $S(H)$ be the set of all bounded linear operators on Hilbert space H, the set of all self-adjoint operators and the set of all density operators. For $A, B \in B(H)_s$, we define commutator and anti-commutator by $[A,B] = AB - BA$ and $\{A,B\} = AB + BA$, respectively. q-commutator and q-anti-commutator are defined by $[A,B]_q = AB - qBA$ and $\{A,B\}_q = AB + qBA$, respectively, where $q \in \mathbb{R}$ and $A, B \in B(H)_s$. q-commutator is a generalization of commutator $[A,B]$.

Definition 2.1. Let $\rho \in S(H)$, $A, B \in B(H)_s$, $0 \leq \alpha \leq 1$ and $q > 0$.

(1) $I^q_{\rho,\alpha}(A) = \frac{1}{2}Tr[(i[\rho^\alpha, A_0]_q)(i[\rho^{1-\alpha}, A_0]_{1/q})]$.

(2) $J^q_{\rho,\alpha}(A) = \frac{1}{2}Tr[\{\rho^\alpha, A_0\}_q\{\rho^{1-\alpha}, A_0\}_{1/q}]$.

(3) $U^q_{\rho,\alpha}(A) = \sqrt{I^q_{\rho,\alpha}(A)J^q_{\rho,\alpha}(A)}$.

Now we state the properties of $I^q_{\rho,\alpha}(A)$ and $J^q_{\rho,\alpha}(A)$.

Proposition 2.2. *Let $\rho = \sum_{i=1}^\infty \lambda_i|\phi_i\rangle\langle\phi_i|$ be the spectral decomposition of ρ and let $a_{ij} = \langle\phi_i|A_0|\phi_j\rangle$*

(1) $I^q_{\rho,\alpha}(A) = \frac{1}{2}\sum_{i,j}\left\{\left(q+\frac{1}{q}\right)\frac{\lambda_i+\lambda_j}{2} - (\lambda_i^\alpha\lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^\alpha)\right\}|a_{ij}|^2$.

(2) $J^q_{\rho,\alpha}(A) = \frac{1}{2}\sum_{i,j}\left\{\left(q+\frac{1}{q}\right)\frac{\lambda_i+\lambda_j}{2} + (\lambda_i^\alpha\lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^\alpha)\right\}|a_{ij}|^2$.

Proof. (1)

$$\begin{aligned} I^q_{\rho,\alpha}(A) &= \frac{1}{2}Tr[(i[\rho^\alpha, A_0]_q)(i[\rho^{1-\alpha}, A_0]_{1/q})] \\ &= -\frac{1}{2}Tr\left[(\rho^\alpha A_0 - qA_0\rho^\alpha)\left(\rho^{1-\alpha}A_0 - \frac{1}{q}A_0\rho^{1-\alpha}\right)\right] \\ &= -\frac{1}{2}Tr\left[\rho^\alpha A_0\rho^{1-\alpha}A_0 - \frac{1}{q}\rho^\alpha A_0^2\rho^{1-\alpha} - qA_0\rho A_0 + A_0\rho^\alpha A_0\rho^{1-\alpha}\right] \end{aligned}$$

$$
\begin{aligned}
&= Tr\left[\frac{1}{2}\left(q+\frac{1}{q}\right)\rho A_0^2 - \rho^\alpha A_0 \rho^{1-\alpha} A_0\right] \\
&= \sum_{i,j} \frac{1}{2}\left(q+\frac{1}{q}\right)\lambda_i |a_{ij}|^2 - \sum_{i,j} \lambda_i^\alpha \lambda_j^{1-\alpha} |a_{ij}|^2 \\
&= \frac{1}{2}\sum_{i,j}\left\{\left(q+\frac{1}{q}\right)\frac{\lambda_i+\lambda_j}{2} - (\lambda_i^\alpha\lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^\alpha)\right\}|a_{ij}|^2.
\end{aligned}
$$

(2)

$$
\begin{aligned}
J_{\rho,\alpha}^q(A) &= \frac{1}{2}Tr[\{\rho^\alpha, A_0\}_q\{\rho^{1-\alpha}, A_0\}_{1/q}] \\
&= \frac{1}{2}Tr\left[(\rho^\alpha A_0 + qA_0\rho^\alpha)\left(\rho^{1-\alpha}A_0 + \frac{1}{q}A_0\rho^{1-\alpha}\right)\right] \\
&= \frac{1}{2}Tr\left[\rho^\alpha A_0 \rho^{1-\alpha} A_0 + \frac{1}{q}\rho^\alpha A_0^2 \rho^{1-\alpha} + qA_0\rho A_0 + A_0\rho^\alpha A_0 \rho^{1-\alpha}\right] \\
&= Tr\left[\frac{1}{2}\left(q+\frac{1}{q}\right)\rho A_0^2 + \rho^\alpha A_0 \rho^{1-\alpha} A_0\right] \\
&= \sum_{i,j} \frac{1}{2}\left(q+\frac{1}{q}\right)\lambda_i |a_{ij}|^2 + \sum_{i,j} \lambda_i^\alpha \lambda_j^{1-\alpha} |a_{ij}|^2 \\
&= \frac{1}{2}\sum_{i,j}\left\{\left(q+\frac{1}{q}\right)\frac{\lambda_i+\lambda_j}{2} + (\lambda_i^\alpha\lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^\alpha)\right\}|a_{ij}|^2.
\end{aligned}
$$

□

We need the following lemma in order to prove the main theorem.

Lemma 2.3. *For $0 \le \alpha \le 1$, $t > 0$ and $q > 0$, the following inequality holds.*

$$
(1-2\alpha)^2(qt-1)^2 - (q^\alpha t^\alpha - q^{1-\alpha}t^{1-\alpha})^2 \ge 0. \tag{2.1}
$$

Proof. Let $F(t) = (1-2\alpha)^2(qt-1)^2 - (q^\alpha t^\alpha - q^{1-\alpha}t^{1-\alpha})^2$. We have

$$
\begin{aligned}
F'(t) &= 2(1-2\alpha)^2 q(qt-1) - 2(q^\alpha t^\alpha - q^{1-\alpha}t^{1-\alpha})(\alpha q^\alpha t^{\alpha-1} - (1-\alpha)q^{1-\alpha}t^{-\alpha}) \\
&= 2(1-2\alpha)^2 q(qt-1) - 2\alpha q^{2\alpha}t^{2\alpha-1} + 2(1-\alpha)q \\
&\quad + 2\alpha q - 2(1-\alpha)q^{2(1-\alpha)}t^{1-2\alpha}. \\
F''(t) &= 2(1-2\alpha)^2 q^2 - 2\alpha(2\alpha-1)q^{2\alpha}t^{2\alpha-2} - 2(1-\alpha)(1-2\alpha)q^{2(1-\alpha)}t^{-2\alpha}. \\
F'''(t) &= -2\alpha(2\alpha-1)(2\alpha-2)q^{2\alpha}t^{2\alpha-3} + 4(1-\alpha)(1-2\alpha)\alpha q^{2(1-\alpha)}t^{-2\alpha-1} \\
&= 4\alpha(1-\alpha)(1-2\alpha)\left\{\frac{q^{2(1-\alpha)}}{t^{2\alpha+1}} - \frac{q^{2\alpha}}{t^{3-2\alpha}}\right\}.
\end{aligned}
$$

When $1-2\alpha > 0$, $F'''(t) \ge 0$ is equivalent to $t \ge q^{-1}$ and $F'''(t) < 0$ is equivalent to $t < q^{-1}$. Since $F''(q^{-1}) = 0$, we have $F''(t) > 0$ for $t > 0$. When $1-2\alpha < 0$,

$F^{'''}(t) \geq 0$ is equivalent to $t \geq q^{-1}$ and $F^{'''}(t) < 0$ is equivalent to $t < q^{-1}$. Since $F^{''}(q^{-1}) = 0$, we have $F^{''}(t) > 0$ for $t > 0$. And since $F^{'}(q^{-1}) = 0$, we have $F^{'}(t) < 0$ for $t < q^{-1}$ and $F^{'}(t) > 0$ for $t > q^{-1}$. Since $F(q^{-1}) = 0$, we have $F(t) \geq 0$ for $t > 0$. Then we have the result. □

We obtain the main theorem.

Theorem 2.4. *Let $\rho \in S(H)$, $A, B \in B(H)_s$ and $0 \leq \alpha \leq 1$. If $0 < q \leq 1$, then*

$$\alpha(1-\alpha)|Tr[\rho[A_0, B_0]_q]|^2 \leq U^q_{\rho,\alpha}(A)U^q_{\rho,\alpha}(B),$$

where $A_0 = A - Tr[\rho A]I$ and $B_0 = B - Tr[\rho B]I$. If $q > 1$, then

$$\alpha(1-\alpha)|Tr[\rho[A_0, B_0]_q]|^2 \leq q^2 U^q_{\rho,1-\alpha}(A)U^q_{\rho,1-\alpha}(B).$$

Proof. We put $t = \frac{\lambda_j}{\lambda_i}$ in (2.1). Then we have

$$(1-2\alpha)^2\left(q\frac{\lambda_j}{\lambda_i} - 1\right)^2 - \left(q^\alpha\left(\frac{\lambda_j}{\lambda_i}\right)^\alpha - q^{1-\alpha}\left(\frac{\lambda_j}{\lambda_i}\right)^{1-\alpha}\right)^2 \geq 0.$$

And we get

$$(1-2\alpha)^2(q\lambda_j - \lambda_i)^2 - (q^\alpha\lambda_j^\alpha\lambda_i^{1-\alpha} - q^{1-\alpha}\lambda_j^{1-\alpha}\lambda_i^\alpha)^2 \geq 0$$

and

$$(q\lambda_j - \lambda_i)^2 - (q^\alpha\lambda_j^\alpha\lambda_i^{1-\alpha} - q^{1-\alpha}\lambda_j^{1-\alpha}\lambda_i^\alpha)^2 \geq 4\alpha(1-\alpha)(q\lambda_j - \lambda_i)^2$$

and

$$(q\lambda_j + \lambda_i)^2 - (q^\alpha\lambda_j^\alpha\lambda_i^{1-\alpha} + q^{1-\alpha}\lambda_j^{1-\alpha}\lambda_i^\alpha)^2 \geq 4\alpha(1-\alpha)(q\lambda_j - \lambda_i)^2.$$

Then

$$\begin{aligned} 2\sqrt{\alpha(1-\alpha)}|\lambda_i - q\lambda_j| &\leq \{(\lambda_i + q\lambda_j)^2 - (q^\alpha\lambda_j^\alpha\lambda_i^{1-\alpha} + q^{1-\alpha}\lambda_j^{1-\alpha}\lambda_i^\alpha)^2\}^{1/2} \\ &= (\lambda_i + q\lambda_j - q^\alpha\lambda_j^\alpha\lambda_i^{1-\alpha} - q^{1-\alpha}\lambda_j^{1-\alpha}\lambda_i^\alpha)^{1/2} \\ &\qquad (\lambda_i + q\lambda_j + q^\alpha\lambda_j^\alpha\lambda_i^{1-\alpha} + q^{1-\alpha}\lambda_j^{1-\alpha}\lambda_i^\alpha)^{1/2}. \end{aligned}$$

Since

$$Tr[\rho[A_0, B_0]_q] = \sum_{i,j}(\lambda_i - q\lambda_j)\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle,$$

we have

$$\begin{aligned} &\alpha(1-\alpha)|Tr[\rho[A_0, B_0]_q]|^2 \\ &\leq \alpha(1-\alpha)\left\{\sum_{i,j}|\lambda_i - q\lambda_j||a_{ij}||b_{ji}|\right\}^2 \\ &= \frac{1}{4}\left\{\sum_{i,j}2\sqrt{\alpha(1-\alpha)}|\lambda_i - q\lambda_j||a_{ij}||b_{ji}|\right\}^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{4}\left\{\sum_{i,j}\{(\lambda_i+q\lambda_j)^2-(q^{\alpha}\lambda_j^{\alpha}\lambda_i^{1-\alpha}+q^{1-\alpha}\lambda_j^{1-\alpha}\lambda_i^{\alpha})^2\}^{1/2}|a_{ij}||b_{ji}|\right\}^2 \\
&\leq \frac{1}{2}\left\{\sum_{i,j}(\lambda_i+q\lambda_j-q^{\alpha}\lambda_j^{\alpha}\lambda_i^{1-\alpha}-q^{1-\alpha}\lambda_j^{1-\alpha}\lambda_i^{\alpha}))|a_{ij}|^2\right\} \\
&\quad\times\frac{1}{2}\left\{\sum_{i,j}(\lambda_i+q\lambda_j+q^{\alpha}\lambda_j^{\alpha}\lambda_i^{1-\alpha}+q^{1-\alpha}\lambda_j^{1-\alpha}\lambda_i^{\alpha}))|b_{ji}|^2\right\} \\
&= \frac{1}{2}\left\{\sum_{i,j}\left((1+q)\frac{\lambda_i+\lambda_j}{2}-\frac{q^{\alpha}+q^{1-\alpha}}{2}(\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha})\right)|a_{ij}|^2\right\} \\
&\quad\times\frac{1}{2}\left\{\sum_{i,j}(1+q)\frac{\lambda_i+\lambda_j}{2}+\frac{q^{\alpha}+q^{1-\alpha}}{2}(\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha}))|b_{ji}|^2\right\} \\
&\leq \frac{1}{2}\left\{\sum_{i,j}\left(\left(q+\frac{1}{q}\right)\frac{\lambda_i+\lambda_j}{2}-(\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha})\right)|a_{ij}|^2\right\} \\
&\quad\times\frac{1}{2}\left\{\sum_{i,j}\left(\left(q+\frac{1}{q}\right)\frac{\lambda_i+\lambda_j}{2}+(\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha})\right)|b_{ji}|^2\right\}.
\end{aligned}
$$

Because the last inequality is given by the following inequality. Since $0<q\leq 1$,

$$
\begin{aligned}
&\left(q+\frac{1}{q}-1-q\right)\frac{\lambda_i+\lambda_j}{2}-\left(1-\frac{q^{\alpha}+q^{1-\alpha}}{2})(\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha}\right) \\
&\geq \left(\frac{1}{q}-1\right)\frac{\lambda_i+\lambda_j}{2}-\left(1-\frac{q^{\alpha}+q^{1-\alpha}}{2}\right)(\lambda_i+\lambda_j) \\
&= \left(\frac{1}{q}-1\right)\frac{\lambda_i+\lambda_j}{2}-\left(2-q^{\alpha}-q^{1-\alpha}\right)\frac{\lambda_i+\lambda_j}{2} \\
&= \left(\frac{1}{q}-3+q^{\alpha}+q^{1-\alpha}\right)\frac{\lambda_i+\lambda_j}{2} \\
&\geq \left(\frac{1}{q}-3+2q^{1/2}\right)\frac{\lambda_i+\lambda_j}{2} \\
&= \frac{(q^{1/2}-1)^2(2q^{1/2}+1)}{q}\frac{\lambda_i+\lambda_j}{2}\geq 0.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
&\left(q+\frac{1}{q}-1-q\right)\frac{\lambda_i+\lambda_j}{2}+\left(1-\frac{q^{\alpha}+q^{1-\alpha}}{2}\right)(\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha}) \\
&= \left(\frac{1}{q}-1\right)\frac{\lambda_i+\lambda_j}{2}+\left(1-\frac{q^{\alpha}+q^{1-\alpha}}{2}\right)(\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha})\geq 0.
\end{aligned}
$$

Then we have

$$\alpha(1-\alpha)|Tr[\rho[A_0,B_0]_q]|^2 \leq I^q_{\rho,\alpha}(A)J^q_{\rho,\alpha}(B).$$

We also have

$$\alpha(1-\alpha)|Tr[\rho[A_0,B_0]_q]|^2 \leq J^q_{\rho,\alpha}(A)I^q_{\rho,\alpha}(B).$$

Hence when $0 < q \leq 1$, we obtain

$$\alpha(1-\alpha)|Tr[\rho[A_0,B_0]_q]|^2 \leq U^q_{\rho,\alpha}(A)U^q_{\rho,\alpha}(B).$$

On the other hand when $q > 1$, we have

$$\alpha(1-\alpha)|Tr[\rho[A_0,B_0]_{1/q}]|^2 \leq U^{1/q}_{\rho,\alpha}(A)U^{1/q}_{\rho,\alpha}(B) = U^q_{\rho,1-\alpha}(A)U^q_{\rho,1-\alpha}(B).$$

Since $[A_0,B_0]_q = -q[B_0,A_0]_{1/q}$, we have

$$\begin{aligned}\alpha(1-\alpha)|Tr[\rho[A_0,B_0]_q]|^2 &= q^2\alpha(1-\alpha)|Tr[\rho[B_0,A_0]_{1/q}]|^2 \\ &\leq q^2U^q_{\rho,1-\alpha}(A)U^q_{\rho,1-\alpha}(B).\end{aligned}$$

□

When $q = 1$ in Theorem 2.4, we get Theorem 1.3. And by putting $\alpha = \frac{1}{2}$, we have the following.

Corollary 2.5. *Let $\rho \in S(H)$, $A, B \in B(H)_s$. For $q > 0$ we have the following.*

$$\frac{1}{4}|Tr[\rho[A_0,B_0]_q]|^2 \leq \max\{1,q^2\}U^q_{\rho,\frac{1}{2}}(A)U^q_{\rho,\frac{1}{2}}(B).$$

By defining $[[A,B]] = \frac{1}{2}\{[A_0,B_0]_q - [B_0,A_0]_q\} = \frac{1+q}{2}[A_0,B_0]$, we have the following theorem. We omit the proof.

Theorem 2.6. *For $0 \leq \alpha \leq 1$ and $q > 0$, the following inequality holds.*

$$\alpha(1-\alpha)|Tr[\rho[[A_0,B_0]]]|^2 \leq \left(\frac{1+q}{2}\right)^2 U_{\rho,\alpha}(A)U_{\rho,\alpha}(B).$$

Remark 2.7. We define $I^q_{\rho,\alpha}(A)$, $J^q_{\rho,\alpha}(A)$ and $U^q_{\rho,\alpha}(A)$ as the following. For $\rho \in S(H)$, $A, B \in B(H)_s$, $0 \leq \alpha \leq 1$ and $q > 0$,

(1) $I^q_{\rho,\alpha}(A) = \frac{1}{2}Tr[(i[\rho^\alpha, A_0]_q)(i[\rho^{1-\alpha}, A_0]_q)]$.

(2) $J^q_{\rho,\alpha}(A) = \frac{1}{2}Tr[\{\rho^\alpha, A_0\}_q\{\rho^{1-\alpha}, A_0\}_q]$.

(3) $U^q_{\rho,\alpha}(A) = \sqrt{I^q_{\rho,\alpha}(A)J^q_{\rho,\alpha}(A)}$.

Then the corresponding uncertainty relation does not hold even if we choose the appropreate $q > 0$.

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