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# WEIGHTED PSEUDO ALMOST PERIODIC SOLUTIONS TO SOME PARTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we obtain the existence and uniqueness of weighted pseudo almost periodic solutions to some partial neutral functional-differential equations. Our abstract results are, subsequently, applied to studying the existence of weighted pseudo almost periodic solutions to an integro-partial differential equation arising in control systems, a scalar reaction-diffusion equation with delay as well as a partial differential system related to the heat conduction.

## 1. Introduction

In Diagana [11], a new generalization of Bohr almost periodic functions was introduced. Such a new concept is called *weighted* pseudo almost periodicity and implements in a natural fashion the notion of *pseudo* almost periodicity introduced in the literature in the early nineties by Zhang [38, 39, 40]. To construct those new spaces, the main idea consists of enlarging the so-called *ergodic* component, utilized in the Zhang's definition of pseudo almost periodicity, with the help of a weighted measure $d\mu(x) = \rho(x)dx$, where $\rho : \mathbb{R} \mapsto (0, \infty)$ is a locally integrable function over $\mathbb{R}$, which is commonly called *weight*. Unlike [11], in which the main results on those weighted pseudo almost periodic functions were announced without proofs, here we provide with all complete proofs. We basically take a closer look into properties of weighted pseudo almost periodic functions and study their relationship with the Zhang's pseudo almost periodicity. For that, we consider a binary equivalence relation, $\prec$, on $\mathbb{U}_\infty$, the collection of all weights $\rho$, which enables us to classify those weights into different equivalence classes. Among other things, if two weights $\rho_1$ and $\rho_2$ are equivalent, that is, $\rho_1 \prec \rho_2$, then their corresponding weighted pseudo almost periodic spaces coincide. In particular, when a weight $\rho$ is bounded with $\liminf_{x\to\infty} \rho(x) > 0$, it is then equivalent to the constant function 1, and hence the weighted pseudo almost periodic space with weight $\rho$ coincides with the Zhang's spaces (Corollary 3.8). In addition to the above, a composition result of weighted pseudo almost periodic functions is obtained (Corollary 3.11).

The existence of almost periodic, asymptotically almost periodic, almost automorphic, asymptotically almost automorphic, and pseudo almost periodic solutions is one of the most attracting topics in the qualitative theory of differential equations due to their significance and applications in physics, mathematical biology, control theory, and others. Some recent contributions on almost periodic, asymptotically

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almost periodic, and pseudo almost periodic solutions to abstract differential and partial differential equations have been made in [2, 5, 8, 9, 10, 12, 21, 22, 23, 24]. Existence results concerning almost periodic and asymptotically almost periodic solutions to ordinary neutral differential equations and abstract partial neutral differential equations have recently been established in [20, 27, 32]. However, the existence of weighted pseudo almost periodic to functional-differential equations with delay, especially, abstract partial neutral differential equations, is an untreated topic. Thus applications to the previous theory will include the search of weighted pseudo almost periodic solutions to ordinary differential equations, abstract partial differential equations, abstract partial functional differential equations and abstract partial neutral differential equations. In order to obtain existence results which are applicable to theses models, in this paper we study the existence of weighted pseudo almost periodic solutions for the abstract neutral functional-differential equations

$$\frac{d}{dt}\left(u(t) + f(t, u_t)\right) = Au(t) + g(t, u_t), \tag{1.1}$$

$$\frac{d}{dt}\mathcal{D}(t, u_t) = A\mathcal{D}(t, u_t) + g(t, u_t), \tag{1.2}$$

where $A$ is the infinitesimal generator of an uniformly exponentially stable semigroup of linear operators on a Banach space $\mathbb{X}$; the history $u_t \in C([-p, 0]; \mathbb{X})$ with $p > 0$ ($u_t$ being defined by $u_t(\theta) = u(t+\theta)$ for each $\theta \in [-p, 0]$); $D\psi = \psi(0) + f(\psi)$ and $f, g$ are some appropriate functions.

Note that neutral differential equations arise in many areas of applied mathematics. For this reason, those equations have been of a great interest during the last few decades. The literature relative to ordinary neutral differential equations is quite extensive; for more on this topic and related applications we refer the reader to Hale [15], which contains a comprehensive presentation on those equations.

Partial neutral differential equations with finite delay arise, for instance, in transmission line theory. Wu and Xia have shown in [35] that a ring array of identical resistively coupled lossless transmission lines leads to a system of neutral functional differential equations with discrete diffusive coupling, which exhibit various types of discrete waves. By taking a natural limit, they did obtain from this system of neutral equations a scalar partial neutral functional differential equation with finite delay defined on the unit circle. Such a partial neutral functional differential equation is also investigated by Hale in [16] under the more general form

$$\begin{aligned} \frac{d}{dt}\mathcal{D}u_t(x) &= \frac{\partial^2}{\partial x^2}\mathcal{D}u_t(x) + f(u_t)(x), \qquad t \geq 0, \\ u_0 &= \varphi \in C([-r, 0]; C(\mathbb{S}^1; \mathbb{R})), \end{aligned}$$

where $k$ is a constant,

$$\mathcal{D}(\psi)(s) := \psi(0)(s) - \int_{-r}^{0} [d\eta(\theta)]\psi(\theta)(s)$$

for $s \in \mathbb{S}^1$, $\psi \in C([-r, 0]; C(\mathbb{S}^1; \mathbb{R}))$ and $\eta$ is a function of bounded variation and nonatomic at zero.

Partial neutral differential equation also arise in the theory development in Gurtin & Pipkin [14] and Nunziato [29] for the description of heat conduction in materials with fading memory. In the classic theory of heat conduction, it is assumed that the internal energy and the heat flux dependent linearly on the temperature $u(\cdot)$ and on its gradient $\nabla u(\cdot)$. Under these conditions, the classic heat equation describes sufficiently well the evolution of the temperature in different type of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [14, 29], the internal energy and the heat flux are described as functionals of $u$ and $u_x$. The next system, see for instance [6, 26, 31], has been frequently used to describe this phenomena,

$$\begin{aligned}
(1.3)\quad \frac{d}{dt}\left[c_0 u(t,x) + \int_{-\infty}^{t} k_1(t-s)u(s,x)ds\right] &= c_1\triangle u(t,x) \\
&\quad + \int_{-\infty}^{t} k_2(t-s)\triangle u(s,x)ds, \\
(1.4)\qquad\qquad\qquad\qquad u(t,x) &= 0, \qquad x \in \partial\Omega.
\end{aligned}$$

In this system, $\Omega \subset \mathbb{R}^n$ is open, bounded with smooth boundary; $(t,x) \in [0,\infty)\times\Omega$; $u(t,x)$ represents the temperature in $x$ at the time $t$; $c_1, c_2$ are physical constants and $k_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are the internal energy and the heat flux relaxation respectively. By assuming that the solution $u(\cdot)$ is known on $(-\infty, 0]$ and that $k_1 = k_2$, we can transform this system into an abstract neutral functional differential equation. For more on partial neutral functional differential equations we refer to Hale [16], Wu et al. [34, 35, 36], Adimy [1] for finite delay equations, and Hernández and Henriquez et al. [17, 18, 19] for unbounded delays.

## 2. Preliminaries

In what follows we recall some definitions, notations, and new notions of pseudo almost periodicity that we need in the sequel.

Let $(\mathbb{X}, \|\cdot\|)$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces. The collection of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ with be denoted $B(\mathbb{X},\mathbb{Y})$. This is simply denoted $B(\mathbb{X})$ when $\mathbb{X} = \mathbb{Y}$. In addition to the above, $B_r(x,\mathbb{X})$ denotes an open ball in $\mathbb{X}$ centered at $x$ with radius $r > 0$.

Let $\mathbb{U}$ denote the collection of all functions (weights) $\rho : \mathbb{R} \mapsto (0,\infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho(x) > 0$ for almost each $x \in \mathbb{R}$. From now on, for $\rho \in \mathbb{U}$ and $r > 0$, we use the notation

$$m(r,\rho) := \int_{-r}^{r} \rho(x)dx.$$

As in the particular case when $\rho(x) = 1$ for each $x \in \mathbb{R}$, we are exclusively interested in those weights, $\rho$, for which, $\lim_{r\to\infty} m(r,\rho) = \infty$. Throughout the rest of the paper, the notations $\mathbb{U}_\infty, \mathbb{U}_B$ stands for the sets of weights functions

$$\mathbb{U}_\infty := \left\{\rho \in \mathbb{U} : \lim_{r\to\infty} m(r,\rho) = \infty \text{ and } \liminf_{x\to\infty} \rho(x) > 0\right\},$$

$$\mathbb{U}_B := \{\rho \in \mathbb{U}_\infty : \ \rho \ \text{ is bounded } \}.$$

Obviously, $\mathbb{U}_B \subset \mathbb{U}_\infty \subset \mathbb{U}$, with strict inclusions.

Let $BC(\mathbb{R},\mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) denote the collection of all $\mathbb{X}$-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$). The space $BC(\mathbb{R}, \mathbb{X})$ equipped with its natural norm, that is, the sup norm defined by

$$\|u\|_\infty = \sup_{t \in \mathbb{R}} \|u(t)\|,$$

is a Banach space. Furthermore, $C(\mathbb{R}, \mathbb{Y})$ (respectively, $C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denotes the class of continuous functions from $\mathbb{R}$ into $\mathbb{Y}$ (respectively, the class of jointly continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$).

**Definition 2.1.** A function $f \in C(\mathbb{R}, \mathbb{X})$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number $\tau$ with the property that

$$\|f(t + \tau) - f(t)\| < \varepsilon \quad \text{for each} \quad t \in \mathbb{R}.$$

The number $\tau$ above is called an *$\varepsilon$-translation* number of $f$, and the collection of all such functions will be denoted $AP(\mathbb{X})$. The next Lemma is also a characterization of almost periodic functions.

**Lemma 2.2.** [37, p. 25] *A function $f \in C(\mathbb{R}, \mathbb{Z})$ is almost periodic if and only if the set of functions $\{\sigma_\tau f : \tau \in \mathbb{R}\}$, where $(\sigma_\tau f)(t) = f(t + \tau)$, is relatively compact in $C(\mathbb{R}, \mathbb{Z})$.*

**Definition 2.3.** A function $F \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called (Bohr) almost periodic in $t \in \mathbb{R}$ uniformly in $y \in \mathbb{Y}$ if for each $\varepsilon > 0$ and any compact $K \subset \mathbb{Y}$ there exists $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a number $\tau$ with the property that

$$\|F(t + \tau, y) - F(t, y)\| < \varepsilon \quad \text{for each} \quad t \in \mathbb{R}, \quad y \in K.$$

The collection of those functions is denoted by $AP(\mathbb{Y}, \mathbb{X})$.

To introduce those weighted pseudo almost periodic functions, we need to define the "weighted ergodic" space $PAP_0(\mathbb{X}, \rho)$. Weighted pseudo almost periodic functions will then appear as perturbations of almost periodic functions by elements of $PAP_0(\mathbb{X}, \rho)$.

Let $\rho \in \mathbb{U}_\infty$. Define

$$PAP_0(\mathbb{X}, \rho) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r \to \infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(\sigma)\| \, \rho(\sigma) \, d\sigma = 0 \right\}.$$

Obviously, when $\rho(x) = 1$ for each $x \in \mathbb{R}$, one retrieves the so-called ergodic space of Zhang, that is, $PAP_0(\mathbb{X})$, defined by

$$PAP_0(\mathbb{X}) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \|f(\sigma)\| \, d\sigma = 0 \right\}.$$

Clearly, the spaces $PAP_0(\mathbb{X}, \rho)$ are richer than $PAP_0(\mathbb{X})$ and give rise to an enlarged space of pseudo almost periodic functions. In Corollary 3.8, some sufficient condition on the weight $\rho \in \mathbb{U}_\infty$ are given so that $PAP_0(\mathbb{X}, \rho) = PAP_0(\mathbb{X})$.

In the same way, we define $PAP_0(\mathbb{Y}, \mathbb{X}, \rho)$ as the collection of jointly continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ such that $F(\cdot, y)$ is bounded for each $y \in \mathbb{Y}$ and

$$\lim_{r \to \infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|F(s, y)\| \, \rho(s) \, ds = 0$$

uniformly in compact subset of $\mathbb{Y}$.

We are now ready to define the notion of weighted pseudo almost periodicity.

**Definition 2.4.** Let $\rho \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{R}, \mathbb{X})$ is called weighted pseudo almost periodic (or $\rho$-pseudo almost periodic) if it can be expressed as $f = g + \phi$, where $g \in AP(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X}, \rho)$. The collection of such functions will be denoted by $PAP(\mathbb{X}, \rho)$.

**Lemma 2.5.** *Let $\rho \in \mathbb{U}_\infty$. Then the space $(PAP(\mathbb{X}, \rho), \|\cdot\|_\infty)$ is a Banach space.*

*Proof.* It suffices to prove that $PAP(\mathbb{X}, \rho)$ is a closed subspace of $BC(\mathbb{R}, \mathbb{X})$. So let $f_n = g_n + \phi_n \in PAP(\mathbb{X}, \rho)$ with $g_n \in AP(\mathbb{X})$ and $\phi_n \in PAP_0(\mathbb{X}, \rho)$ and such that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Using along the same lines as the proof of [38, Lemma 1.3], one can easily see that $g_n(\mathbb{R}) \subset \overline{f_n(\mathbb{R})}$ and $\|f_n\| \geq \|g_n\|$ for each $n \in \mathbb{N}$. Consequently, there exists $g \in AP(\mathbb{X})$ such that $\|g_n - g\|_\infty \to 0$ as $n \to \infty$.

Now $f_n - g_n = \phi_n \to \phi := f - g$ as $n \to \infty$. Thus writing $\phi = (\phi - \phi_n) + \phi_n$ we obtain the following inequality:

$$\frac{1}{m(r,\rho)} \int_{-r}^{r} \|\phi(\sigma)\|\, \rho(\sigma) d\sigma \leq \|\phi - \phi_n\|_\infty + \frac{1}{m(r,\rho)} \int_{-r}^{r} \|\phi_n(\sigma)\|\, \rho(\sigma) d\sigma.$$

Thus one completes the proof by letting respectively $r \to \infty$ and $n \to \infty$. □

**Remark 2.6.**

(i) The functions $g$ and $\phi$ appearing in Definition 2.4 are respectively called the almost periodic and the weighted ergodic perturbation components of $f$.

(ii) Let $\rho \in \mathbb{U}_\infty$ and assume that the limits $\limsup_{s\to\infty} \left[\frac{\rho(s+\tau)}{\rho(s)}\right]$ and $\limsup_{r\to\infty} \left[\frac{m(r+\tau,\rho)}{m(r,\rho)}\right]$ are finite for every $\tau \in \mathbb{R}$. Then the space $PAP(\mathbb{X}, \rho)$ is translation invariant.

**Theorem 2.7.** *Fix $\rho \in \mathbb{U}_\infty$. The decomposition of a weighted pseudo almost periodic function $f = g + \phi$, where $g \in AP(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X}, \rho)$, is unique.*

*Proof.* Let $f = g_1 + \phi_1$ where $g_1 \in AP(\mathbb{X})$ and $\phi_1 \in PAP_0(\mathbb{X}, \rho)$. Proceeding as in the proof of [38, Lemma 1.3], it easily follows that $g_1(\mathbb{R}) \subset \overline{f(\mathbb{R})}$. Thus, if $f = g_2 + \phi_2$ where $g_2 \in AP(\mathbb{X})$ and $\phi_2 \in PAP_0(\mathbb{X}, \rho)$, then

$$0 = f - f = (g_1 - g_2) + (\phi_1 - \phi_2) \in PAP(\mathbb{X}, \rho),$$

where $(g_1 - g_2) \in AP(\mathbb{X})$ and $(\phi_1 - \phi_2) \in PAP_0(\mathbb{X}, \rho)$. Hence, using the argument above, it follows that $(g_1 - g_2)(\mathbb{R}) \subset \{0\}$, and therefore, $g_1 = g_2$ and $\phi_1 = \phi_2$. The proof is complete □

Let $\rho \in \mathbb{U}_\infty$. To study issues related to delayed differential equations we need to introduce the new space of functions $PAP_0(\mathbb{X}, p, \rho)$ defined for each $p > 0$ as the collection of all functions $f \in BC(\mathbb{R}, \mathbb{X})$ such that

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \left( \sup_{\theta \in [t-p,t]} \|f(\theta)\| \right) \rho(t) dt = 0.$$

In addition to the above-mentioned spaces, the present setting requires the introduction of the following function spaces

$$PAP_0(\mathbb{X},\mathbb{Y},\rho) = \left\{ f \in BC(\mathbb{R}\times\mathbb{X},\mathbb{Y}) : \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \parallel f(t,z) \parallel_{\mathbb{Y}} \rho(t)dt = 0 \right\},$$

and $PAP_0(\mathbb{X},\mathbb{Y},p,\rho)$ defined as the collection of all functions $f \in BC(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ such that

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \left( \sup_{\theta\in[t-p,t]} \parallel f(\theta,z) \parallel_{\mathbb{Y}} \right) \rho(t)dt = 0,$$

where in both cases the limit as $r \mapsto \infty$ is uniform in the second variable $z$ in compact subset of $\mathbb{Y}$.

**Definition 2.8.** A function $F \in BC(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ is called weighted pseudo almost periodic if $F = G + \Phi$, where $G \in AP(\mathbb{X},\mathbb{Y},\rho)$ and $\Phi \in PAP_0(\mathbb{X},\mathbb{Y},\rho)$. The class of such functions will be denoted by $PAP(\mathbb{X},\mathbb{Y},\rho)$.

We need to introduce the following new notions of weighted pseudo almost periodicity that we will use in the sequel.

**Definition 2.9.** A function $F \in BC(\mathbb{R},\mathbb{X})$ is called weighted pseudo almost periodic of class p if $F = G + \varphi$, where $G \in AP(\mathbb{X})$ and $\varphi \in PAP_0(\mathbb{X},p,\rho)$. The class of such functions will be denoted by $PAP(\mathbb{X},p,\rho)$.

**Definition 2.10.** A function $F \in BC(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ is called weighted pseudo almost periodic of class p if $F = G+\varphi$, where $G \in AP(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ and $\varphi \in PAP_0(\mathbb{X},\mathbb{Y},p,\rho)$. The class of such functions will be denoted by $PAP(\mathbb{X},\mathbb{Y},p,\rho)$.

The rest of the paper is organized as follows. In the section 3 we study some basic properties of weighted pseudo almost periodic functions. The existence of weighted pseudo almost periodic solutions for the neutral systems (1.1) and (1.2) is investigated in Section 4.1. Finally, Section 5 considers some applications.

## 3. Properties of weighted pseudo almost periodic functions

In this section we consider some basic properties of Weighted pseudo almost periodic functions. In particular, we establish a result concerning the composition of Weighted pseudo almost periodic functions which is basic to obtain our result on existence of Weighted pseudo almost periodic solutions for functional differential equations. Into the follows, $\rho$ is a function in $\mathbb{U}_\infty$.

**Proposition 3.1.** *Let $f \in PAP_0(\mathbb{R},\rho)$, $g \in L^1(\mathbb{R})$ and assume that $\rho$ verifies the conditions in Remark 2.6(ii). Then $f * g$, the convolution of $f$ and $g$ on $\mathbb{R}$, belongs to $PAP_0(\mathbb{R},\rho)$.*

*Proof.* From $f \in PAP_0(\mathbb{R},\rho)$ and $g \in L^1(\mathbb{R})$ it is clear that $f * g \in BC(\mathbb{R})$. Moreover, for $r > 0$ we see that

$$\begin{aligned} \frac{1}{m(r,\rho)} \int_{-r}^{r} |(f*g)(t)|\rho(t)dt &\leq \int_{-\infty}^{+\infty} |g(s)| \left( \frac{1}{m(r,\rho)} \int_{-r}^{r} |f(t-s)|\rho(t)dt \right) ds \\ &= \int_{-\infty}^{+\infty} |g(s)|\phi_r(s)ds, \end{aligned}$$

where $\phi_r(s) = \dfrac{1}{m(r,\rho)} \displaystyle\int_{-r}^{r} |f(t-s)|\rho(t)dt$. Since $PAP_0(\mathbb{R},\rho)$ is translation invariant, it follows that $\phi_r(s) \mapsto 0$ as $r \mapsto \infty$. Next, using the boundedness of $\phi_r$ ($|\phi_r(s)| \leq \|f\|_\infty$) and the fact that $g \in L^1(\mathbb{R})$, the Lebesgue dominated convergence theorem yields

$$\lim_{r\to\infty} \int_{-\infty}^{+\infty} |g(s)|\phi_r(s)ds = 0,$$

which prove that $f * g \in PAP_0(\mathbb{R},\rho)$. The proof is complete □

It is clear that if $h \in AP(\mathbb{R})$ and $\psi \in L^1(\mathbb{R})$, then the convolution $h*\psi \in AP(\mathbb{R})$. Combining those results one obtains.

**Corollary 3.2.** *Let $f \in PAP(\mathbb{R},\rho)$ and $g \in L^1(\mathbb{R})$. If $\rho$ verifies the conditions in Remark 2.6(ii), then $f * g$ belongs to $PAP(\mathbb{R},\rho)$.*

**Example 3.3.** Assume $\rho \in \mathbb{U}_\infty$ satisfying the conditions in Remark 2.6(ii). Define the function $W(\cdot)$ by

$$W(x) = \int_{-\infty}^{\infty} K(x-y)f(y)dy,$$

where $K \in L^1(\mathbb{R})$ and $f \in PAP(\mathbb{R},\rho)$. Then $W \in PAP(\mathbb{R},\rho)$, by Corollary 3.2.

**Proposition 3.4.** *Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$, $q > 1$, and assume that*

$$\limsup_{r\to\infty} \frac{m(r,\rho_2)}{m(r,\rho_1)} \left( \int_{-r}^{r} \left( \frac{\rho_2(s)}{\rho_1(s)} \right)^{q'} ds \right)^{\frac{1}{q'}} < \infty,$$

*where $\dfrac{1}{q'} + \dfrac{1}{q} = 1$. Then $PAP^q(\mathbb{R},\rho_1) = \{f \in BC(\mathbb{R},\mathbb{X}) : |f|^q \in PAP_0(\mathbb{R},\rho_1)\} \subset PAP_0(\mathbb{R},\rho_2)$.*

*Proof.* Let $f \in PAP^q(\mathbb{R},\rho_1)$. Passing to the limit as $r \to \infty$ in the inequality

$$\frac{1}{m(r,\rho_2)} \int_{-r}^{r} |f(t)|\rho_2(t)dt \leq$$

$$\frac{m(r,\rho_1)^{\frac{1}{q}}}{m(r,\rho_2))} \left( \frac{1}{m(r,\rho_1)} \int_{-r}^{r} |f(t)|^q \rho_1(t)dt \right)^{1/q} \left( \int_{-r}^{r} \left( \frac{\rho_2(s)}{\rho_1(s)} \right)^{q'} ds \right)^{\frac{1}{q'}}$$

we obtain the desired result. □

If $f, g \in PAP(\mathbb{X},\rho)$ and let $\lambda \in \mathbb{R}$, then $f + \lambda g$ is also in $PAP(\mathbb{X},\rho)$. Moreover, if $|f(\cdot)|$ is not even and $x \to \frac{\rho(-x)}{\rho(x)} \in L^\infty(\mathbb{R})$ then the function $\tilde{f}(x) := f(-x)$ for $x \in \mathbb{R}$ is also in $PAP(\mathbb{X},\rho)$. In particular, if $\rho$ is even, then $\tilde{f}$ belongs to $PAP(\mathbb{X},\rho)$.

**Definition 3.5.** Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. One says that $\rho_1$ is equivalent to $\rho_2$ and denote it $\rho_1 \prec \rho_2$, if the following limits exist $\liminf_{t\to\infty} \dfrac{\rho_1}{\rho_2}(t)$ and $\limsup_{t\to\infty} \dfrac{\rho_2}{\rho_1}(t)$.

Let $\rho_1, \rho_2, \rho_3 \in \mathbb{U}_\infty$. It is clear that $\rho_1 \prec \rho_1$ (reflexivity); if $\rho_1 \prec \rho_2$, then $\rho_2 \prec \rho_1$ (symmetry); and if $\rho_1 \prec \rho_2$ and $\rho_2 \prec \rho_3$, then $\rho_1 \prec \rho_3$ (transitivity). So, $\prec$ is a binary equivalence relation on $\mathbb{U}_\infty$. The equivalence class of a given weight $\rho \in \mathbb{U}_\infty$ will be denoted by $\breve{\rho} = \{\varpi \in \mathbb{U}_\infty : \rho \prec \varpi\}$. It is then clear that $\mathbb{U}_\infty = \bigcup_{\rho\in\mathbb{U}_\infty} \breve{\rho}$.

**Theorem 3.6.** *If $\rho_1, \rho_2 \in \breve{\rho}$, then $PAP_0(\mathbb{X}, \rho_1) = PAP_0(\mathbb{X}, \rho_2)$.*

*Proof.* From $\rho_1 \prec \rho_2$, there exist constants $K, K', r_0 > 0$ such that $K'\rho_2(x) \leq \rho_1(x) \leq K\rho_2(x)$ for each $| x |> r_0$. Consequently, for $r > r_0$ we have that

$$\begin{aligned} m(r,\rho_1) = \int_{-r}^{r} \rho_1(s)ds &\leq \int_{-r}^{-r_0} \frac{\rho_1(s)}{\rho_2(s)}\rho_2(s)ds + \int_{-r_0}^{r_0} \rho_1(s)ds + \int_{r_0}^{r} \frac{\rho_1(s)}{\rho_2(s)}\rho_2(s)ds \\ &\leq K\int_{-r}^{-r_0} \rho_2(s)ds + \int_{-r_0}^{r_0} \rho_1(s)ds + K\int_{r_0}^{r} \rho_2(s)ds \\ &\leq \int_{-r_0}^{r_0} \rho_1(s)ds + Km(r,\rho_2), \end{aligned}$$

so that

$$\frac{1}{m(r,\rho_2)} \leq \frac{K}{m(r,\rho_1) - \int_{-r_0}^{r_0} \rho_1(s)ds}, \qquad r \geq r_0.$$

Similarly, we can prove that

$$\frac{1}{m(r,\rho_1)} \leq \frac{1}{K'(m(r,\rho_2) - \int_{-r_0}^{r_0} \rho_2(s)ds)}, \qquad r \geq r_0.$$

Let $\phi \in PAP_0(\mathbb{X}, \rho_2)$. In view of the above it easily follows that for $r > r_0$

$$\begin{aligned} &\frac{1}{m(r,\rho_1)}\int_{-r}^{r} \|\phi(s)\|\rho_1(s)ds \\ &= \frac{1}{m(r,\rho_1)}\int_{-r}^{-r_0} \|\phi(s)\|\left(\frac{\rho_1}{\rho_2}\right)(s)\rho_2(s)ds + \frac{1}{m(r,\rho_1)}\int_{-r_0}^{r_0} \|\phi(s)\|\rho_1(s)ds \\ &\quad + \frac{1}{m(r,\rho_1)}\int_{r_0}^{r} \|\phi(s)\|\left(\frac{\rho_1}{\rho_2}\right)(s)\rho_2(s)ds \\ &\leq \frac{K}{m(r,\rho_1)}\int_{-r}^{-r_0} \|\phi(s)\|\rho_2(s)ds + \frac{1}{m(r,\rho_1)}\int_{-r_0}^{r_0} \|\phi(s)\|\rho_1(s)ds \\ &\quad + \frac{K}{m(r,\rho_1)}\int_{r_0}^{r} \|\phi(s)\|\rho_2(s)ds \\ &\leq \frac{K}{K'(m(r,\rho_2) - \int_{-r_0}^{r_0} \rho_2(s)ds)}\int_{-r}^{r} \|\phi(s)\|\rho_2(s)ds + \frac{1}{m(r,\rho_1)}\int_{-r_0}^{r_0} \|\phi(s)\|\rho_1(s)ds, \end{aligned}$$

which implies that

$$\lim_{r\mapsto\infty} \frac{1}{m(r,\rho_1)}\int_{-r}^{r} \|\phi(s)\|\rho_1(s)ds = 0,$$

and hence $PAP_0(\mathbb{X}, \rho_2) \subset PAP_0(\mathbb{X}, \rho_1)$. Similarly, we can show that $PAP_0(\mathbb{X}, \rho_1) \subset PAP_0(\mathbb{X}, \rho_2)$. The proof is complete. □

In view of the above, the proof of the next corollary is quite immediate.

**Corollary 3.7.** *If $\rho_1 \prec \rho_2$, then (i) $PAP(\mathbb{X}, \rho_1+\rho_2) = PAP(\mathbb{X}, \rho_1) = PAP(\mathbb{X}, \rho_2)$, and (ii) $PAP(\mathbb{X}, \frac{\rho_1}{\rho_2}) = PAP(\mathbb{X}, \breve{1}) = PAP(\mathbb{X})$.*

Another immediate consequence of Theorem 3.6 is that $PAP(\mathbb{X},\rho) = PAP(\mathbb{X},\breve{\rho})$. This enables us to identify the Zhang's space $PAP(\mathbb{X})$ with a weighted pseudo almost periodic class $PAP(\mathbb{X},\rho)$.

**Corollary 3.8.** *If $\rho \in \mathbb{U}_B$, then $PAP(\mathbb{X},\rho) = PAP(\mathbb{X},\breve{1}) = PAP(\mathbb{X})$.*

The next theorem is a generalization of the theorems of composition of pseudo almost periodic functions and pseudo almost periodic functions of class $p$ given in [5] and [24] respectively.

**Theorem 3.9.** *Let $\rho \in \mathbb{U}_\infty$, $p \geq 0$, $F \in PAP(\mathbb{X},\mathbb{Y},p,\rho)$ and $h \in PAP(\mathbb{Y},p,\rho)$. Assume that there exists a function $L_F : \mathbb{R} \mapsto [0,\infty)$ satisfying*

$$(3.1) \qquad \|F(t,z_1) - F(t,z_2)\|_{\mathbb{Y}} \;\leq\; L_F(t)\,\|z_1 - z_2\|\,, \quad \forall t \in \mathbb{R},\ \forall z_1, z_2 \in \mathbb{X}.$$

*If*

$$(3.2) \qquad \limsup_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \left( \sup_{\theta\in[t-p,t]} L_F(\theta) \right) \rho(t)dt < \infty, \quad \textit{and}$$

$$(3.3) \qquad \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \left( \sup_{\theta\in[t-p,t]} L_F(\theta) \right) \xi(t)\rho(t)dt = 0$$

*for each $\xi \in PAP_0(\mathbb{R},\rho)$, then the function $t \mapsto F(t,h(t))$ belongs to $PAP(\mathbb{Y},p,\rho)$.*

*Proof.* Assume that $F = F_1 + \varphi$, $h = h_1 + h_2$, where $F_1 \in AP(\mathbb{X},\mathbb{Y})$, $\varphi \in PAP_0(\mathbb{X},\mathbb{Y},p)$, $h_1 \in AP(\mathbb{X})$ and $h_2 \in PAP_0(\mathbb{X},p)$. Consider the decomposition

$$F(t,h(t)) \;=\; F_1(t,h_1(t)) + [F(t,h(t)) - F(t,h_1(t)))] + \varphi(t,h_1(t)).$$

Since $F_1(\cdot,h_1(\cdot)) \in AP(\mathbb{Y})$, it remains to prove that both $[F(\cdot,h(\cdot)) - F(\cdot,h_1(\cdot)))]$ and $\varphi(\cdot,h_1(\cdot))$ belong to $PAP_0(\mathbb{Y},p)$. Indeed, using (3.1) above it follows that

$$\begin{aligned}
&\frac{1}{m(r,\rho)} \int_{-r}^{r} \left( \sup_{\theta\in[t-p,t]} \| F(\theta,h(\theta)) - F(\theta,h_1(\theta))) \| \right) \rho(t)dt \\
&\qquad \leq \frac{1}{m(r,\rho)} \int_{-r}^{r} \left( \sup_{\theta\in[t-p,t]} L_F(\theta)\|h_2(\theta)\| \right) \rho(t)dt \\
&\qquad \leq \frac{1}{m(r,\rho)} \int_{-r}^{r} \left( \sup_{\theta\in[t-p,t]} L_F(\theta) \right) . \left( \sup_{\theta\in[t-p,t]} \|h_2(\theta)\| \right) \rho(t)dt,
\end{aligned}$$

which implies that $[F(\cdot,h(\cdot)) - F(\cdot,h_1(\cdot)))] \in PAP_0(\mathbb{Y},p,\rho)$, by (3.3).

Since $h_1(\mathbb{R})$ is relatively compact in $\mathbb{X}$ and $F_1$ is uniformly continuous on sets of the form $\mathbb{R} \times K$ where $K \subset \mathbb{X}$ is a compact subset, for $\varepsilon > 0$ there exists $0 < \delta \leq \varepsilon$ such that

$$\| F_1(t,z) - F_1(t,\bar{z}) \| \leq \varepsilon, \qquad z, \bar{z} \in h_1(\mathbb{R})$$

for every $z, \bar{z} \in h_1(\mathbb{R})$ with $\| z - \bar{z} \| < \delta$. Now, fix $z_1, \ldots, z_n \in h_1(\mathbb{R})$ such that

$$h_1(\mathbb{R}) \subset \bigcup_{i=1}^{n} B_\delta(z_i,\mathbb{Z}).$$

Obviously, the sets $E_i = h_1^{-1}(B_\delta(z_i))$ form an open covering of $\mathbb{R}$, and therefore using the sets

$$B_1 = E_1, \quad B_2 = E_2 \setminus E_1, \quad \text{and} \quad B_i = E_i \setminus \bigcup_{j=1}^{i-1} E_j,$$

one obtains a covering of $\mathbb{R}$ by disjoint open sets.

For $t \in B_i$ with $h_1(t) \in B_\delta(z_i)$

$$\begin{aligned}
\| \varphi(t, h_1(t)) \| &\leq \| F(t, h_1(t)) - F(t, z_i) \| + \| -F_1(t, h_1(t)) + F_1(t, z_i) \| \\
&+ \| -\varphi(t, z_i) \| \\
&\leq L_F(t) \| h_1(t) - z_i \| + \varepsilon + \| \varphi(t, z_i) \| \\
&\leq L_F(t)\varepsilon + \varepsilon + \| \varphi(t, z_i) \| .
\end{aligned}$$

Now, using this we see that

$$\begin{aligned}
&\frac{1}{m(r,\rho)} \int_{-r}^{r} \left( \sup_{\theta \in [t-p,t]} \| \varphi(t, h_1(t)) \| \right) \rho(t) dt \\
&\leq \frac{1}{m(r,\rho)} \sum_{i=1}^{n} \int_{B_i \cap [-r,r]} \left( \sup_{j=1,..n} \left[ \sup_{\theta \in [t-p,t] \bigcap B_j} \| \varphi(\theta, h_1(\theta)) \| \right] \right) \rho(t) dt \\
&\leq \frac{1}{m(r,\rho)} \int_{-r}^{r} \left[ \sup_{\theta \in [t-p,t]} L_F(\theta)\varepsilon + \varepsilon \right] \rho(t) dt \\
&\quad + \sum_{i=1}^{n} \frac{1}{m(r,\rho)} \int_{-r}^{r} \left[ \sup_{\theta \in [t-p,t]} \| \varphi(\theta, z_j) \| \right] \rho(t) dt.
\end{aligned}$$

In view of the above it is clear that $\varphi(\cdot, h_1(\cdot))$ belongs to $PAP_0(\mathbb{Y}, p, \rho)$. □

**Remark 3.10.** Note that assumptions (3.2) and (3.3) are verified by many functions. Concrete examples include constants functions, and functions in $PAP(\mathbb{R}, p, \rho)$, among others.

**Corollary 3.11.** *Let $f \in BC(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ satisfying the Lipschitz condition*

$$\|f(t,u) - f(t,v)\| \leq L\|u - v\|_{\mathbb{Y}} \quad \text{for all} \quad u, v \in \mathbb{Y}, \ t \in \mathbb{R}.$$

- **(a)** *If $h \in PAP(\mathbb{Y}, \breve{\rho})$, then $f(\cdot, h(\cdot)) \in PAP(\mathbb{X}, \breve{\rho})$.*
- **(b)** *Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$ with $\rho_2 \in \breve{\rho}_1$. If $f \in PAP(\mathbb{Y}, \mathbb{X}, \breve{\rho}_1)$ and $h \in PAP(\mathbb{Y}, \rho_2)$, then $f(\cdot, h(\cdot)) \in PAP_0(\mathbb{X}, \breve{\rho}_1)$.*
- **(c)** *If $\rho \in \mathbb{U}_B$, $f \in PAP(\mathbb{Y}, \mathbb{X}, \breve{\rho})$ and $h \in PAP(\mathbb{Y}, \rho)$, then $f(\cdot, h(\cdot)) \in PAP(\mathbb{X})$.*

To complete this section, we establish conditions under which the history function is weighted pseudo almost periodic.

**Theorem 3.12.** *Let $\rho \in \mathbb{U}_\infty$, $p > 0$ and $u \in PAP(\mathbb{X}, p, \rho)$. If $\limsup_{r\to\infty} \dfrac{m(r+p,\rho)}{m(r,\rho)} < \infty$ and $z(t) = \dfrac{\rho(t-p)}{\rho(t)} \in L^\infty(\mathbb{R})$, then the function $t \to u_t$ belongs to $PAP(C([-p,0], \mathbb{X}), p, \rho)$.*

*Proof.* Suppose that $u = h+g$, where $h \in AP(\mathbb{X})$ and $g \in PAP_0(\mathbb{X}, p, \rho)$. Obviously, $u_t = h_t + g_t$ and from Lemma 2.2, we infer that $t \to h_t \in AP(C([-p,0],\mathbb{X}))$. On the other hand, for $r > 0$ we see that

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}\left[\sup_{\theta\in[t-p,t]}\left(\sup_{\xi\in[-p,0]}\|g(\theta+\xi)\|\right)\right]\rho(t)dt\\
&\quad\leq \frac{1}{m(r,\rho)}\int_{-r}^{r}\left(\sup_{\theta\in[t-2p,t]}\| g(\theta) \|\right)\rho(t)dt\\
&\quad\leq \frac{1}{m(r,\rho)}\int_{-r-p}^{r-p}\left(\sup_{\theta\in[t-p,t]}\|g(\theta)\| + \sup_{\theta\in[t,t+p]}\|g(\theta)\|\right)\rho(t)dt\\
&\quad\leq \frac{1}{m(r,\rho)}\int_{-r-p}^{r-p}\left(\sup_{\theta\in[t-p,t]}\|g(\theta)\|\right)\rho(t)dt\\
&\qquad+\frac{1}{m(r,\rho)}\int_{-r-p}^{r-p}\left(\sup_{\theta\in[t,t+p]}\|g(\theta)\|\right)\rho(t)dt\\
&\quad\leq \frac{m(r+p,\rho)}{m(r,\rho)}\frac{1}{m(r+p,\rho)}\int_{-r-p}^{r+p}\left(\sup_{\theta\in[t-p,t]}\|g(\theta)\|\right)\rho(t)dt\\
&\qquad+\frac{1}{m(r,\rho)}\int_{-r}^{r}\left(\sup_{\theta\in[t-p,t]}\|g(\theta)\|\right)\rho(t)\frac{\rho(t-p)}{\rho(t)}dt,
\end{aligned}$$

which enables to complete the proof. □

## 4. Weighted pseudo almost periodic solutions to neutral systems

In this section we discuss the existence of weighted pseudo almost periodic for some abstract Neutral differential systems. Throughout the rest of the paper, $\rho \in \mathbb{U}_\infty$, $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is the infinitesimal generator of an uniformly asymptotically stable semigroup of linear operators $(T(t))_{t\geq 0}$ and $M, w$ are positive constants such that $\| T(t) \|\leq Me^{-wt}$ for all $t \geq 0$. Additionally, we introduce the following condition

$$\mathbf{H}_\omega \ \gamma(\omega) := \sup_{r>0, s<r}\left[\int_s^r e^{-\omega(t-s)}\rho(t)dt\right] < \infty \text{ and}$$

$$\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-\infty}^{-r}\int_{-r}^{r} e^{-\omega(t-s)}\rho(t)dtds = 0$$

.

**Remark 4.1.** Note that in the particular case when $\rho \in \mathbb{U}_B$, that is, $PAP(\mathbb{X}, \breve{\rho}) = PAP(\mathbb{X})$ by Corollary 3.8, we retrieve the "non-weighted" situation, since assumption "$\gamma(\omega) < \infty$" is always achieved in that event.

**Remark 4.2.** Its is interesting to note that the condition $\mathbf{H}_\omega$ is verified, for instance, for functions which behave like polynomial at infinitum. In particular, we retrieve the case $\rho(t) = 1$, the "non-weighted" situation.

**Theorem 4.3.** *Let $u \in PAP_0(\mathbb{X}, p, \rho)$ and assume that assumption $\mathbf{H}_\omega$ holds. If $v$ is the function defined by*

$$v(t) := \int_{-\infty}^{t} T(t-s)u(s)ds, \quad \forall t \in \mathbb{R},$$

*then $v \in PAP_0(\mathbb{X}, p, \rho)$.*

*Proof.* Let $K, r_0$ positive constants such that $\rho(t) \geq K$ for every $\mid t \mid \geq r_0$. Then, for $r > r_0$ we have that

$$\begin{aligned}
&\frac{1}{m(r,\rho)} \int_{-r}^{r} \left( \sup_{\theta \in [t-p,t]} v(\theta) \right) \rho(t)dt \\
\leq \quad & \frac{1}{m(r,\rho)} \int_{-r}^{r} \left( \sup_{\theta \in [t-p,t]} \int_{-\infty}^{\theta} e^{-\omega(\theta - s)} \parallel u(s) \parallel ds \right) \rho(t)dt \\
\leq \quad & \frac{1}{m(r,\rho)} \int_{-r}^{r} \left( \sup_{\theta \in [t-p,t]} e^{\omega p} \int_{-\infty}^{\theta} e^{-\omega(t-s)} \|u(s)\| ds \right) \rho(t)dt \\
\leq \quad & \frac{e^{\omega p}}{m(r,\rho)} \int_{-r}^{r} \int_{-\infty}^{t} e^{-\omega(t-s)} \|u(s)\| \rho(t) ds dt \\
\leq \quad & \frac{e^{\omega p}}{m(r,\rho)} \int_{-\infty}^{-r} \int_{-r}^{r} e^{-\omega(t-s)} \|u(s)\| \rho(t) dt ds \\
& + \frac{K e^{\omega p}}{m(r,\rho)} \int_{-r}^{-r_0} \|u(s)\| \rho(s) \int_{s}^{r} e^{-\omega(t-s)} \rho(t) dt ds \\
& + \frac{e^{\omega p}}{m(r,\rho)} \int_{-r_0}^{r_0} \|u(s)\| \int_{s}^{r} e^{-\omega(t-s)} \rho(t) dt ds + \\
& \frac{K e^{\omega p}}{m(r,\rho)} \int_{r_0}^{r} \|u(s)\| \rho(s) \int_{s}^{r} e^{-\omega(t-s)} \rho(t) dt ds \\
\leq \quad & \frac{e^{\omega p} \|u\|_\infty}{m(r,\rho)} \int_{-\infty}^{-r} \int_{-r}^{r} e^{-\omega(t-s)} \rho(t) dt ds + \frac{\gamma_\omega K e^{\omega p}}{m(r,\rho)} \int_{-r}^{-r_0} \|u(s)\| \rho(s) ds \\
& + \frac{\gamma_\omega e^{\omega p}}{m(r,\rho)} \int_{-r_0}^{r_0} \|u(s)\| ds + \frac{\gamma_\omega K e^{\omega p}}{m(r,\rho)} \int_{r_0}^{r} \|u(s)\| \rho(s) ds \\
\leq \quad & \frac{e^{\omega p} \|u\|_\infty}{m(r,\rho)} \int_{-\infty}^{-r} \int_{-r}^{r} e^{-\omega(t-s)} \rho(t) dt ds + \frac{\gamma_\omega K e^{\omega p}}{m(r,\rho)} \int_{-r}^{r} \|u(s)\| \rho(s) ds \\
& + \frac{\gamma_\omega e^{\omega p}}{m(r,\rho)} \int_{-r_0}^{r_0} \|u(s)\| ds,
\end{aligned}$$

which permit to conclude that

$$\frac{1}{m(r,\rho)} \int_{-r}^{r} \left( \sup_{\theta \in [t-p,t]} v(\theta) \right) \rho(t)dt \to 0 \quad \text{as} \quad r \to \infty.$$

The proof is complete. $\square$

4.1. **Existence of pseudo almost periodic solutions to a neutral system.** In the sequel, we discuss the existence and uniqueness of a weighted pseudo almost periodic solution of class $p > 0$ to the neutral system

$$\frac{d}{dt}(u(t) + f(t, u_t)) = Au(t) + g(t, u_t), \qquad t \in [\sigma, \sigma + a), \tag{4.1}$$

$$u_\sigma = \phi \in \mathcal{C} = C([-p, 0]; \mathbb{X}). \tag{4.2}$$

In the next definition we adopt the notion of mild solution to (4.1)-(4.2) from the one given in Hernández and Henríquez [17].

**Definition 4.4.** A continuous function $u : [\sigma, \sigma + a) \to \mathbb{X}$, $a > 0$, is a mild solution of the neutral system (4.1)-(4.2) on $[\sigma, \sigma + a)$, if the function $s \to AT(t-s)f(s, u_s)$ is integrable on $[0, t)$ for every $\sigma < t < \sigma + a$, and

$$\begin{aligned} u(t) &= T(t-\sigma)(\varphi(\sigma) + f(\sigma, \varphi)) - f(t, u_t) - \int_\sigma^t AT(t-s)f(s, u_s)ds \\ &\quad + \int_\sigma^t T(t-s)g(s, u_s)ds, \quad t \in [\sigma, \sigma + a). \end{aligned}$$

**Remark 4.5.** In the rest of this paper, we always assume that $\rho \in \mathbb{U}_\infty$ is a weight which verifies the conditions in Theorem 3.12 and condition $\mathbf{H}_\omega$.

In the sequel, $\mathcal{C}$ is the space $C([-p, 0]; \mathbb{X})$ endowed with the sup norm $\|\psi\|_\mathcal{C}$ on $[-r, 0]$, and $[D(A)]$ denotes the domain of $A$ when it is endowed with graph norm, $\| x \|_{[D(A)]} = \| x \| + \| Ax \|$ for each $x \in D(A)$.

To discuss the existence of weighted pseudo almost periodic solutions to (4.1)-(4.2) we need to set some assumptions on $f$ and $g$.

- $\mathbf{H_1}$ The functions $f, g : \mathbb{R} \times \mathcal{C} \to \mathbb{X}$ are continuous, $f$ is $D(A)$-valued and there exist a positive constant $L_f$ and a continuous functions $L_g : \mathbb{R} \to [0, \infty)$ such that

$$\begin{aligned} \| f(t, \psi_1) - f(t, \psi_2) \|_{[D(A)]} &\leq L_f \| \psi_1 - \psi_2 \|_\mathcal{C}, \\ \| g(t, \psi_1) - g(t, \psi_2) \| &\leq L_g(t) \| \psi_1 - \psi_2 \|_\mathcal{C}, \end{aligned}$$

  for all $t \in \mathbb{R}$, $\psi_i \in \mathcal{C}$.

**Remark 4.6.** The assumption on $f$ is linked to the integrability of the function $s \to AT(t-s)f(s, u_s)$ over $[0, t)$. In general, except trivial cases, the operator function $t \to AT(t)$ is not integrable over $[0, a]$. If $f$ satisfies $\mathbf{H_1}$, then from the Bochner's criterion for integrable functions and the estimate

$$\| AT(t-s)f(s, u_s) \| \leq Me^{-w(t-s)} \| Af(s, u_s) \|,$$

it follows that the function $s \mapsto AT(t-s)f(s, u_s)$ is integrable over $(-\infty, t)$ for each $t > 0$. For additional remarks related this type of conditions in partial neutral differential equations, see, e.g., [17, 18], and specially, [19].

**Definition 4.7.** A function $u \in BC(\mathbb{R}, \mathbb{X})$ is a mild weighted pseudo almost periodic solution to the neutral system (4.1)-(4.2) provided that the function $s \to AT(t-s)f(s, u_s)$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$, and

$$u(t) = -f(t, u_t) - \int_{-\infty}^t AT(t-s)f(s, u_s)ds + \int_{-\infty}^t T(t-s)g(s, u_s)ds, \quad t \in \mathbb{R}.$$

**Theorem 4.8.** *Let $\rho \in \mathbb{U}_\infty$ be a weight which verifies the conditions in Theorem 3.12, and suppose that assumptions $\mathbf{H_1}$ and $\mathbf{H}_\omega$ hold. If*

$$\Theta := \left( L_f \left[ 1 + \frac{M}{\omega} \right] + M \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\omega(t-s)} L_g(s) ds \right) < 1, \tag{4.3}$$

*then there exist a unique weighted pseudo almost periodic solution to (4.1)-(4.2).*

*Proof.* In $PAP(\mathbb{X}, p, \rho)$ define the operator $\Gamma : PAP(\mathbb{X}, p, \rho) \to C(\mathbb{R}, \mathbb{X})$ by setting

$$\Gamma u(t) := -f(t, u_t) - \int_{-\infty}^{t} AT(t-s) f(s, u_s) ds + \int_{-\infty}^{t} T(t-s) g(s, u_s) ds, \quad t \in \mathbb{R}.$$

From previous assumptions one can easily see that $\Gamma u$ is well-defined and continuous. Moreover, from Theorems 3.9, 3.12 and 4.3 we infer that $\Gamma u \in PAP(\mathbb{X}, p, \rho)$, that is, $\Gamma : PAP(\mathbb{X}, p, \rho) \mapsto PAP(\mathbb{X}, p, \rho)$.

On the other hand, for $u, v \in PAP(\mathbb{X}, p, \rho)$ we get

$$\begin{aligned} \| \Gamma u(t) - \Gamma v(t) \| &\leq L_f \| u_t - v_t \|_{\mathcal{C}} + M L_f \int_{-\infty}^{t} e^{-\omega(t-s)} \| u_s - v_s \|_{\mathcal{C}} \, ds \\ &\quad + M \int_{-\infty}^{t} e^{-\omega t-s)} L_g(s) \| u_s - v_s \|_{\mathcal{C}} \, ds \\ &\leq \left( L_f \left[ 1 + \frac{M}{\omega} \right] + M \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\omega(t-s)} L_g(s) ds \right) \| u - v \|_\infty \\ &\leq \Theta \| u - v \|_\infty . \end{aligned}$$

which prove that $\Gamma$ is a contraction.

Finally, from (4.3) and the contraction mapping principle it follows that the system (4.1)-(4.2) has a unique weighted pseudo almost periodic mild solution. The proof is complete. □

To complete this subsection, we study briefly the existence of weighted pseudo almost periodic mild solution for a class of neutral abstract system described in the form

$$\frac{d}{dt}\mathcal{D}(t, u_t) = A\mathcal{D}(t, u_t) + f(t, u_t), \qquad t \in [\sigma, \sigma + a), \tag{4.4}$$

$$u_\sigma = \varphi \in \mathcal{C} = C([-r, 0]; X)), \tag{4.5}$$

where $D(t, \psi) = \psi(0) + f(t, \psi)$ and $f, g$ are appropriate functions.

In general, the next definitions and results are similar to those for the system (4.1)-(4.2). For this reason and for the sake of brevity we will omit details.

**Definition 4.9.** A function $u : (-\infty, \sigma + a] \to X$, $a > 0$, is a mild solution of (4.4)-(4.5) on $[\sigma, \sigma + a]$, if $u \in C([\sigma, \sigma + a] : X)$; $u_\sigma = \varphi$ and

$$u(t) = T(t-\sigma)(\varphi(0) + f(\sigma, \varphi)) - f(t, u_t) + \int_{\sigma}^{t} T(t-s) g(s, u_s) ds, \qquad t \in [\sigma, \sigma + a].$$

**Definition 4.10.** A function $u \in BC(\mathbb{R}, \mathbb{X})$ is called a mild weighted pseudo almost periodic solution of (4.4)- (4.5) if

$$u(t) = -f(t, u_t) + \int_{-\infty}^{t} T(t-s) g(s, u_s) ds, \quad t \in \mathbb{R}.$$

**Theorem 4.11.** *Assume that the functions $f, g : \mathbb{R} \times \mathcal{C} \to \mathbb{X}$ are continuous and that there exist a positive constant $L_f$ and a continuous function $L_g : \mathbb{R} \to [0, \infty)$ such that*

$$\begin{aligned} \| f(t, \psi_1) - f(t, \psi_2) \| &\leq L_f \| \psi_1 - \psi_2 \|_{\mathcal{C}}, \\ \| g(t, \psi_1) - g(t, \psi_2) \| &\leq L_g(t) \| \psi_1 - \psi_2 \|_{\mathcal{C}}, \end{aligned}$$

*for all $t \in \mathbb{R}$, $\psi_i \in \mathcal{C}$. If $\Theta := L_f + M \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\omega(t-s)} L_g(s) ds < 1$, then there exist a unique weighted pseudo almost periodic solution of (4.4)- (4.5).*

## 5. Examples

In this section we consider some applications to illustrate our previous abstract results. For that, we first introduce the required background needed in the sequel.

Throughout the rest of this section, we take $\mathbb{X} = L^2([0, \pi])$ and let $A$ be the operator given by $Af = f''$ with domain $D(A) := \{f \in X : f'' \in X,\ f(0) = f(\pi) = 0\}$. It is well known that $A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on $\mathbb{X}$. Furthermore, $A$ has a discrete spectrum with eigenvalues of the form $-n^2, n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(\xi) := \sqrt{\frac{2}{\pi}} \sin(n\xi)$. In addition to the above, the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathbb{X}$;
(b) For $f \in \mathbb{X}$, $T(t)f = \sum_{n=1}^{\infty} e^{-n^2 t \langle f, z_n\rangle z_n}$ and $Af = -\sum_{n=1}^{\infty} n^2 \langle f, z_n\rangle z_n$, for every $f \in D(A)$.
(c) For $f \in \mathbb{X}$ and $\alpha \in (0, 1)$, $(-A)^{-\alpha} f = \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \langle f, z_n\rangle z_n$;
(d) For $\alpha > 0$, the operator $(-A)^{\alpha} : D((-A)^{\alpha}) \subseteq \mathbb{X} \to \mathbb{X}$ is given by

$$(-A)^{\alpha} f = \sum_{n=1}^{\infty} n^{2\alpha} \langle f, z_n\rangle z_n, \quad \forall f \in D((-A)^{\alpha}),$$

where $D((-A)^{\alpha}) = \{f(\cdot) \in \mathbb{X} : \sum_{n=1}^{\infty} n^{2\alpha} \langle f, z_n\rangle z_n \in \mathbb{X}\}$.

5.1. **Reaction-diffusion equations with delay.** Most of different differential equations, reaction diffusion equations with delay, wave equations, age-dependent population equations, can be described through abstract semilinear functional-differential equations, see Wu [36].

Here, we make use of Theorem 4.8 to study the existence and uniqueness of pseudo almost periodic solutions to the scalar reaction-diffusion equation with delay given by

$$\frac{\partial}{\partial t}(t, \xi) = \frac{\partial^2}{\partial \xi^2} u(t, \xi) + g(t, u(t - p, \xi)), \tag{5.1}$$

$$u(t, 0) = u(t, \pi) = 0, \tag{5.2}$$

$$u(\tau, \xi) = \varphi(\tau, \xi), \quad \tau \in [-p, 0], \xi \in [0, \pi]. \tag{5.3}$$

We have

**Theorem 5.1.** *Assume that $g : \mathbb{R} \times \mathbb{R} \to R$ is continuous and the existence of a positive and integrable function $L_g : \mathbb{R} \to \mathbb{R}$ such that*

$$\|g(t, \psi_1) - g(t, \psi_2)\| \leq L_g(t) \|\psi_1 - \psi_2\|_{\infty},$$

*for every* $t \in \mathbb{R}$ *and all* $\psi_1, \psi_2 \in C([-p,0];\mathbb{X})$.

*If* $\Theta := \sup_{t\in\mathbb{R}} \left( \int_{-\infty}^{t} e^{-(t-s)} L_g(s)ds \right) < 1$, *then there exists a unique weighted pseudo almost periodic mild solution to the problem (5.1)-(5.2)- (5.3).*

*Proof.* This is a straightforward consequence of Theorem 4.8.

5.2. **A neutral equation in the theory of heat conduction.** Next, we consider the problem of the existence of weighted pseudo almost periodic solutions for a particular case of the partial differential system (1.3)-(1.3). Consider the differential equation

$$\begin{aligned} \frac{\partial}{\partial t}[u(t,\xi) + \int_{-r}^{t} a_1(t-s)u(s,\xi)ds] &= \frac{\partial^2}{\partial \xi^2}[u(t,\xi) + \int_{-r}^{t} a_1(t-s)u(s,\xi)ds] \\ &\quad + a_2(t,\xi)u(t-r,\xi) \\ &\quad + \int_{-r}^{t} a_3(t-s)u(s,\xi)ds + a_4(t,\xi), \end{aligned} \tag{5.4}$$

$$u(t,0) = u(t,\pi) = 0, \tag{5.5}$$

for $(t,\xi) \in \mathbb{R} \times [0,\pi]$.

To transform this system into the abstract Cauchy problem (4.4)-(4.5), we introduce the functions $f,\ g : C([-r,0];X) \to \mathbb{X}$ defined by

$$\begin{aligned} f(t,\psi)(\xi) &:= \int_{-r-t}^{0} a_1(-s)\psi(s,\xi)ds, \\ g(t,\psi)(\xi) &:= a_2(t,\xi)\psi(-r,\xi) + \int_{-r-t}^{0} a_3(-s)\psi(s,\xi)ds + a_4(t,\xi). \end{aligned}$$

The next result is a direct consequence of Theorem 4.11. We omit the details of the proof. □

**Theorem 5.2.** *Assume that the functions* $a_i(\cdot)$ *for* $i = 1,2,3$ *and* $4$, *are continuous and bounded. If*

$$\sup_{t\in\mathbb{R}} \left( \int_{-r-t}^{0} a_1^2(-s)ds \right)^{\frac{1}{2}} + \| a_2(\cdot) \|_\infty + \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-(t-s)} \left( \int_{-r-t}^{0} a_3^2(-s)ds \right)^{\frac{1}{2}} < 1,$$

*then there exists a unique weighted pseudo almost periodic to (5.4)-(5.5).*

5.3. **On a first-order neutral differential equation.** To finish his section, we consider the first-order boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t}\left[u(t,\xi) + \int_{-r}^{0}\int_{0}^{\pi} b(s,\eta,\xi)u(t+s,\eta)d\eta ds\right] &= \frac{\partial^2}{\partial \xi^2}u(t,\xi) + a_0(\xi)u(t,\xi) \\ &\quad + \int_{-r}^{0} a_1(s)u(t+s,\xi)ds, \end{aligned} \tag{5.6}$$

$$u(t,0) = u(t,\pi) = 0, \tag{5.7}$$

for $t \in \mathbb{R}$ and $\xi \in I = [0,\pi]$.

Note that equations of type (5.6)-(5.7) arise in control systems described by abstract retarded functional-differential equations with feedback control governed by proportional integro-differential law, see [17, Examples 4.2] for details.

To study (5.6)-(5.7) we suppose that the functions $a_0, a_1$ are continuous and that the following holds:

**(i)** The functions $b(\cdot)$, $\dfrac{\partial^i}{\partial\zeta^i}b(\tau,\eta,\zeta)$, $i = 1,2$, are ( Lebesgue ) measurable, $b(\tau,\eta,\pi) = 0$, $b(\tau,\eta,\,0) = 0$ for every $(\tau,\eta)$ and

$$N_1 := \max\{\int_0^\pi \int_{-r}^0 \int_0^\pi \left(\frac{\partial^i}{\partial\zeta^i}b(\tau,\eta,\zeta)\right)^2 d\eta d\tau d\zeta : i = 0,1,2\} < \infty.$$

Under these conditions, we define the functions $f,\, g : C([-r,0];\mathbb{X})$ by setting

$$\begin{aligned} f(t,\psi)(\xi) &:= \int_{-r}^0 \int_0^\pi b(s,\eta,\xi)\psi(s,\eta)d\eta ds \\ g(t,\psi)(\xi) &:= a_0(\xi)\psi(0,\xi) + \int_{-r}^0 a_1(s)\psi(s,\xi)ds. \end{aligned}$$

In view of the above, it is clear that the system (5.6)-(5.7) can be rewritten as an abstract system of the form (1.1). By a straightforward estimation that uses **(i)** one can show that $f$ has values in $D(A)$ and that $f(t,\cdot) : C([-r,0];\mathbb{X}) \to [D(A)]$ is a bounded linear operator with $\| Af(t,\cdot) \| \leq (N_1 r)^{\frac{1}{2}}$ for each $t \in \mathbb{R}$. Furthermore, $g$ is a bounded linear operator on $\mathbb{X}$ with

$$\| g(t,\cdot) \| \leq \|a_0\|_\infty + r^{\frac{1}{2}} \left(\int_{-r}^0 a_1^2(s)ds\right)^{\frac{1}{2}}, \qquad t \in \mathbb{R}.$$

The next result is a consequence of Theorem 4.8.

**Theorem 5.3.** *Under the previous assumptions, the system (5.6)-(5.7) has a unique weighted pseudo almost periodic solution whenever*

$$2\sqrt{N_1 r} \;+\; \|a_0\|_\infty \;+\; r^{\frac{1}{2}} \left(\int_{-r}^0 a_1^2(s)ds\right)^{\frac{1}{2}} < 1.$$

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