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RELAXATION AND BOLZA PROBLEM INVOLVING A SECOND ORDER EVOLUTION INCLUSION

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ABSTRACT. We present a relaxation problem in control theory when the dynamics are governed by a second order evolution inclusion with antiperiodic boundary conditions and its application to a Bolza type problem. A related variational convergence result is also investigated.

1. INTRODUCTION

This paper is devoted to a relaxation problem in control theory when the dynamics are governed by a second order evolution inclusion with antiperiodic boundary conditions and its application to a Bolza type problem. In the first order evolution inclusions, some related results are given in [21] [12], [11], [16], [17]. In [6], [13] the authors present some results in this framework when the dynamic is given by a second order ordinary differential equation with two points boundary conditions. Existence and uniqueness of antiperiodic solution appeared in a series of works [2], [3], [4], [5], [19], [26], [27], [15]. In this paper, we present some relaxation results in a second order evolution inclusion with antiperiodic boundary conditions. Our proofs rely on some results on existence and uniqueness of antiperiodic solution of a second order evolution inclusion with convex compact valued upper semicontinuous perturbation and the stable convergence of Young measures [12]. In section 3 we state some existence and uniqueness results of anti-periodic solutions for these classes of second order evolution inclusions. In section 4 we present main results in relaxation problems for a second order evolution inclusion with antiperiodic boundary conditions via Young measures and its application to a Bolza problem. A variational convergence result in this class of evolution inclusion is also investigated.

2. PRELIMINARIES AND BACKGROUND

In this section, $(\Omega, \mathcal{S}, P)$ is a complete probability space, S and T are two Polish spaces and E is a separable Banach space. By $L^1_E(\Omega, \mathcal{S}, P)$ we denote the space of all Lebesgue-Bochner integrable E-valued functions defined on Ω. For the sake of completeness, we summarize some useful facts concerning Young measures. Let X be a completely regular Suslin space and let $\mathcal{C}^b(X)$ be the space of all bounded continuous functions defined on X. Let $\mathcal{M}^1_+(X)$ be the set of all Borel probability measures on X equipped with the narrow topology. A Young measure $\lambda : \Omega \to \mathcal{M}^1_+(X)$ is, by definition, a *scalarly measurable* mapping from Ω into $\mathcal{M}^1_+(X)$,

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that is, for every $f \in \mathcal{C}^b(X)$, the mapping $\omega \mapsto \langle f, \lambda_\omega \rangle := \int_X f(x)\, d\lambda_\omega(x)$ is $\mathcal{S}$-measurable. A sequence (λ^n) in the space of Young measures $\mathcal{Y}(\Omega, \mathcal{S}, P; \mathcal{M}^1_+(X))$ *stably converges* to a Young measure $\lambda \in \mathcal{Y}(\Omega, \mathcal{S}, P; \mathcal{M}^1_+(X))$ if the following holds

$$\lim_n \int_A [\int_X f(x)\, d\lambda^n_\omega(x)]\, dP(\omega) = \int_A [\int_X f(x)\, d\lambda_\omega(x)]\, dP(\omega)$$

for every $A \in \mathcal{S}$ and for every $f \in \mathcal{C}^b(X)$. We recall and summarize some results for Young measures.

Proposition 2.1 ([12, Theorem 3.3.1])**.** *Assume that S and T are Polish spaces. Let (μ^n) be a sequence in $\mathcal{Y}(\Omega, \mathcal{S}, P; \mathcal{M}^1_+(S))$ and let (ν^n) be a sequence in $\mathcal{Y}(\Omega, \mathcal{S}, P; \mathcal{M}^1_+(T))$. Assume that*

(i) *(μ^n) converges in probability to $\mu^\infty \in \mathcal{Y}(\Omega, \mathcal{S}, P; \mathcal{M}^1_+(S))$,*
(ii) *(ν^n) stably converges to $\nu^\infty \in \mathcal{Y}(\Omega, \mathcal{S}, P; \mathcal{M}^1_+(T))$.*

Then $(\mu^n \otimes \nu^n)$ stably converges to $\mu^\infty \otimes \nu^\infty$.

Proposition 2.2 ([12, Theorem 6.3.5])**.** *Assume that X and Z are Polish spaces. Let (u^n) be sequence of $\mathcal{S}$-measurable mappings from Ω into X such that (u^n) converges in probability to a $\mathcal{S}$-measurable mapping u^∞ from Ω into X and (v^n) be a sequence of $\mathcal{S}$-measurable mappings from Ω into Z such that (v^n) stably converges to $\nu^\infty \in \mathcal{Y}(\Omega, \mathcal{S}, P; \mathcal{M}^1_+(Z))$. Let $h : \Omega \times X \times Z \to \mathbf{R}$ be a Carathéodory integrand such that the sequence $(h(., u_n(.), v_n(.))$ is uniformly integrable. Then the following holds*

$$\lim_{n\to\infty} \int_\Omega h(\omega, u^n(\omega), v^n(\omega))\, dP(\omega) = \int_\Omega [\int_Z h(\omega, u^\infty(\omega), z)\, d\nu^\infty_\omega(z)]\, dP(\omega).$$

3. Some existence theorem in second order evolution inclusions

We begin with a second order evolution inclusion with upper semicontinuous convex weakly compact valued perturbation in a separable Hilbert space. For this purpose we need first a closure type lemma that is a particular form of a similar result given in ([13], Lemma 4.1).

Lemma 3.1. *Let H be a separable Hilbert space. Let φ be a convex lower semicontinuous function defined on H with values in $]-\infty, +\infty]$. Let $(u_n)_{n\in\mathbf{N}\cup\{\infty\}}$ be a sequence of measurable mapppings from $[0,T]$ into H. such that $u_n \to u_\infty$ pointwisely with respect to the norm topology. Assume that $\varphi(u_n(t))$ is finite for every $n \in \mathbf{N}\cup\{\infty\}$ and for every $t \in [0,T]$ and $(\zeta_n)_{n\in\mathbf{N}}$ is a sequence in $L^1_H([0,T])$ satisfying*

$$\zeta_n(t) \in \partial\varphi(u_n(t)) \quad \text{a.e.} \quad t \in [0,T]$$

for each $n \in \mathbf{N}$ and $\sigma(L^1_H, L^\infty_H)$ converging to $\zeta_\infty \in L^1_H([0,T])$. Then we have

$$\zeta_\infty(t) \in \partial\varphi(u_\infty(t)) \quad \text{a.e.} \quad t \in [0,T].$$

Proof. We will use Komlós techniques. See ([20] , [18]). Namely we may assume that (ζ_n) Komlós converges to ζ_∞ and $(|\zeta_n|)$ Komlós converges to $\rho_\infty \in L^1_{\mathbf{R}}([0,T])$, because the sequence (ζ_n) (resp. $(|\zeta_n|)$) is bounded in $L^1_H([0,T])$ (resp. $L^1_{\mathbf{R}}([0,T])$).

Accordingly there are a Lebesgue negligible set $\mathcal{M}$ in $[0,T]$ and subsequences (ζ'_m), $(|\zeta'_m|)$ such that

$$\lim_n \frac{1}{n}\sum_{m=1}^{n} \zeta'_m(t) = \zeta_\infty(t),$$

$$\lim_n \frac{1}{n}\sum_{m=1}^{n} |\zeta'_m|(t) = \rho_\infty(t),$$

for all $t \in [0,T] \setminus \mathcal{M}$. Let $\varepsilon > 0$ and let $t \in [0,T] \setminus \mathcal{M}$. By lower semicontinuity of φ and pointwise convergence of u_m to u_∞, there is $N_\varepsilon \in \mathbf{N}$ such that $||u_m(t) - u_\infty(t)|| \leq \varepsilon$ and that $\varphi(u_m(t)) \geq \varphi(u_\infty(t)) - \varepsilon$ for all $m \geq N_\varepsilon$. Then we have the estimate

$$\varphi(x) \geq \varphi(u_\infty(t)) - \varepsilon + \langle x - u_\infty(t), \zeta'_m(t)\rangle - |\zeta'^m|(t)\varepsilon$$

for all $x \in H$, using the classical definition of subdifferential in convex analysis and the preceding estimate. Applying the previous convergences in the last inequality gives

$$\varphi(x) \geq \varphi(u_\infty(t)) - \varepsilon + \langle x - u_\infty(t), \zeta_\infty(t)\rangle - \rho_\infty(t)\varepsilon$$

As ε is arbitrary > 0 we finally get

$$\varphi(x) \geq \varphi(u_\infty(t)) + \langle \zeta_\infty(t), x - u_\infty(t)\rangle$$

for all $x \in H$. Whence we have $\zeta_\infty(t) \in \partial\varphi(u_\infty(t))$ a.e.. $\square$

Let us recall and summarize a classical closure type lemma.

Lemma 3.2. *Let H be a separable Hilbert space. Let φ be a convex lower semicontinuous function defined on H with values in $]-\infty, +\infty]$. Let $(u_n)_{n\in\mathbf{N}} \cup \{\infty\}$ be a sequence in $L^2_H([0,T])$ such that $(u_n)_{n\in\mathbf{N}}$ strongly converges to $u_\infty \in L^2_H([0,T])$. Assume that $\varphi(u_n(t))$ is finite for every $n \in \mathbf{N} \cup \{\infty\}$ and for every $t \in [0,T]$ and $(\zeta_n)_{n\in\mathbf{N}}$ is a sequence in $L^2_H([0,T])$ satisfying*

$$\zeta_n(t) \in \partial\varphi(u_n(t)) \quad \text{a.e.} \quad t \in [0,T]$$

for each $n \in \mathbf{N}$ and converging weakly to $\zeta_\infty \in L^2_H([0,T])$. Then we have

$$\zeta_\infty(t) \in \partial\varphi(u(t)) \quad \text{a.e.} \quad t \in [0,T].$$

Proof. It is well-known that $A := \partial\varphi$ is an maximal monotone operator in H. Let us denote by $\mathcal{A}$ its extension to $L^2_H([0,T])$ defined by $v \in \mathcal{A}u \iff v(t) \in Au(t)$ a.e $\forall u, v \in L^2_H([0,T])$. It is easy to see that $\mathcal{A}$ is monotone in $L^2_H([0,T])$. Let us check that $\mathcal{A}$ is maximal monotone. Let $g \in L^2_H([0,T])$. It is enough to show that there exists $v \in L^2_H([0,T]$ such that

$$g \in v + \mathcal{A}v$$

As A is maximal monotone, the function $v(t) = [I + A]^{-1}g(t)$ satisfies

$$g(t) \in [I_H + A]v(t) \iff g \in [I_{L^2_H([0,T])} + \mathcal{A}]v.$$

So it remains to check that $v \in L^2_H([0,T])$. Indeed, we have $v(t) = g(t) - prox_\gamma g(t)$, here γ is the conjugate of φ and $prox_\gamma$ is the prox mapping associated with γ. Since $prox_\gamma$ is a contraction, it is easy to check that the mapping $t \mapsto prox_\gamma g(t)$ belongs to $L^2_H([0,T])$. So $\mathcal{A}$ is maximal monotone in $L^2_H([0,T])$. As its graph is sequentially strong-weak closed, (u_n) strongly converges in $L^2_H([0,T]$ to u_∞, and (ζ_n) converges weakly in $L^2_H([0,T]$ to ζ, we have $\zeta \in \mathcal{A}u \Longleftrightarrow v(t) \in \partial\varphi(u(t))$ a.e. □

Remarks. Lemma 3.2 is well-known. However, we would like to mention the differences of techniques occuring in the two preceding results. The use of Komlós convergence or Mazur convergence appeared first in the recent works dealing with the proximal subdifferential of nonconvex lower semicontinuous functions [23], [13], [24]. In particular, it can be applied to nonconvex lower semicontinuous functions pln functions [22]. Komlós argument allows to prove the validity of Lemma 3.1 when (u_n) pointwisely converges to u_∞. An inspection of the proof of Lemma 3.1 shows that this technique is completetely independent of convex analysis and maximal monotone operators and can be used only in the case when φ_t does not depend on the parameter $t \in [0,T]$. Coming back to the proof of Lemma 3.2, we see that, it can be applied to the case when φ_t depends on a parameter $t \in [0,T]$, provided that the mapping $t \mapsto prox_{\gamma_t}x$ is Borel, or Lebesque mesurable for each fixed $x \in H$, γ_t being the conjugate of φ_t. See [25] for details concerning the theory of prox mappings in Hilbert spaces. These considerations allow to extend Lemma 3.2 to the case when φ_t depends on a parameter, by introducing the extension of the operator $A(t) = \partial\varphi_t$ to $L^2_H([0,T]$ by $v \in \mathcal{A}v \Longleftrightarrow v(t) \in A(t)u(t)$ a.e. The details are left to the reader. At this point let us mention some results related to Lemma 3.2, see [8] dealing of a family $A(t)$ of m-accretive operators in a separable reflexive Banach spaces such that its strong dual is uniformly convex and [28], [29] dealing with $A(t) := \partial\varphi_t$ in Hilbert spaces. In view of applications, it is worthwhile to mention that Lemma 3.2 is valid when $H = \mathbf{R}^d$ and $\partial\varphi$ is replaced by any m-accretive operator $A : H \rightrightarrows H$. Indeed, the graph of A is closed. So A is Borel, that is $A^- B := \{x \in H : A(x) \cap B \neq \emptyset\}$ is Borel for any closed subset of H so that $D(A)$ is Borel. It follows that $x \mapsto [I_H + A]^{-1}x$ is Borel, so that, for any Lebesgue measurable mapping $g : [0,T] \to H$ the mapping $t \mapsto [I_H + A]^{-1}g(t)$ is Lebesgue-measurable. See [8], [9] for details.

The following deal with a convex compact valued perturbation of a second order evolution governed by subdifferential operators of convex lower semicontinuous functions with anti-periodic boundary conditions. Compare with Lemma 3.4 in [2] dealing with single valued continuous perturbations and [5] dealing with first order evolution inclusions. For simplicity, we will assume that $H = \mathbf{R}^d$.

Theorem 3.3. *Let $H = \mathbf{R}^d$, $\gamma \in \mathbf{R}$, $\varphi : H \to]-\infty, +\infty]$ be a proper, convex, l.s.c and even function. Let $F : [0,T] \times H \rightrightarrows H$ be a convex compact valued mapping, separately scalarly measurable on $[0,T]$, separately scalarly upper semicontinuous on H satisfying: there is $r \in L^2_{\mathbf{R}^+}([0,T])$ such that $F(t,x) \subset \Gamma(t) := r(t)\overline{B}_H(0,1)$ for all $(t,x) \in [0,T] \times H$.*
Then the problem

$$\begin{cases} \ddot{u}(t) + \gamma\dot{u}(t) \in F(t,u(t)) + \partial\varphi(u(t)), & a.e. \quad t \in [0,T], \\ u(T) = -u(0), \quad \dot{u}(T) = -\dot{u}(0). \end{cases}$$

has at least an anti-periodic $W^{2,2}_H([0,T])$ solution.

Proof. Recall that a $W_H^{2,2}([0,T])$ function $u : [0,T] \to H$ is solution of the problem under consideration if there exists a function $h \in L^2_H([0,T])$ such that

$$\begin{cases} \ddot{u}(t) + \gamma \dot{u}(t) \in h(t) + \partial\varphi(u(t)), \quad a.e. \quad t \in [0,T], \\ h(t) \in F(t, u(t)) \quad a.e. \quad t \in [0,T], \\ u(T) = -u(0), \quad \dot{u}(T) = -\dot{u}(0). \end{cases}$$

Let us denote by $\mathcal{S}^2_\Gamma$ the set of all $L^2_H([0,T])$-selection of Γ

$$\mathcal{S}^2_\Gamma := \{f \in L^2_H([0,T]) : f(t) \in \Gamma(t) \quad a.e. \quad t \in [0,T]\}.$$

By virtue of ([2], Theorem 2.1) , for each $f \in \mathcal{S}^2_\Gamma$, there is a unique $W_H^{2,2}([0,T])$-solution u_f of

$$\begin{cases} \ddot{u}_f(t) + \gamma \dot{u}_f(t) \in f(t) + \partial\varphi(u_f(t)), \quad a.e. \quad t \in [0,T], \\ u_f(T) = -u_f(0), \quad \dot{u}_f(T) = -\dot{u}_f(0). \end{cases}$$

For each $f \in \mathcal{S}^2_\Gamma$, let us define the multifunction

$$\Psi(f) := \{g \in L^2_H([0,T]) : g(t) \in F(t, u_f(t)) \quad a.e. \quad t \in [0,T]\}.$$

Then is is clear that $\Psi(f)$ is a nonempty convex weaky compact subset of $\mathcal{S}^2_\Gamma$, here the nonemptiness follows from ([14], Theorem VI-6). From the above consideration, we need to prove that the convex weakly compact valued mapping $\Psi : \mathcal{S}^2_\Gamma \rightrightarrows \mathcal{S}^2_\Gamma$ admits a fixed point. By virtue of Kakutani-Ky Fan theorem, it is enough to prove that Ψ is upper semicontinuous when $\mathcal{S}^2_\Gamma$ is endowed with the weak topology of $L^2_H([0,T])$. As $L^2_H([0,T])$ is separable, $\mathcal{S}^2_\Gamma$ is compact metrizable with respect to the weak topology of $L^2_H([0,T])$. So it turns out to check that the graph $Gr(\Psi)$ is sequentially weakly closed in $\mathcal{S}^2_\Gamma \times \mathcal{S}^2_\Gamma$. Let $(f_n, g_n) \in Gr(\Psi)$ weakly converging to $(f,g) \in \mathcal{S}^2_\Gamma \times \mathcal{S}^2_\Gamma$. From the definition of Ψ, that means u_{f_n} is the $W_H^{2,2}([0,T])$ solution of

$$\begin{cases} \ddot{u}_{f_n}(t) + \gamma \dot{u}_{f_n}(t) \in f_n(t) + \partial\varphi(u_{f_n}(t)), \quad a.e. \quad t \in [0,T], \\ u_{f_n}(T) = -u_{f_n}(0), \quad \dot{u}_{f_n}(T) = -\dot{u}_{f_n}(0) \end{cases}$$

with $f_n \in \mathcal{S}^2_\Gamma$ and $g_n(t) \in F(t, u_{f_n}(t))$ a.e. $t \in [0,T]$. Taking account into the anti-periodicity of $\dot{u}_{f_n}$ and u_{f_n} and using the estimate ([2], Lemma 2.2)

$$||\ddot{u}_{f_n}||_{L^2_H([0,T])} \leq ||f_n||_{L^2_H([0,T])} \leq ||r||_{L^2_{\mathbf{R}^+}([0,T])}, \quad \forall n \in \mathbf{N},$$

we may conclude that

$$\sup_{n\geq 1} ||\dot{u}_{f_n}||_{\mathcal{C}_H([0,T])} < +\infty \quad \text{and} \quad \sup_{n\geq 1} ||u_{f_n}||_{\mathcal{C}_H([0,T])} < +\infty.$$

We may assume that $(\ddot{u}_{f_n})$ converges weakly in $L^2_H([0,T])$ to a function $w \in L^2_H([0,T])$ and $(\dot{u}_{f_n})$ pointwisely converges to a function v, namely

$$\begin{aligned} v(t) := \lim_n \dot{u}_{f_n}(t) = \lim_n [\dot{u}_{f_n}(0) + \int_0^t \ddot{u}_{f_n}(s)ds] \\ = \lim_n \dot{u}_{f_n}(0) + \int_0^t w(s)ds, \forall t \in [0,T]. \end{aligned}$$

and hence

$$u(t) := \lim_n u_{f_n}(t) = \lim_n [u_{f_n}(0) + \int_0^t \dot{u}_{f_n}(s)ds]$$
$$= \lim_n u_{f_n}(0) + \int_0^t v(s)ds, \forall t \in [0,T].$$

We conclude that $u \in W_H^{2,2}([0,T])$ with $\dot{u} = v$ and $\ddot{u} = w$ and satisfies the anti-periodic conditions: $u(T) = -u(0); \dot{u}(T) = -\dot{u}(0)$. Furthermore, it is easy to see that (u_{f_n}) converges pointwisely to u and $(\dot{u}_{f_n})$ converges to v with respect to the weak topology of $L_H^2([0,T])$. Combining these facts and applying Lemma 3.1 to the inclusion

$$\ddot{u}_{f_n}(t) + \gamma \dot{u}_{f_n}(t) - f_n(t) \in \partial\varphi(u_{f_n}(t))$$

yields

$$\ddot{u}(t) + \gamma \dot{u}(t) - f(t) \in \partial\varphi(u(t)) \quad a.e.$$

because $(\ddot{u}_{f_n} + \gamma \dot{u}_{f_n} - f_n)$ weakly converges to $\ddot{u} + \gamma\dot{u} - f$ and (u_{f_n}) pointwisely converges to u. By uniqueness, we have $u = u_f$. Further using the inclusion $g_n(t) \in F(t, u_{f_n}(t))$ a.e. and invoking the closure type lemma in ([14], Theorem VI-4), we have $g(t) \in F(t, u_f(t))$ a.e. The proof is therefore complete. □

A more general version of the preceding result is available by introducing some inf-compactness assumption ([2], page 396) on the function φ.

Proposition 3.4. *Let H be a separable Hilbert space, $\gamma \in \mathbf{R}$, $\varphi : H \to [0, +\infty]$ is proper, convex, l.s.c, even satisfying: $\varphi(0) = 0$ and for each $M, L > 0$, the set $\{x \in D(\varphi) : ||x|| \le M, \varphi(x) \le L\}$ is compact. Let $F : [0,T] \times H \rightrightarrows H$ be a convex weakly compact valued mapping, separately scalarly measurable on $[0,T]$, separately scalarly upper semicontinuous on H satisfying: there is a L^2-integrably bounded convex weakly compact valued multfunction Γ such that $F(t,x) \subset \Gamma(t)$ for all $(t,x) \in [0,T] \times H$. Then the problem*

$$\begin{cases} \ddot{u}(t) + \gamma\dot{u}(t) \in F(t, u(t)) + \partial\varphi(u(t)), & a.e. \quad t \in [0,T], \\ u(T) = -u(0), \quad \dot{u}(T) = -\dot{u}(0). \end{cases}$$

has at least an anti-periodic $W_H^{2,2}([0,T])$ solution.

Proof. By our assumption, the scalar function $|\Gamma|$ belongs to $L_R^2([0,T])$ with $|\Gamma|(t) = \sup\{||x|| : x \in \Gamma(t)\}$ so that $\Gamma(t) \subset |\Gamma|(t)\overline{B}_H(0,1)$. Furthermore, using the notations of the proof of Theorem 3.3,

$$\ddot{u}_{f_n}(t) + \gamma\dot{u}_{f_n}(t) - f_n(t) \in \partial\varphi(u_{f_n}(t))$$

for every $f_n \in \mathcal{S}_\Gamma^2$, the absolute continuity of $\varphi(u_{f_n}(.))$ and the chain rule theorem [7], yields

$$\langle \ddot{u}_{f_n}(t), \dot{u}_{f_n}(t)\rangle + \langle \gamma\dot{u}_{f_n}(t) - f_n(t), \dot{u}_{f_n}(t)\rangle = \frac{d}{dt}\varphi(u_{f_n}(t))$$

for every $f_n \in \mathcal{S}^2_\Gamma$ so that

$$+\infty > \sup_{n\geq 1} \int_0^T |\langle \ddot{u}_{f_n}(t), \dot{u}_{f_n}(t)\rangle + \langle \gamma \dot{u}_{f_n}(t) - f_n(t), \dot{u}_{f_n}(t)\rangle| dt$$

$$= \sup_{n\geq 1} \int_0^T |\frac{d}{dt}\varphi(u_{f_n}(t))| dt.$$

Further apply the classical definition of the subdifferential to convex funtion φ yields

$$0 = \varphi(0)) \geq \varphi(u_{f_n}(t)) + \langle -u_{f_n}(t), \ddot{u}_{f_n}(t) + \gamma \dot{u}_{f_n}(t) - f_n(t)\rangle$$

or

$$0 \leq \varphi(u_{f_n}(t) \leq \langle u_{f_n}(t), \ddot{u}_{f_n}(t) + \gamma \dot{u}_{f_n}(t) - f_n(t)\rangle.$$

Hence $\sup_{n\geq 1} |\varphi(u_{f_n})|_{L^1_{\mathbf{R}}([0,T])} < +\infty$. Now we assert that $|\varphi(u_{f_n}(t))| \leq L$ for every $t \in [0,T]$, here L is a positive constant. Indeed for all $t \in [0,T]$ we have

$$\varphi(u_{f_n}(0)) \leq |\varphi(u_{f_n}(t)) - \varphi(u_{f_n}(0))| + \varphi(u_{f_n}(t))$$

$$\leq \int_0^T |\frac{d}{dt}\varphi(u_{f_n}(t))| dt + \varphi(u_{f_n}(t)).$$

Hence

$$\varphi(u_{f_n}(0)) \leq \sup_{n\geq 1} \int_0^T |\frac{d}{dt}\varphi(u_{f_n}(t))| dt + \frac{1}{T} \sup_{n\geq 1} \int_0^T \varphi(u_{f_n}(t)) dt < +\infty.$$

Whence we have

$$M := \sup_{n\geq 1} \sup_{t\in[0,T]} ||u_{f_n}(t)|| < +\infty, \quad L = \sup_{n\geq 1} \sup_{t\in[0,T]} \varphi(u_{f_n}(t)) < +\infty$$

so that $(u_{f_n}(t))$ is relatively compact with respect to the norm topology of H using the inf-compactness assumption on φ. The proof can be therefore achieved as Theorem 3.3 by invoking Ascoli theorem, Lemma 3.1 or Lemma 3.2 and a closure type lemma ([14], Theorem VI-4). □

Here is an existence and uniqueness result related to Theorem 3.3 when the perturbation is single-valued. For this purpose, we need a useful result.

Lemma 3.5. *Let $H = \mathbf{R}^d$. Let $w : [0,T] \to \mathbf{R}^d$ satisfying:*

$$w(t) = w(0) + \int_0^t \dot{w}(s) ds, \quad t \in [0,T]; \quad w(T) = -w(0),$$

$$\dot{w}(t) = \dot{w}(0) + \int_0^t \ddot{w}(s) ds, \quad t \in [0,T],$$

$$\dot{w}(T) = -\dot{w}(0); \quad \ddot{w} \in L^2_H([0,T]).$$

Then the following inequalities hold

$$(a) \qquad ||w||_{\mathcal{C}_H([0,T])} \leq \frac{\sqrt{T}}{2} ||\dot{w}||_{L^2_H([0,T])}.$$

$$(b) \qquad \int_0^T ||w(t)||^2 dt \leq \frac{T^2}{\pi^2} \int_0^T ||\dot{w}(t)||^2 dt.$$

Proof. (a) Since $w(t) = w(0) + \int_0^t \dot{w}(s)ds$ and $w(t) = w(T) - \int_t^T \dot{w}(s)ds$, $\forall t \in [0,T]$, by adding these equalities, we get, by anti-periodicity

$$2w(t) = \int_0^t \dot{w}(s)ds - \int_t^T \dot{w}(s)ds, \quad \forall t \in [0,T].$$

Hence we have

$$2||w(t)|| \leq \int_0^t ||\dot{w}(s)||ds + \int_t^T ||\dot{w}(s)||ds = \int_0^T ||\dot{w}(s)||ds \quad \forall t \in [0,T]$$

and so, by Holder inequality

$$||w||_{\mathcal{C}_H([0,T])} = \sup_{t\in[0,T]} ||w(t)|| \leq \frac{\sqrt{T}}{2}||\dot{w}||_{L^2_H([0,T])}.$$

(b) Extend w and $\dot{w}$ by anti-periodicity by putting

$$w(t+T) = -w(t) \quad \text{and } \dot{w}(t+T) = -\dot{w}(t) \quad \forall t \in \mathbf{R}.$$

Then w is $2T$-periodic. Indeed, it is $w(t+2T) = w(t+T+T) = -w(t+T) = w(t)$, similarly, so is $\dot{w}$. Now, as w is $2T$-periodic, T-anti-periodic, by invoking ([1], page 10) we infer that w has the Fourier expansion

$$w(t) = \sum_{n\in\mathbf{Z}} w_n^1 \cos(\frac{(2n-1)\pi}{T}t) + w_n^2 \sin(\frac{(2n-1)\pi}{T}t)$$

for all $t \in [0,2T]$, here w_n^1, w_n^2 are the (constant) Fourier coefficients. Hence we have

$$\dot{w}(t) = \frac{\pi}{T}\sum_{n\in\mathbf{Z}} -(2n-1)w_n^1 \sin(\frac{(2n-1)\pi}{T}t) + (2n-1)w_n^2 \cos(\frac{(2n-1)\pi}{T}t)$$

for all $t \in [0,2T]$. By virtue of Parseval equality, we have

$$\frac{1}{2T}\int_0^{2T} ||w(t)||^2 dt = \sum_{n\in\mathbf{Z}} (||w_n^1||^2 + ||w_n^2||^2)$$

and

$$\frac{1}{2T}\int_0^{2T} ||\dot{w}(t)||^2 dt = \frac{\pi^2}{T^2}\sum_{n\in\mathbf{Z}} (2n-1)^2(||w_n^1||^2 + ||w_n^2||^2).$$

Further we have the estimate

$$2T\sum_{n\in\mathbf{Z}} (||w_n^1||^2 + ||w_n^2||^2) \leq 2T\sum_{n\in\mathbf{Z}} (2n-1)^2(||w_n^1||^2 + ||w_n^2||^2).$$

Whence

$$\int_0^{2T} ||w(t)||^2 dt \leq \frac{T^2}{\pi^2}\int_0^{2T} ||\dot{w}(t)||^2 dt.$$

Let observe that $||w(t)||^2$ and $||\dot{w}(t)||^2$ are T-periodic because $||w(t+T)||^2 = ||-w(t)||^2 = ||w(t)||^2$ and similarly $||\dot{w}(t+T)||^2 = ||-\dot{w}(t)||^2 = ||\dot{w}(t)||^2$. Hence we deduce that

$$\int_0^{2T} ||w(t)||^2 dt = 2\int_0^T ||w(t)||^2 dt \quad \text{and} \quad \int_0^{2T} ||\dot{w}(t)||^2 dt = 2\int_0^T ||\dot{w}(t)||^2 dt.$$

Finally we get the required inequality

$$\int_0^T ||w(t)||^2 dt \le \frac{T^2}{\pi^2} \int_0^T ||\dot{w}(t)||^2 dt.$$

□

The following is a uniqueness result.

Theorem 3.6. *Let $H = \mathbf{R}^d$, $\gamma \in \mathbf{R}$. Assume that $\varphi : H \to]-\infty, +\infty]$ is proper, convex, l.s.c, even and $f : \mathbf{R} \times H \to H$ is a Carathéodory mapping satisfying (H_1): $||f(t,x) - f(t,y)|| \le L||x-y||$ for all $(t,x) \in \mathbf{R} \times H$, for some positive constant $L > 0$ and (H_2): there is a $L^2_{\mathbf{R}}$ integrable function $r : \mathbf{R} \to \mathbf{R}^+$ such that $||f(t,x)|| \le r(t)$ for all $(t,x) \in \mathbf{R} \times H$. If $0 < T < \frac{\pi}{\sqrt{L}}$, then the inclusion*

$$\begin{cases} \ddot{u}(t) + \gamma \dot{u}(t) \in f(t, u(t)) + \partial\varphi(u(t)), & a.e. \quad t \in [0,T], \\ u(T) = -u(0), \quad \dot{u}(T) = -\dot{u}(0). \end{cases}$$

admits a unique $W^{2,2}_H([0,T])$-anti-periodic solution.

Proof. Existence of at least an $W^{2,2}_H([0,T])$-anti-periodic solution is ensured by Theorem 3.3. Assume that (u_1) and (u_2) are two solutions of the inclusion under consideration.

$$\begin{cases} \ddot{u}_1(t) + \gamma \dot{u}_1(t) \in f(t, u_1(t)) + \partial\varphi(u_1(t)), & a.e. \quad t \in [0,T], \\ u_1(T) = -u_1(0), \quad \dot{u}_1(T) = -\dot{u}_2(0). \end{cases}$$

$$\begin{cases} \ddot{u}_2(t) + \gamma \dot{u}_2(t) \in f(t, u_2(t)) + \partial\varphi(u_2(t)), & a.e. \quad t \in [0,T], \\ u_2(T) = -u_2(0), \quad \dot{u}_2(T) = -\dot{u}_2(0). \end{cases}$$

For simplicity, let us set

$$v_1(t) = \ddot{u}_1(t) + \gamma \dot{u}_1(t) - f(t, u_1(t)) \quad \forall t \in [0,T],$$

$$v_2(t) = \ddot{u}_2(t) + \gamma \dot{u}_2(t) - f(t, u_2(t)) \quad \forall t \in [0,T],$$

$$w_{1,2}(t) = u_1(t) - u_2(t) \quad \forall t \in [0,T].$$

Then we have

$$(*) \qquad \ddot{w}_{1,2}(t) + \gamma \dot{w}_{1,2}(t) - f(t, u_1(t)) + f(t, u_2(t)) = v_1(t) - v_2(t) \quad \forall t \in [0,T]$$

with $w_{1,2}(T) = -w_{1,2}(0)$ and $\dot{w}_{1,2}(T) = -\dot{w}_{1,2}(0)$. Multiplying scalarly $(*)$ by $w_{1,2}$ and integrating on $[0,T]$ yields

$$\begin{aligned} (^{**}) \quad \langle w_{1,2}, \ddot{w}_{1,2} \rangle + \gamma \langle w_{1,2}, \dot{w}_{1,2} \rangle + \int_0^T \langle w_{1,2}(t), f(t, u_2(t)) - f(t, u_1(t)) \rangle dt \\ = \langle w_{1,2}(T), \dot{w}_{1,2}(T) \rangle - \langle w_{1,2}(0), \dot{w}_{1,2}(0) \rangle - \int_0^T \langle \dot{w}_{1,2}, \dot{w}_{1,2} \rangle dt \\ + \gamma \int_0^T \langle w_{1,2}, \dot{w}_{1,2} \rangle dt + \int_0^T \langle w_{1,2}(t), f(t, u_2(t)) - f(t, u_1(t)) \rangle dt \\ = \int_0^T \langle v_1(t) - v_2(t), u_1(t) - u_2(t) \rangle dt \end{aligned}$$

As $\int_0^T \langle v_1(t) - v_2(t), u_1(t) - u_2(t)\rangle dt \geq 0$ by monotonicity and

$$\langle w_{1,2}(T), \dot{w}_{1,2}(T)\rangle - \langle w_{1,2}(0), \dot{w}_{1,2}(0\rangle = 0 \quad \text{and} \quad \int_0^T \langle w_{1,2}, \dot{w}_{1,2}\rangle dt = 0$$

by antiperiodiocity, (**) implies

$$||\dot{w}_{1,2}||^2_{L^2_H([0,T])} \leq \int_0^T \langle w_{1,2}(t), f(t, u_2(t)) - f(t, u_1(t))\rangle dt$$

$$\leq L \int_0^T ||w_{1,2}(t)||^2 dt < L\frac{T^2}{\pi^2}||\dot{w}_{1,2}||^2_{L^2_H([0,T])} < ||\dot{w}_{1,2}||^2_{L^2_H([0,T])}$$

using the estimation (b) in Lemma 3.2 and the choice of T. It follows that $||\dot{w}_{1,2}||^2_{L^2_H([0,T])} = 0$. By inequality (a) in Lemma 3.2, (or by antiperiodicity), we conclude that $w_{1,2}(t) = u_1(t) - u_2(t) = 0$ for all $t \in [0, T]$. □

4. Relaxation, Bolza problem and variational convergence

In this section we present a relaxation problem in control theory and some variational convergence results related to the second order of evolution inclusion presented in the preceding section. We need some notations and backgrounds on Young measures in this special context. Let Y and Z two compact metric spaces, $\mathcal{M}^1_+(Y)$ and $\mathcal{M}^1_+(Z)$ are the spaces of all probability Radon measures on Y and Z respectively. We will endowed $\mathcal{M}^1_+(Y)$ and $\mathcal{M}^1_+(Z)$ with the vague topology so that $\mathcal{M}^1_+(Y)$ and $\mathcal{M}^1_+(Z)$ are compact metrizable spaces. Let us denote by $\mathcal{Y}([0,T]; \mathcal{M}^1_+(Z))$ the space of all Young measures (alias relaxed controls) defined on $[0, T]$ endowed with the stable topology so that $\mathcal{Y}([0,T]; \mathcal{M}^1_+(Z))$ is a compact metrizable space with respect to this topology. By its definition, a sequence (ν^n) in $\mathcal{Y}([0,T]; \mathcal{M}^1_+(Z))$ stably converges to $\nu \in \mathcal{Y}([0,T]; \mathcal{M}^1_+(Z))$ if

$$\lim_{n\to\infty} \int_0^T [\int_Z h_t(z) d\nu^n_t(z)] dt = \int_0^T [\int_Z h_t(z) d\nu_t(z)] dt$$

for all $h \in L^1_{\mathcal{C}(Z)}([0,T])$, here $\mathcal{C}(Z)$ denotes the space of all continuous real valued functions defined on Z endowed with the norm of uniform convergence. Finally let us denote by $\mathcal{Z}$ the set of all Lebesgue measurable mappings (alias original controls) $z : [0, T] \to Z$ and $\mathcal{R} := \mathcal{Y}([0,T]; \mathcal{M}^1_+(Z))$ the set of all relaxed controls (alias Young measures) associated with Z.

Theorem 4.1. *Let $H = \mathbf{R}^d$, Y and Z two compact metric spaces, $\gamma \in \mathbf{R}$, and let $\varphi : H \times Y \to [0, +\infty[$ be a lower semicontinuous function satisfying (i) : $\varphi(x, .)$ is continuous on Y for each fixed $x \in H$, (ii) : $\varphi(., y)$ is even on H for each fixed $y \in Y$, (iii) : $0 \leq \varphi(x, y) \leq \alpha(1 + ||x||)$ for all $(x, y) \in H \times Y$, for some positive constant α. Let $f : \mathbf{R} \times H \times Z \to H$ be a Carathéodory mapping on $\mathbf{R} \times [H \times Z]$ satisfying*
(H_1): $||f(t, x, z) - f(t, y, z)|| \leq L||x - y||$ for all $(t, x, z) \in \mathbf{R} \times H \times Z$, for some positive constant $L > 0$ and
(H_2): there is a positive $L^2_{\mathbf{R}}$ function r such that $||f(t, x, z)|| \leq r(t)$ for all $(t, x, z) \in \mathbf{R} \times H \times Z$.

Assume that $0 < T < \frac{\pi}{\sqrt{L}}$. *For each* $(\mu,\nu) \in \mathcal{M}_+^1(Y) \times \mathcal{Y}([0,T];\mathcal{M}_+^1(Z))$, *let* $u_{\mu,\nu}$ *be the* $W_H^{2,2}([0,T])$*-anti-periodic solution of*

$$(\mathcal{S}_\mathcal{R}) \begin{cases} \ddot{u}_{\mu,\nu}(t) + \gamma \dot{u}_{\mu,\nu}(t) \in \int_Z f(t, u_{\mu,\nu}(t), z) d\nu_t(z) + \partial(\int_Y \varphi(.,y) d\mu(y))(u_{\mu,\nu}(t)) \\ u_{\mu,\nu}(T) = -u_{\mu,\nu}(0), \quad \dot{u}_{\mu,\nu}(T) = -\dot{u}_{\mu,\nu}(0). \end{cases}$$

and for each $(\mu, z) \in \mathcal{M}_+^1(Y) \times \mathcal{Z}$, *let* $u_{\mu,z}$ *be the* $W_H^{2,2}([0,T])$*-anti-periodic solution of*

$$(\mathcal{S}_\mathcal{O}) \begin{cases} \ddot{u}_{\mu,z}(t) + \gamma \dot{u}_{\mu,z}(t) \in f(t, u_{\mu,z}(t), z(t)) + \partial(\int_Y \varphi(.,y) d\mu(y))(u_{\mu,z}(t)) \\ u_{\mu,z}(T) = -u_{\mu,z}(0), \quad \dot{u}_{\mu,z}(T) = -\dot{u}_{\mu,z}(0) \end{cases}$$

Then the set $\{u_{\mu,\nu} : (\mu,\nu) \in \mathcal{M}_+^1(Y) \times \mathcal{Y}([0,T];\mathcal{M}_+^1(Z))\}$ *of all solutions of* $(\mathcal{S}_\mathcal{R})$ *is compact and the set* $\{u_{\mu,z} : (\mu,z) \in \mathcal{M}_+^1(Y) \times \mathcal{Z}\}$ *of all solutions of* $(\mathcal{S}_\mathcal{O})$ *is dense in the compact set* $\{u_{\mu,\nu} : (\mu,\nu) \in \mathcal{M}_+^1(Y) \times \mathcal{Y}([0,T],\mathcal{M}_+^1(Z))\}$ *of all solutions of* $(\mathcal{S}_\mathcal{R})$.

Proof. For each $\mu \in \mathcal{M}_+^1(Y)$, the function φ_μ

$$\varphi_\mu(.) = \int_Y \varphi(.,y) d\mu(y)$$

is nonnegative, finite, l.s.c and even on H. For each $\nu \in \mathcal{Y}([0,T],\mathcal{M}_+^1(Z))$, the mapping f_ν

$$f_\nu(t,x) := \int_Z f(t,x,z) d\nu_t(z)$$

inherits the properties of f, that is, f_ν is separately measurable on $\mathbf{R}$, continuous on H and satisfies the Lipschitz condition $||f_\nu(t,x) - f_\nu(t,y)|| \leq L||x-y||$ for all $(t,x,y) \in \mathbf{R} \times H \times H$. By virtue of Theorem 3.6, $(\mathcal{S}_\mathcal{R})$ admits a unique solution $u_{\mu,\nu}$ associated to $(\mu,\nu) \in \mathcal{M}_+^1(Y) \times \mathcal{Y}([0,T];\mathcal{M}_+^1(Z))$, respectively $(\mathcal{S}_\mathcal{O})$ admits a unique solution $u_{\mu,z}$ associated to $(\mu,z) \in \mathcal{M}_+^1(Y) \times \mathcal{Z}$. Let (μ^n,ν^n) be a sequence in $\mathcal{M}_+^1(Y) \times \mathcal{Y}([0,T];\mathcal{M}_+^1(Z))$. By compactness, we may assume that (μ^n) converges vaguely to $\mu^\infty \in \mathcal{M}_+^1(Y)$ and (ν^n) stably converges to $\nu^\infty \in \mathcal{Y}([0,T];\mathcal{M}_+^1(Z))$. Since $||\ddot{u}_{\mu^n,\nu^n}||_{L_H^2} \leq ||r||_{L_\mathbf{R}^2}$ for all $n \geq 1$, we may argue as in Theorem 3.3 by assuming that $(\ddot{u}_{\mu^n,\nu^n})$ weakly converging to $\ddot{u}$ in $L_H^2([0,T])$, $(\dot{u}_{\mu^n,\nu^n})$ pointwisely converging to $\dot{u}$, and (u_{μ^n,ν^n}) pointwisely converges to $u \in W_H^{2,2}([0,T])$ with $u(T) = -u(0)$ and $\dot{u}(T) = -\dot{u}(0)$.

Main fact: u coincides with the $W_H^{2,2}([0,T])$-anti-periodic solution $u_{\mu^\infty,\nu^\infty}$ associated with $(\mu^\infty,\nu^\infty) \in \mathcal{M}_+^1(Y) \times \mathcal{Y}([0,T];\mathcal{M}_+^1(Z))$ of the inclusion

$$\begin{cases} \ddot{u}_{\mu^\infty,\nu^\infty}(t) + \gamma \dot{u}_{\mu^\infty,\nu^\infty}(t) \in f_{\nu^\infty}(t, u_{\mu^\infty,\nu^\infty}(t)) + \partial\varphi_{\mu^\infty}(u_{\mu^\infty,\nu^\infty}(t)) \\ u_{\mu^\infty,\nu^\infty}(T) = -u_{\mu^\infty,\nu^\infty}(0), \quad \dot{u}_{\mu^\infty,\nu^\infty}(T) = -\dot{u}_{\mu^\infty,\nu^\infty}(0) \end{cases}$$

Let $v \in L_H^2([0,T])$, by the definition of the subdifferential of a convex lower semi continuous function and by integrating

$$\int_0^T [\int_Y \varphi(v(t),y) d\mu^n(y)] dt \geq \int_0^T [\int_Y \varphi(u_{\mu^n,\nu^n}(t),y) d\mu^n(y)] dt$$
$$+ \int_0^T \langle v(t) - u_{\mu^n,\nu^n}(t), \ddot{u}_{\mu^n,\nu^n}(t) + \gamma \dot{u}_{\mu^n,\nu^n}(t) \rangle dt$$

$$-\int_0^T \langle v(t) - u_{\mu^n,\nu^n}(t), \int_Z f(t, u_{\mu^n,\nu^n}(t), z) d\nu_t^n(z)\rangle dt.$$

As the fiber product

$$(\delta_{u_{\mu^n,\nu^n}} \otimes \mu^n)$$

stably converges to $(\delta_u \otimes \mu^\infty)$, and φ is l.s.c on $H \times Y$ by hypothesis, applying Lemma 3.4 in [13] yields

$$(^*) \qquad \liminf_n \int_0^T [\int_Y \varphi(u_{\mu^n,\nu^n}(t), y) d\mu^n(y)]dt \geq \int_0^T [\int_Y \varphi(u(t), y) d\mu^\infty(y)]dt.$$

As we already note that $\ddot{u}_{\mu^n,\nu^n} + \gamma \dot{u}_{\mu^n,\nu^n}$ converges weakly to $\ddot{u} + \gamma\dot{u}$ in $L^2_H([0,T])$ and the sequence (u_{μ^n,ν^n}) converges pointwisely to u with

$$||u_{\mu^n,\nu^n}(t)|| \leq C := \sup_{t\in[0,T]} ||u_{\mu^n,\nu^n}(t)|| < +\infty$$

by dominated convergence theorem, (u_{μ^n,ν^n}) strongly converges to u in $L^2_H([0,T])$. Whence we have

$$(^{**}) \quad \lim_n \int_0^T \langle v(t) - u_{\mu^n,\nu^n}(t), \ddot{u}_{\mu^n,\nu^n}(t) + \gamma\dot{u}_{\mu^n,\nu^n}(t)\rangle dt$$
$$= \int_0^T \langle v(t) - u(t), \ddot{u}(t) + \gamma\dot{u}(t)\rangle dt.$$

Further the fiber product $(\delta_{u_{\mu^n,\nu^n}} \otimes \nu^n)$ stably converges to towards $(\delta_u \otimes \nu^\infty)$. Now observe that for every $v \in L^2_H([0,T])$, the integrand $g_v(t,x,z) : \langle v(t), f(t,x,z)\rangle$ is L^1-bounded , namely

$$|g_v(t,x,z)| = |\langle v(t), f(t,x,z)\rangle| \leq ||v(t)||r(t)$$

with $t \mapsto ||v(t)||r(t) \in L^1_{\mathbf{R}^+}([0,T])$, we conclude that

$$\lim_{n\to\infty} \int_0^T [\int_Z g_v(t, u_{\mu^n,\nu^n}(t), z)\, \nu_t^n(dz)]dt = \int_0^T [\int_Z g_v(t, u(t), z)\, \nu_t^\infty(dz)]dt.$$

In other words, the sequence (w^n) in $L^2_H([0,T])$ given by

$$w^n(t) = \int_Z f(t, u_{\mu^n,\nu^n}(t), z)\, \nu_t^n(dz), \quad \forall t \in [0,T],$$

weakly converges to the function $w^\infty \in L^2_H([0,T])$ given by

$$w^\infty(t) = \int_Z f(t, u(t), z)\, \nu_t^\infty(dz), \quad \forall t \in [0,T].$$

It follows that

$$(^{***}) \qquad \lim_n \int_0^T \langle w^n(t), v(t) - u_{\mu^n,\nu^n}(t)\rangle\, dt = \int_0^T \langle w^\infty(t), v(t) - u(t)\rangle\, dt.$$

Combining $(^*) - -(^{***})$ we conclude that

$$\limsup_n \int_0^T [\int_Y \varphi(v(t), y) d\mu^n(y)]dt \geq \int_0^T [\int_Y \varphi(u(t), y) d\mu^\infty(y)]dt$$
$$\int_0^T \langle v(t) - u(t), \ddot{u}(t) + \gamma\dot{u}(t) - \int_Z f(t, u(t), z)\, \nu_t^\infty(dz)\rangle dt.$$

As $0 \leq \varphi(v(t), y) \leq \alpha(1 + ||v(t)||)$ for all $v \in L^2_H([0, T])$ and for all $t \in [0, T]$, it follows that

$$\limsup_n \int_0^T [\int_Y \varphi(v(t), y) d\mu^n(y)] dt \leq \lim_n \int_0^T [\int_Y \varphi(v(t), y) d\mu^n(y)] dt$$
$$\int_0^T \lim_n [\int_Y \varphi(v(t), y) d\mu^n(y)] dt = \int_0^T [\int_Y \varphi(v(t), y) d\mu^\infty] dt.$$

Finally

$$\int_0^T [\int_Y \varphi(v(t), y) d\mu^\infty(y)] dt \geq \int_0^T [\int_Y \varphi(u(t), y) d\mu^\infty(y)] dt$$
$$+ \int_0^T \langle v(t) - u(t), \ddot{u}(t) + \gamma \dot{u}(t) - \int_Z f(t, u(t), z)\, \nu_t^\infty(dz) \rangle dt$$

for all $v \in L^2_H([0, T])$. In other words,

$$\ddot{u} + \gamma \dot{u} - f_{\nu^\infty}(., u(.)) \in \partial I_{\varphi_{\mu^\infty}}(u)$$

Here $\partial I_{\varphi_{\mu^\infty}}$ denotes the subdifferential of the convex integral functional defined on $L^2_H([0, T])$ by

$$I_{\varphi_{\mu^\infty}}(u) = \begin{cases} \int_0^T \varphi_{\mu^\infty}(u(t)) dt & \text{if} \quad \int_0^T \varphi_{\mu^\infty}(u(t)) dt \quad \text{is finite} \\ +\infty \quad \text{otherwise.} \end{cases}$$

By uniqueness of solutions we get $u = u_{\mu^\infty, \nu^\infty}$. This proves the first part of the theorem, while the second part follows by continuity and density since $\mathcal{R}$ is dense in $\mathcal{Y}([0, T], \mathcal{M}^1_+(Z))$ with respect to the stable topology ([12], Lemma 7.1.1). □

Open question. It is worthwhile to prove the validity of Theorem 4.1 when (H_2) is replaced by the following weaker condition (H_2'): there is a positive $L^2_{\mathbf{R}}$ function r such that $||f(t, x, z)|| \leq r(t)(1 + ||x||)$ for all $(t, x, z) \in \mathbf{R} \times H \times Z$.

The following is an application to a Bolza type problem.

Theorem 4.2. *With the hypotheses and notations of Theorem 4.1, let* $J : [0, T] \times R^d \times R^d \times Z \to \mathbf{R}$ *be a Carathéodory integrand such that for any sequence* (μ^n) *in* $\mathcal{M}^1_+(Y)$ *and for any sequence* (ζ^n) *in* $\mathcal{Z}$*, the sequence*

$$(t \to J(t, u_{\mu^n, \zeta^n}(t), \dot{u}_{\mu^n, \zeta^n}(t), \zeta^n(t))$$

is uniformly integrable, here u_{μ^n, ζ^n} *denotes the* $W^{2,2}_H([0, T])$*-anti-periodic solution associated with* $(\mu^n, \zeta^n) \in \mathcal{M}^1_+(Y) \times \mathcal{Z}$ *of the inclusion*

$$\begin{cases} \ddot{u}_{\mu^n, \zeta^n}(t) + \gamma \dot{u}_{\mu^n, \zeta^n}(t) \in f(t, u_{\mu^n, \zeta^n}(t), \zeta^n(t)) + \partial(\int_Y \varphi(., y) d\mu^n(y))(u_{\mu^n, \zeta^n}(t)) \\ u_{\mu^n, \zeta^n}(T) = -u_{\mu^n, \zeta^n}(0), \quad \dot{u}_{\mu^n, \zeta^n}(T) = -\dot{u}_{\mu^n, \zeta^n}(0) \end{cases}$$

Let $\mathcal{D}_Y$ *be a dense subset of* $\mathcal{M}^1_+(Y)$ *and let us consider the problem*

$$(\mathcal{P}_{\mathcal{D}, \mathcal{Z}}) = \inf_{(\mu, \zeta) \in \mathcal{D} \times \mathcal{Z}} \int_0^T J(t, u_{\mu, \zeta}(t), \dot{u}_{\mu, \zeta}(t), \zeta(t)) dt$$

$$(\mathcal{P}_{\mathcal{M}^1_+(Y), \mathcal{R}}) = \inf_{(\mu, \nu) \in \mathcal{M}^1_+(Y) \times \mathcal{R}} \int_0^T [\int_Z J(t, u_{\mu, \nu}(t), \dot{u}_{\mu, \nu}(t), z) d\nu_t(z)] dt$$

here $u_{\mu,\zeta}$ is the $W^{2,2}_H([0,T])$-anti-periodic solution associated with $(\mu,\zeta)\in\mathcal{D}_Y\times\mathcal{Z}$ of the inclusion

$$\begin{cases} \ddot u_{\mu,\zeta}(t)+\gamma\dot u_{\mu,\zeta}(t)\in f(t,u_{\mu,\zeta}(t),\zeta(t))+\partial(\int_Y\varphi(.,y)d\mu(y))(u_{\mu,\zeta}(t))\\ u_{\mu,\zeta}(T)=-u_{\mu,\zeta}(0),\quad \dot u_{\mu,\zeta}(T)=-\dot u_{\mu,\zeta}(0)\end{cases}$$

and $u_{\mu,\nu}$ is the $W^{2,2}_H([0,T])$-anti-periodic solution associated with $(\mu,\nu)\in\mathcal{M}^1_+(Y)\times\mathcal{R}$ of the inclusion

$$\begin{cases} \ddot u_{\mu,\nu}(t)+\gamma\dot u_{\mu,\nu}(t)\in \int_Z f(t,u_{\mu,\nu}(t),z)d\nu_t(z)+\partial(\int_Y\varphi(.,y)d\mu(y))(u_{\mu,\nu}(t))\\ u_{\mu,\nu}(T)=-u_{\mu,\nu}(0),\quad \dot u_{\mu,\nu}(T)=-\dot u_{\mu,\nu}(0)\end{cases}$$

Then we have

$$\inf(\mathcal{P}_{\mathcal{D}_Y,\mathcal{Z}})=\min(\mathcal{P}_{\mathcal{M}^1_+(Y),\mathcal{R}})$$

Proof. Let us recall that $\mathcal{Z}$ is dense in $\mathcal{R}$ with respect to the stable topology ([12], Lemma 7.1.1) and $\mathcal{D}_Y$ is dense in $\mathcal{M}^1_+(Y)$ by hypothesis. Now let $(\mu,\nu)\in\mathcal{M}^1_+(Y)\times\mathcal{R}$. Let (μ^n,ζ^n) be a sequence in $\mathcal{D}_Y\times\mathcal{Z}$ stably converging to (μ,ν). Let us denote by u_{μ^n,ζ^n} (resp. $u_{\mu,\nu}$) the $W^{2,2}$-antiperiodic solution associated with (μ^n,ζ^n) and (μ,ν) respectively, namely

$$\begin{cases} \ddot u_{\mu^n,\zeta^n}(t)+\gamma\dot u_{\mu^n,\zeta^n}(t)\in f(t,u_{\mu^n,\zeta^n}(t),\zeta^n(t))+\partial(\int_Y\varphi(.,y)d\mu^n(y))(u_{\mu^n,\zeta^n}(t))\\ u_{\mu^n,\zeta^n}(T)=-u_{\mu^n,\zeta^n}(0),\quad \dot u_{\mu^n,\zeta^n}(T)=-\dot u_{\mu^n,\zeta^n}(0)\end{cases}$$

$$\begin{cases} \ddot u_{\mu,\nu}(t)+\gamma\dot u_{\mu,\nu}(t)\in \int_Z f(t,u_{\mu,\nu}(t),z)d\nu_t(z)+\partial(\int_Y\varphi(.,y)d\mu(y))(u_{\mu,\nu}(t))\\ u_{\mu,\nu}(T)=-u_{\mu,\nu}(0),\quad \dot u_{\mu,\nu}(T)=-\dot u_{\mu,\nu}(0)\end{cases}$$

Arguing as in the proof of Theorem 4.1 we may assume that $(\dot u_{\mu^n,\zeta^n})$ and (u_{μ^n,ζ^n}) pointwisely converges to $\dot u_{\mu,\nu}$ and $u_{\mu,\nu}$ respectively. Using the fiber product of Young measures, see Proposition 2.1 or ([12], Theorem 2.3.1), we conclude that

$$(\delta_{u_{\mu^n,\zeta^n}}\otimes\delta_{\dot u_{\mu^n,\zeta^n}}\otimes\delta_{\zeta^n})$$

stably converges to

$$(\delta_{u_{\mu,\nu}}\otimes\delta_{\dot u_{\mu,\nu}}\otimes\nu).$$

By our assumption, the sequence

$$(t\to J(t,u_{\mu^n,\zeta^n}(t),\dot u_{\mu^n,\zeta^n}(t),\zeta_n(t))$$

is uniformly integrable, so that, in view of the preceding convergences and Proposition 2.2 or Theorem 6.3.5 in [12], we get

$$\lim_{n\to\infty}\int_0^T J(t,u_{\mu^n,\zeta^n}(t),\dot u_{\mu^n,\zeta^n}(t),\zeta_n(t)dt$$
$$=\int_0^T[\int_Z J(t,u_{\mu,\nu}(t),\dot u_{\mu,\nu}(t),z)d\nu_t(z)]dt.$$

As

$$\int_0^T J(t,u_{\mu^n,\zeta^n}(t),\dot u_{\mu^n,\zeta^n}(t),\zeta_n(t)dt\geq\inf(\mathcal{P}_{\mathcal{D},\mathcal{Z}})$$

for all $n\in\mathbf{N}$, it follows that

$$\int_0^T[\int_Z J(t,u_{\mu,\nu}(t),\dot u_{\mu,\nu}(t),z)d\nu_t(z)]dt\geq\inf(\mathcal{P}_{\mathcal{D},\mathcal{Z}}).$$

As $(\mu,\nu) \in \mathcal{M}^1_+(Y) \times \mathcal{R}$ is arbitrary given in $\mathcal{M}^1_+(Y) \times \mathcal{R}$ we get

$$\inf(\mathcal{P}_{\mathcal{M}^1_+(Y),\mathcal{R}}) \geq \inf(\mathcal{P}_{\mathcal{D},\mathcal{Z}}).$$

But

$$\inf(\mathcal{P}_{\mathcal{D},\mathcal{Z}}) \geq \inf(\mathcal{P}_{\mathcal{M}^1_+(Y),\mathcal{R}})$$

consequently

$$\inf(\mathcal{P}_{\mathcal{D},\mathcal{Z}}) = \inf(\mathcal{P}_{\mathcal{M}^1_+(Y),\mathcal{R}}).$$

We complete the proof by observing that the set of solutions

$$\{u_{\mu,\nu} : (\mu,\nu) \in \mathcal{M}^1_+(Y) \times \mathcal{R}\}$$

is compact by repeating the arguments of Theorem 4.1 so that

$$\inf(\mathcal{P}_{\mathcal{M}^1_+(Y),\mathcal{R}}) = \min(\mathcal{P}_{\mathcal{M}^1_+(Y),\mathcal{R}}).$$

□

We finish our paper by proceeding to a variational convergence result. Compare with Theorem 2.3 in [2].

Theorem 4.3. *Let $H = \mathbf{R}^d$, $\gamma \in \mathbf{R}$, $f_n \in L^2_{\mathbf{R}^d}([0,T])$, $\varphi_n, \varphi : \mathbf{R}^d \to [0,+\infty]$ are proper, convex, l.s.c, even with $\varphi_n(0) = \varphi(0) = 0, \forall n \in \mathbf{N}$. Let u_n denote the unique $W^{2,2}_{\mathbf{R}^d}([0,T])$ anti-periodic solution of*

$$\begin{cases} \ddot{u}_n(t) + \gamma\dot{u}_n(t) \in f_n(t) + \partial\varphi_n(u_n(t)), \quad a.e. \quad t \in [0,T], \\ u_n(T) = -u_n(0), \quad \dot{u}_n(T) = -\dot{u}_n(0). \end{cases}$$

Assume that
(H_1): (f_n) weakly converges to $f \in L^2_{\mathbf{R}^d}([0,T])$.
(H_2): (φ_n) epiconverges to φ.
Then, up to extracted subsequences, (u_n) converges pointwisely to an anti-periodic $W^{2,2}_{\mathbf{R}^d}([0,T])$ solution u of the inclusion

$$\begin{cases} \ddot{u} + \gamma\dot{u} - f \in \partial I_\varphi(u), \\ u(T) = -u(0), \quad \dot{u}(T) = -\dot{u}(0). \end{cases}$$

with $\int_0^T \varphi(u(t))dt < +\infty$, here ∂I_φ denotes the subdifferential of the convex integral functional I_φ defined on $L^2_{\mathbf{R}^d}([0,T])$ by

$$I_\varphi(u) = \begin{cases} \int_0^T \varphi(u(t))dt & \text{if} \quad \int_0^T \varphi(u(t))dt \quad \text{is finite} \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Step 1 Thanks to the estimate $||\ddot{u}_n||_{L^2_{\mathbf{R}^d}} \leq ||f_n||_{L^2_{\mathbf{R}^d}}$ and according to (H_1) and the anti-periodicity of (u_n) and $\dot{u}_n$, we have that

$$\sup_{n\geq 1} ||\dot{u}_n||_{\mathcal{C}_{\mathbf{R}^d}([0,T])} < +\infty \quad \text{and} \quad \sup_{n\geq 1} ||u_n||_{\mathcal{C}_{\mathbf{R}^d}([0,T])} < +\infty.$$

Further using the inclusion

$$\ddot{u}_n(t) + \gamma\dot{u}_n(t) \in f_n(t) + \partial\varphi_n(u_n(t))$$

and repeating the computations given in the proof of Proposition 3.4 involving the chain rule formula, we get the estimate,

$$(*) \qquad \sup_{n\geq 1}\sup_{t\in[0,T]} \varphi_n(u_n(t)) \leq L(= constant).$$

We may assume that $\lim_{n\to\infty} u_n(0) = u_0 \in \mathbf{R}^d$, $\lim_{n\to\infty} \dot{u}_n(0) = v_0 \in \mathbf{R}^d$ and there is $w \in L^2_{\mathbf{R}^d}([0,T])$ such that

$$\lim_{n\to\infty} \int_0^T \langle h, \ddot{u}_n\rangle dt = \int_0^T \langle h, w\rangle dt$$

for every $h \in L^2_{\mathbf{R}^d}([0,T])$. Let us set $v(t) := v_0 + \int_0^t w(s)ds$ for all $t \in [0,T]$. Then

$$\int_0^t w(s)ds = \lim_{n\to\infty} \int_0^t \ddot{u}_n(s)ds = \lim_{n\to\infty} [\dot{u}_n(t) - \dot{u}_n(0)] = \lim_{n\to\infty} \dot{u}_n(t) - v_0$$

Hence $\lim_{n\to\infty} \dot{u}_n(t) = \int_0^t w(s)ds + v_0 = v(t)$ for all $t \in [0,T]$. In particular we have

$$-v(0) = -v_0 = \lim_{n\to\infty} \dot{u}_n(T) = \int_0^T w(s)ds + v_0 = v(T).$$

As $u_n(t) = u_n(0) + \int_0^t \dot{u}_n(s)ds$ for all $t \in [0,T]$, using the boundedness of $(\dot{u}_n)$, and the dominated convergence theorem, we have

$$\lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} u_n(0) + \lim_{n\to\infty} \int_0^t \dot{u}_n(s)ds = u_0 + \int_0^t v(s)ds \quad \forall t \in [0,T].$$

Let us set $u(t) = u_0 + \int_0^t v(s)ds$ for all $t \in [0,T]$. Then

$$-u(0) = -u_0 = \lim_{n\to\infty} u_n(T) = \int_0^T v(s)ds + u_0 = u(T).$$

Hence (u_n) converges pointwisely to the antiperiodic $W^{2,2}_{\mathbf{R}^d}([0,T])$ fonction u with $\ddot{u} = w$ and $\dot{u} = v$. And we have

$$\int_0^T \varphi(u(t))dt \leq \liminf_n \int_0^T \varphi_n(u_n(t))dt \leq LT < +\infty$$

taking account into (H_2) and (*).
Step 2 u is solution of

$$\begin{cases} \ddot{u} + \gamma\dot{u} - f \in \partial I_\varphi(u), \\ u(T) = -u(0), \quad \dot{u}(T) = -\dot{u}(0). \end{cases}$$

with $\int_0^T \varphi(u(t))dt \leq LT < +\infty$, ∂I_φ being the subdifferential of the convex integral functional I_φ defined on $L^2_{\mathbf{R}^d}([0,T])$ by

$$I_\varphi(u) = \begin{cases} \int_0^T \varphi(u(t))dt & \text{if} \quad \int_0^T \varphi(u(t))dt \quad \text{is finite} \\ +\infty & \text{otherwise.} \end{cases}$$

For simplicity let $z_n := \ddot{u}_n + \gamma\dot{u}_n - f_n$ and $z := \ddot{u} + \gamma\dot{u} - f$. Then

$$(**) \qquad z_n(t) \in \varphi_n(u_n(t))$$

a.e. Further it is not difficult to show that $(\dot{u}_n)$ converges weakly to $\dot{u}$ in $L^2_{\mathbf{R}^d}([0,T])$, hence (z_n) converges weakly in $L^2_{\mathbf{R}^d}([0,T])$ to z. The proof will be achieved by using

some facts developed in ([13], Lemma 3.4 and Lemma 3.7).
Fact 1 If h_n, h are measurable mappings $h_n, h : [0,T] \to \mathbf{R}^d$ such that (h_n) pointwisely converges to h. Then

$$\liminf_{n\to\infty} \int_B \varphi_n(h_n(t))dt \geq \int_B \varphi(h(t))dt$$

for every measurable subset B of $[0,T]$, using (H_2).
Fact 2 Let $v \in L^\infty_{\mathbf{R}^d}([0,T])$. Then there exists a bounded sequence (v_n) in $L^\infty_{\mathbf{R}^d}([0,T])$ which pointwisely converges to v and such that

$$\limsup_{n\to\infty} \int_B \varphi_n(v_n(t))dt \leq \int_B \varphi(v(t)dt$$

for every measurable subset B of $[0,T]$, using (H_2) and the estimate (*). From Fact 1 and the result obtained in Step 1, we have

$$+\infty > LT \geq \liminf_{n\to\infty} \int_0^T \varphi_n(u_n(t))dt \geq \int_0^T \varphi(u(t))dt.$$

From (**) we have

$$\varphi_n(v(t)) \geq \varphi_n(u_n(t)) + \langle v(t) - u_n(t), z_n(t)\rangle \quad \text{a.e.} \quad t \in [0,T]$$

for every $v \in L^\infty_{\mathbf{R}^d}([0,T])$. By integrating

$$\int_0^T \varphi_n(v(t))dt \geq \int_0^T \varphi_n(u_n(t))dt + \int_0^T \langle v(t) - u_n(t), z_n(t)\rangle dt.$$

For every $v \in L^\infty_{\mathbf{R}^d}([0,T])$, from Fact 2, there is a bounded sequence (v_n) in $L^\infty_{\mathbf{R}^d}([0,T])$ which converges pointwisely to v and such that

$$\limsup_{n\to\infty} \int_0^T \varphi_n(v_n(t))dt \leq \int_0^T \varphi(v(t))dt.$$

Combining this with Fact 2 gives

$$\lim_{n\to\infty} \int_0^T \varphi_n(v_n(t))dt = \int_0^T \varphi(v(t))dt.$$

As

$$\lim_{n\to\infty} \int_0^T \langle v(t) - u_n(t), z_n(t)\rangle dt = \int_0^T \langle v(t) - u(t), z(t)\rangle dt$$

because the sequence $(v_n - u_n)$ is bounded in $L^\infty_{\mathbf{R}^d}([0,T])$ and converges pointwisely to $u - v$ and the sequence (z_n) is uniformly integrable in $L^1_{\mathbf{R}^d}([0,T])$ and converges to z with respect to the weak topology of $L^2_{\mathbf{R}^d}([0,T])$, then a fortiori converges $\sigma(L^1_{\mathbf{R}^d}([0,T]), L^\infty_{\mathbf{R}^d}([0,T])$ to z. Finally by combining these facts and by passing to the limit when $n \to \infty$ in the integral subdifferential inequality

$$\int_0^T \varphi_n(v_n(t))dt \geq \int_0^T \varphi_n(u_n(t))dt + \int_0^T \langle v_n(t) - u_n(t), z_n(t)\rangle dt$$

we get

$$\int_0^T \varphi(v(t))dt \geq \int_0^T \varphi(u(t))dt + \int_0^T \langle v(t) - u(t), z(t)\rangle dt.$$

Hence we conclude that $z = \ddot{u} + \gamma\dot{u} \in \partial I_\varphi(u)$ with $I_\varphi(u) \leq LT < +\infty$. $\square$

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