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A VARIANT EXISTENCE RESULT FOR PERIODIC SOLUTIONS OF A CLASS OF HAMILTONIAN SYSTEMS WITH INDEFINITE POTENTIAL

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Abstract. In this paper we have solved an open problem by providing a variant existence result for periodic solutions to a class of Hamiltonian systems with a sign-indefinite super quadratic potential.

1. Introduction

It is our objective to investigate the existence of periodic solutions of the class of Hamiltonian systems:

$$\text{(P)} \qquad \begin{aligned} &\ddot{x} + A(t)x + b(t)V'(x) = 0 \quad \text{on} \quad [0,T] \\ &x(0) = x(T) \\ &\dot{x}(0) = \dot{x}(T) \end{aligned}$$

where

1.1 $A \in C^0(\mathbb{R}, \mathbb{R}^{N\times N})$ is a symmetric T-periodic matrix function that is not sign definite on [0,T]

1.2 $b \in C^0(\mathbb{R}, \mathbb{R})$ is T-periodic and changes sign on [0,T]

1.3 $V \in C^2(\mathbb{R}^N, \mathbb{R})$ with $V(x) \geqslant V(0) = 0$ for all $x \in \mathbb{R}^N$, and has super quadratic behaviour.

The above problem has been studied by several authors case-by-case:

Case A: $A \equiv 0$. Lassoued [14] obtained existence of T-periodic solutions when V is strictly convex and homogeneous, while Ben Naoum etal [6] provided existence results by relaxing condition on V to only homogeneity.

By using the Alama-Tarantella condition [1] given as follows:

$$\text{(1.4)} \qquad \text{There exist} \quad c > 0, \beta > 2 : |V'(x)x - \beta V(x)| \leqslant c|x|^2 \quad \text{for all} \quad x \in \mathbb{R}^N,$$

Girardi and Matzeu [13] proved some existence and multiplicity results for T-periodic and subharmonic solutions.

Case B: $A \neq 0$. Refer to [3], [2], [10] (with the references contained therein), severally for existence of periodic, homoclinic, and subharmonic solutions where b and matrix function A are sign-definite. Besides, when we assume b changes sign and A is a negative definite matrix,there are some existence and multiplicity results of periodic, subharmonic, and homoclinic solutions.(Refer to [12], [8], and [5]). When $b(.)$ changes sign and A is not sign-definite, the author is aware of only two results,

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namely, Antonacci's [4], and (the most recent) that of Yuang-Tong Xu and Zhi-Ming Guo's [18]. Antonacci proved existence of non-trivial periodic solutions under the assumption that

$$\int_0^T (A(t)\xi,\xi)dt > 0 \quad \text{for all} \quad \xi \in \mathbb{R}^N, \quad |\xi| = 1 \tag{1.5}$$

Antonacci observed the vital role played by assumption (1.5) as he posed two open problems:

Given that $A(.)$ is not negative definite, to:

1. Study the existence of T-periodic solutions of (P) in the case (1.5)holds and$A(.)$is sign indefinite in any interval [0,T].
2. find some existence results in the case that A does not satisfy (1.5).

Antonacci solved problem 1 in [4]. Yuang-Tong Xu and Zhi-Ming Guo, in attempting the solution of open problem 2 in [18], relied among other assumptions, on the weaker version of (1.5) that,

$$\text{There exist} \quad \alpha,\mu > 0, \quad \beta > 2, \quad \gamma : 0 < \gamma < \alpha\left(\int_0^T b(t)dt\right)\beta \quad \text{such that}$$

$$\int_0^T (A(t)\xi,\xi)dt \geqslant -\gamma, \quad \text{for all} \quad \xi \in \mathbb{R}^N, \quad |\xi| = 1 \tag{1.6}$$

$$\text{where} \quad V(x) \geqslant \alpha|x|^\beta - \mu \quad \text{for all sufficiently large } x \in \mathbb{R}^N.$$

It is clear from (1.6) that $\int_0^T (A(t)\xi,\xi)dt$, $\xi \in \mathbb{R}^N$, $|\xi| = 1$ is neither assumed to be definitely positive nor negative, which is a relaxation on condition (1.5.) In this paper we seek to fill in some gaps and provide a variant existence result to those of the above quoted authors [4,18]. For instance, while Yuang-Tong Xu and Zhi-Ming Guo do not assume (1.5) in order to answer the second open problem posed by Antonacci, their restriction to the case $\int_0^T b(t)dt > 0$ leaves the case $\int_0^T b(t)dt < 0$ unresolved, which would indeed break down assumption (1.6) and hence the entire structure of the proof in [18]. We shall resolve this problem.

2. Main result

Theorem 2.1. *Let conditions 1.1-1.3 be verified in addition to the following conditions:*

a.1 $0 < l = \max_{t\in[0,T]} |A(t)| < \dfrac{4\pi^2}{(1+4\pi^2)T^2} : (A(t)x,x) \leqslant l|x|^2.$

for all $x \in \mathbb{R}^N$, $t \in [0,T]$

b.1 $\int_0^T b(t)dt \neq 0.$

m.1 *There exist* $\delta > 0$, $\eta :$ $0 < \eta < l$, $t_0 \in [0,T]$, $R_1 > 0 :$

i) $b(t) > 0$ *for all* $t \in I_\delta$, $I_\delta = [t_0 - \delta, t_0 + \delta] \subset [0,T]$.

ii) $\displaystyle\int_{I_\delta}(A(t)x,x)dt \geqslant \eta \int_{I_\delta}|x|^2 dt$ *for all* $x \in \mathbb{R}^N$.

iii) $\displaystyle\int_{\bar{I}}(A(t)\xi,\xi)dt \geqslant -\eta T$ *for all* $\xi \in \mathbb{R}^N$, $|\xi| = 1$, $(\bar{I} = [0,T] \setminus I_\delta)$.

V.1 *There exist* $\beta > 2, a_1 > 0, R_2 > 0$:

i) $\beta V(x) \leqslant V'(x)x$ *and*

ii) $V(x) \geqslant a_1|x|^\beta$ *for all* $x \in \mathbb{R}^N, |x| \geqslant R_2$.

V.2 *There exists* $R_3 > 0 : -\beta V(x) + (V'(x), x) \leqslant c|x|^2$ *for all* $x \in \mathbb{R}^N, |x| \geqslant R_3$ *where*

$$c < \frac{\beta - 2}{2m}\left[\frac{4\pi^2}{(1+4\pi^2)T^2} - l\right], \quad m = \max_{t\in[0,T]} b(t).$$

V.3 $\displaystyle\lim_{|x|\to 0} \frac{V(x)}{|x|^2} = 0$

Then problem (P) has at least one periodic solution.

Remark. Assumption m.1 indicates that

$$\int_0^T (A(t)\xi,\xi)dt \geqslant \eta(2\delta - T) \quad \text{for all} \quad \xi \in \mathbb{R}^N, \quad |\xi| = 1; 2\delta - T < 0$$

This is an improvement on condition (0.2) in [4].

We shall investigate the periodic solutions of (P) in the Sobolev space $H^1_T = H^1([0,T],\mathbb{R}^N) = \{u : [0,T] \to \mathbb{R}^N, \text{is absolutely continuous}, u(0) = u(T), \dot{u} \in L^2(0,T;\mathbb{R}^N)\}$ with the norm:

$$\|u\| = \left(\int_0^T |\dot{u}|^2 + \int_0^T |u|^2\right)^{\frac{1}{2}} \quad \text{for all} \quad u \in H^1_T.$$

Hamiltonian action:

We reduce problem (P) to finding critical points of the functional:

$$(*) \qquad I(x) = \frac{1}{2}\left[\int_0^T |\dot{x}|^2 - \int_0^T A(t)x.x\right] - \int_0^T b(t)V(x) \quad x \in H^1_T$$

associated with (P). We note that the critical points of this functional correspond to periodic solutions of (P). (xy denotes the inner product of the pair of vectors $x, y \in \mathbb{R}^N$).

Definition 2.2. Given a real Banach space X, we say that $I \in C^1(X,\mathbb{R})$ satisfies the *Cerami-Palais-Smale* condition at level $d \in \mathbb{R}$ (i.e condition $(CPS)_d$ for short) if for any sequence $(x_n) \in X$ such that

$$I(x_n) \to d \quad \text{and} \quad (1 + \|x_n\|)I'(x_n) \to 0,$$

it implies that (x_n) contains a strongly convergent subsequence in X.

Remark. The Cerami-Palais-Smale condition was introduced by G. Cerami [9] as a weaker variant of the classical Palais-Smale condition.

We shall rely on the following linking theorem of Benci- Rabinowitz (see [16], [7]) as an investigative tool:

Theorem 2.3 (Linking theorem)**.** *Let E be a Banach space and $I \in C^1(E,\mathbb{R})$ such that*

a) $E = E_- \oplus E_+$, $\dim E_- < \infty$ *and* E_+ *is closed in* E.

b) There exist $r, R: \ 0 < r < R$, *and* $\nu \in E_+$ *with* $\|\nu\| = 1$:

$$\sup\{I(x) : x \in \partial Q\} \leqslant \inf\{I(x) : x \in D\}$$

$$Q = \{u + \lambda\nu : \lambda \geqslant 0, u \in E_-, \|u + \lambda\nu\| \leqslant R\} \subset E_- \oplus \mathbb{R}\nu;$$

$$D = \{x \in E_+ : \|x\| = r\}$$

c) I *satisfies the Cerami-Palais-Smale condition*$(CPS)_d$,

$$d = \inf_{g\in\Gamma} \sup_{y\in Q} I(g(y)), \quad \Gamma = \{g \in C(Q,E) : g(y) = y \quad \textit{for all} \quad y \in \partial Q\}$$

Then $d \geqslant \inf_D I$ *and d is a critical value of I .Moreover, if* $d = \inf_D I$, *there is a critical point* $x_d \in D : I(x_d) = d$.

Lemma 2.4. *Let conditions* $1.1 - 1.3$, $a.1$, $b.1$, $V.1$, *and*$V.2$ *be verified. Then* $I \in C^1(H^1_T, \mathbb{R})$ *given as in* $(*)$ *satisfies the Cerami-Palais-Smale condition for all real* d.

Proof. We only need to show that given any sequence $(x_n) \in H^1_T$ such that $I(x_n)$ is bounded and $(1 + \|x_n\|)I'(x_n) \to 0$ as $n \to \infty$ (i.e $\sup\{(1 + \|x_n\|)I'(x_n)\theta : \theta \in H^1_T, \|\theta\| = 1\} \to 0$ as $n \to \infty$), then (x_n) contains a strongly convergent subsequence in H^1_T.

Boundedness of $I(x_n)$ implies that there exists $k > 0$ such that

$$\frac{1}{2}\int_0^T |\dot{x}_n|^2 \leqslant \frac{1}{2}\int_0^T (A(t)x_n, x_n) + \int_0^T b(t)V(x_n) + k. \tag{2.1}$$

Since $(1 + \|x_n\|)I'(x_n)$ is linear, it follows from Riesz's representation theorem that there exists a sequence $(z_n) \in H^1_T$ such that

$$\|(1 + \|x_n\|)I'(x_n)\|_{H_T^{-1}} = \|z_n\| = \varepsilon_n \to 0 \quad \text{as} \quad n \to +\infty \quad \text{and}$$

$$(1 + \|x_n\|)I'(x_n)\varphi = \langle z_n, \varphi\rangle_{H^1_T} \quad \text{for all} \quad \varphi \in H^1_T.$$

Thus, we have from Cauchy-Schwartz inequality that

$$\int_0^T |\dot{x}_n|^2 \geqslant \int_0^T (A(t)x_n, x_n) + \int_0^T b^+(t)(V'(x_n), x_n) - \int_0^T b^-(t)(V'(x_n), x_n) - \varepsilon_n$$

where $b^+(t) = \max\{0, b(t)\}$ and $b^-(t) = -\min\{0, b(t)\}$, $t \in [0, T]$.

Clearly, $b(t) = b^+(t) - b^-(t)$ for all $t \in [0, T]$.

Hence,

$$\int_0^T |\dot{x}_n|^2 \geqslant \int_0^T (A(t)x_n, x_n) + \beta\int_0^T b(t)V(x_n) - mc\int_0^T |x_n|^2 - \varepsilon_n.$$

Therefore,

$$(2.2)\quad -\frac{1}{\beta}\int_0^T |\dot{x}_n|^2 \leqslant -\frac{1}{\beta}\int_0^T (A(t)x_n, x_n) - \int_0^T b(t)V(x_n) + \frac{mc}{\beta}\int_0^T |x_n|^2 + \frac{\varepsilon_n}{\beta}.$$

Combining (2.1) and (2.2) results in,

$$\frac{\beta-2}{2\beta}\int_0^T |\dot{x}_n|^2 \leqslant \frac{(\beta-2)l+2mc}{2\beta}\int_0^T |x_n|^2 + \frac{\varepsilon_n}{\beta} + k. \quad \text{That is,}$$

$$(\beta-2)\int_0^T |\dot{x}_n|^2 \leqslant [(\beta-2)l+2mc]\int_0^T |x_n|^2 + \bar{k}, \quad (0 < \bar{k} < \infty).$$

Or, according to Wirtinger's inequality, we have that

$$\left\{(\beta-2) - [(\beta-2)l+2mc]\frac{T^2}{4\pi^2}\right\}\int_0^T |\dot{x}_n|^2 \leqslant T[(\beta-2)l+2mc]|\bar{x}_n|^2 + \bar{k}$$

where

$$\bar{x}_n = \frac{1}{T}\int_0^T x_n(t)dt, \quad n \in \mathbb{N}.$$

Clearly from condition V.2,

$$\bar{d} = (\beta-2) - [(\beta-2)l+2mc]\frac{T^2}{4\pi^2} > 0.$$

Thus, setting

$$d_1 = \frac{T[(\beta-2)l+2mc]}{\bar{d}}(>0) \quad \text{and} \quad d_2 = \frac{\bar{k}}{\bar{d}}(>0), \quad \text{we obtain}$$

$$(2.3)\qquad \int_0^T |\dot{x}_n|^2 \leqslant d_1|\bar{x}_n|^2 + d_2.$$

Moreover, applying Sobolev inequality and from (2.3), it is not difficult to verify that,

$$\min_{t\in[0,T]} |x_n(t)| \geqslant d_3|\bar{x}_n| - d_4, \quad (d_3 = 1 - (Td_1)^{\frac{1}{2}}, \quad d_4 = (Td_2)^{\frac{1}{2}}).$$

We claim that $1 - Td_1 > 0$. Suppose the contrary, that

$$1 - Td_1 \leqslant 0. \quad \text{That is if and only if} \quad 2mc \geqslant (\beta-2)\left[\frac{4\pi^2}{(1+4\pi^2)T^2} - l\right].$$

This is a contradiction. Hence, we have,

$$(2.4)\qquad \min_{t\in[0,T]} |x_n(t)| \geqslant d_3|\bar{x}_n| - d_4, \quad (d_3, d_4 > 0)$$

Clearly, if the sequence $(|\bar{x}_n|)$ of real numbers is bounded,then so is (x_n) in H_T^1. Suppose

$$|\bar{x}_n| \to +\infty \quad \text{as} \quad n \to +\infty. \quad \text{Then}$$

$$V(x_n) \to +\infty. \quad \text{So, we can define a sequence in} \quad H_T^1:$$

$$\psi_n(t) = \frac{x_n(t)}{(1+\|x_n\|)V(x_n(t))}, t \in [0,T] \quad \text{which is well defined at infinity.}$$

$$\dot{\psi}_n = \frac{V(x_n)\dot{x}_n - x_n[V'(x_n)\dot{x}_n]}{(1+\|x_n\|)(V(x_n))^2}; \text{So,}$$

$$|\dot{\psi}_n|^2 \leqslant \frac{const.|\dot{x}_n|^2}{[(1+\|x_n\|)V(x_n)]^2} \quad \text{for sufficiently large n, and}$$

$$\|\psi_n\|^2 \leqslant \frac{[\{\frac{T^2}{4\pi^2} + const.\}d_1 + T]|\bar{x}_n|^2 + const.}{a_1{}^2(1+\|x_n\|)^2(d_3|\bar{x}_n| - d_4)^{2\beta}} \to 0 \quad \text{as} \quad n \to +\infty. \quad \text{Hence,}$$

$$(1+\|x_n\|)I'(x_n)\psi_n \to 0 \quad \text{as} \quad n \to +\infty.$$

It is clear that,

$$i). \quad (1+\|x_n\|)\left|\int_0^T \dot{x}_n\dot{\psi}_n\right| \to 0 \quad \text{as} \quad n \to +\infty.$$

$$ii) \quad (1+\|x_n\|)\left|\int_0^T A(t)x_n\psi_n\right| \to 0 \quad \text{as} \quad n \to +\infty.$$

Now,

$$\begin{aligned}\left|(1+\|x_n\|)\int_0^T b(t)V'(x_n)\psi_n\right| &\geqslant \left|\beta\int_0^T b(t)dt - c\int_0^T b^+(t)\frac{|x_n|^2}{V(x_n)}dt\right| \\ &\geqslant \left|\beta\int_0^T b(t)dt\right| - c\left|\int_0^T b^+(t)\frac{|x_n|^2}{V(x_n)}dt\right|.\end{aligned}$$

Thus,

$$\left|(1+\|x_n\|)\int_0^T b(t)V'(x_n)\psi_n\right| \to L \geqslant \left|\beta\int_0^T b(t)dt\right| \neq 0 \quad \text{as} \quad n \to +\infty$$

since $\int_0^T b(t)dt \neq 0$ and

$$\left|\int_0^T b^+(t)\frac{|x_n|^2}{V(x_n)}dt\right| < \frac{m(\frac{T^2}{4\pi^2}d_1 + T)|\bar{x}_n|^2 + const.}{a_1(d_3|\bar{x}_n| - d_4)^\beta} \to 0 \quad \text{as} \quad n \to +\infty.$$

Note: Either $\int_0^T b(t)dt < 0$ or $\int_0^T b(t)dt > 0$; whichever the case may be, the same conclusion holds.

This contradicts our assumption. Therefore, (x_n) is bounded, and so by a similar proof in [5], admits a strongly convergent subsequence. □

Proof of theorem 2.1. $H_T^1 = \widetilde{H} \oplus \mathbb{R}^N$, $\widetilde{H} = \left\{x \in H_T^1 : \int_0^T x(t)dt = 0\right\}$. Let $x \in \widetilde{H}$. Then,

$$\begin{aligned}I(x) &= \frac{1}{2}\int_0^T |\dot{x}|^2 - \frac{1}{2}\int_0^T (A(t)x, x) - \int_0^T b(t)V(x) \\ &\geqslant \frac{1}{2}\left(1 - \frac{lT^2}{4\pi^2}-\right)\int_0^T |\dot{x}|^2 - m\int_0^T V(x)\end{aligned}$$

According to V.3, we can choose $\epsilon : \quad 0 < \epsilon < \frac{1}{2m}(\frac{4\pi^2}{T^2} - l)$ such that there exists $r'(\epsilon) > 0$ sufficiently small and

$$I(x) \geqslant \frac{1}{2}\left(1 - \frac{lT^2}{4\pi^2}\right)\int_0^T |\dot{x}|^2 - m\epsilon\int_0^T |x|^2$$

$$\geqslant \frac{1}{2}\left(1-\frac{lT^2}{4\pi^2}-m\epsilon\frac{T^2}{2\pi^2}\right)\int_0^T |\dot{x}|^2$$
$$\geqslant \frac{T}{24}\left(1-\frac{lT^2}{4\pi^2}-m\epsilon\frac{T^2}{2\pi^2}\right)\|x\|_\infty^2 \geqslant \omega > 0$$

for all $x \in \widetilde{H}, \|x\|_\infty = r'$. Thus, we can find $r(r') > 0 : I(x) \geqslant \omega$ for all $x \in \widetilde{H}, \|x\| = r$. Furthermore, choose a constant vector $\phi_0 \in \mathbb{R}^N : |\phi_0|^2 = \frac{\delta}{\delta^2+\pi^2}$.

Define,

$$\nu(t) = \begin{cases} \phi_0 \cos\frac{\pi}{\delta}(t+\delta-t_0) & t \in I_\delta \\ 0 & t \in \bar{I} = [0,T]\setminus I_\delta \end{cases}$$

So, we have $\nu(t) \in H_T^1, \|\nu\| = 1$. Besides, $\mathrm{supp}\{\nu\} = I_\delta$.
Set $\overline{H} = \left\{\lambda\nu + z : \lambda \geqslant 0, z \in \mathbb{R}^N\right\}$. Thus, for every $\lambda\nu + z \in \overline{H}$, we have

$$\begin{aligned} I(\lambda\nu+z) &\leqslant \frac{\lambda^2}{2}\int_0^T |\dot{\nu}|^2 - \frac{|z|^2}{2}\int_{\bar{I}}(A(t)\xi,\xi) - \frac{\eta}{2}\int_{I_\delta}|\lambda\nu+z|^2 \\ &\quad - \int_{I_\delta} b(t)V(\lambda\nu+z) - V(z)\int_{\bar{I}} b(t) \\ &\leqslant \frac{\lambda^2}{2}\int_{I_\delta}|\dot{\nu}|^2 + \frac{|z|^2}{2}\eta T - \frac{\eta}{2}\int_{I_\delta}|\lambda\nu+z|^2 \\ &\quad - a_1\tilde{\beta}\int_{I_\delta}|\lambda\nu+z|^\beta - V(z)\int_{\bar{I}} b(t) \end{aligned}$$

where $\tilde{\beta} = \min\limits_{I_\delta} b(t) > 0$ and $\xi \in \mathbb{R}^N, \xi = z/|z|$.

Now, defining some two norms on $\overline{H}$:

$$\|\lambda\nu+z\|_{\delta_1} = \left(\int_{I_\delta}|\lambda\nu+z|^2\right)^{\frac{1}{2}}, \quad \|\lambda\nu+z\|_{\delta_2} = \left(\int_{I_\delta}|\lambda\nu+z|^\sigma\right)^{\frac{1}{\sigma}}, \quad \sigma \geqslant 2$$

it is clear that they are equivalent since $\dim \overline{H} < \infty$. Therefore, for our fixed $\beta > 2$, we have that, there exists $k_1(\delta) > 0$ such that $\|\lambda\nu+z\|_{\delta_2}^\beta \geqslant k_1\|\lambda\nu + z\|_{\delta_1}^\beta$ for all $\lambda \geqslant 0, z \in \mathbb{R}^N$. Besides, we can find some constant $k_2 > 0$ (independent of $z \in \mathbb{R}^N, \lambda \geqslant 0$):

$$I(\lambda\nu+z) \leqslant (k_2 - \frac{\eta}{2})\|\lambda\nu+z\|_{\delta_1}^2 - k_1 a_1\tilde{\beta}\|\lambda\nu+z\|_{\delta_1}^\beta - V(z)\int_{\bar{I}} b(t) \tag{2.5}$$

We have two cases, namely,

$$\text{Case1: } \int_{\bar{I}} b(t) > 0. \quad \left(\text{i.e} \int_0^T b(t) > 0.\right)$$
$$\text{Case 2: } \int_{\bar{I}} b(t) < 0. \quad \left(\text{if and only if} \quad -\int_{\bar{I}} b(t) > 0.\right)$$

Case1: We have from (2.5) that

$$I(\lambda\nu+z) \leqslant k_2\|\lambda\nu+z\|_{\delta_1}^2 - \frac{\eta}{2}\|\lambda\nu+z\|_{\delta_1}^2 - k_1 a_1\tilde{\beta}\|\lambda\nu+z\|_{\delta_1}^\beta$$

Clearly, we can find $R > r$ (sufficiently large) such that,

$$I(\lambda\nu + z) \leqslant 0 \quad \text{for all} \quad \lambda\nu + z \in \overline{H} : \|\lambda\nu + z\| = R.$$

Therefore,

$$\sup\{I(\lambda\nu + z) : \lambda \geqslant 0 \quad z \in \mathbb{R}^N, \quad \|\lambda\nu + z\| = R\} \leqslant \inf\{I(x) : x \in \widetilde{H}, \quad \|x\| = r\}.$$

Hence the conditions of the linking theorem are satisfied. Consequently, I has a non-constant critical point in H_T^1 which is a solution of (P).

Note:$\|\lambda\nu + z\|_{\delta_1} \leqslant \|\lambda\nu + z\|$.

Case2: $\int_{\bar{I}} b(t)dt < 0$

Without loss of generality, we further assume that there exists real numbers, $\alpha, \gamma : \alpha > \gamma > 1$ such that

$$\gamma \int_{I_\delta} b(t)dt + \int_{\bar{I}} b(t)dt > 0$$

and for all $z_1, z_2 \in \mathbb{R}^N, z_1 \neq z_2$, and $|z_1 + z_2| > R_2$, we have

$$V(z_1 + z_2) > \gamma\left(V(z_1) + V(z_2)\right) - \alpha|z_1 - z_2|^2.$$

For any $\lambda \geqslant 0$ and $z \in \mathbb{R}^N, \|\lambda\nu + z\| = \lambda^2 + T|z|^2$. So, $\|\lambda\nu + z\| \to +\infty$ if and only if $\lambda \to +\infty$ or $|z| \to +\infty$.

Hence, for sufficiently large $\lambda > 0$ or $|z|$ we have,

$$\begin{aligned} I(\lambda\nu + z) \leqslant{}& \frac{\lambda^2}{2}\int_0^T |\dot{\nu}|^2 - \frac{|z|^2}{2}\int_{\bar{I}} (A(t)\xi, \xi) - \frac{\eta\lambda^2}{2}\int_{I_\delta} |\nu|^2 - \eta\delta|z|^2 \\ & - \int_{I_\delta} b(t)V(\lambda\nu + z) - V(z)\int_{\bar{I}} b(t)dt. \end{aligned}$$

Or

$$\begin{aligned} I(\lambda\nu + z) &\leqslant \frac{\lambda^2}{2}\left(\int_{I_\delta} (|\dot{\nu}|^2 - \eta|\nu|^2)\right) - \frac{|z|^2}{2}\left(\int_{\bar{I}} (A(t)\xi, \xi) + 2\eta\delta\right) \\ &\quad - \left(\gamma\int_{I_\delta} b(t)dt + \int_{\bar{I}} b(t)dt\right)V(z) - \gamma\int_{I_\delta} V(\lambda\nu)dt + \alpha\int_{I_\delta} b(t)|\lambda\nu - z|^2 \\ &\leqslant \frac{\lambda^2}{2}\left(\int_{I_\delta} |\dot{\nu}|^2 + (2\alpha m - \eta)\int_{I_\delta} |\nu|^2\right) - \frac{|z|^2}{2}\left(\int_{\bar{I}} (A(t)\xi, \xi) + 2\eta\delta - 4m\delta\alpha\right) \\ &\quad - |z|^\beta a_1\left(\gamma\int_{I_\delta} b(t)dt + \int_{\bar{I}} b(t)dt\right) - a_1\lambda^\beta\gamma\int_{I_\delta} |\nu|^\beta dt \end{aligned}$$

Clearly, as $\|\lambda\nu + z\| \to +\infty, I(\lambda\nu + z) \to -\infty$; and so, we can find $R > r$ such that for $\|\lambda\nu + z\| = R, I(\lambda\nu + z) \leqslant 0$.

This verifies the Linking theorem and hence, completes the proof of theorem 2.1 □

Remark. We consider some examples of the potential V that satisfy our assumptions:

Example 1. The first obvious example is the function

$$V(x) = |x|^\beta, \quad \text{for any} \quad \beta > 2.$$

Here

$$V'(x).x = \beta V(x) \quad \text{for all} \quad x \in \mathbb{R}^N.$$

So, for any $c > 0$, $V(x) = |x|^\beta$ satisfies condition V.2. Furthermore,

$$|x_1|^\beta + |x_2|^\beta - \kappa|x_1 - x_2|^2 \leqslant \frac{1}{2}|x_1 + x_2|^\beta, \quad (\kappa > 1 \quad \text{is large enough})$$

for all $x_1, x_2 \in \mathbb{R}^N$, $|x_1 - x_2| \neq 0$, and $|x_1 + x_2|$ sufficiently large.

Example 2. We define

$$V(x) = |x|^\beta \log_e\left(\frac{1 + 2|x|^2}{1 + |x|^2}\right), \quad 2 < \beta < 3.$$

For sufficiently large $|x|$,

$$V(x) \geqslant a|x|^\beta \quad \text{for some} \quad a : 0 < a < \log_e 2$$

since

$$\lim_{|x| \to +\infty} \log_e\left(\frac{1 + 2|x|^2}{1 + |x|^2}\right) = \log_e 2.$$

$$V'(x)x = \beta V(x) + \frac{2|x|^{\beta+2}}{(1 + |x|^2)(1 + 2|x|^2)} \leqslant \beta V(x) + c|x|^2, \quad \text{for some} \quad c > 0,$$

while

$$V'(x)x \geqslant \beta V(x).$$

Remark. The logarithmic operator moderates the growth of the potential in example 2. So, for any $z_1, z_2 \in \mathbb{R}^N, z_1 \neq z_2$, and $|z_1 + z_2|$ sufficiently large, we can find some $\alpha > \gamma > 1, \alpha$, large enough so that

$$|z_1 + z_2|^\beta \log_e\left(\frac{1 + 2|z_1 + z_2|^2}{1 + |z_1 + z_2|^2}\right)$$
$$\geqslant \gamma\left[|z_1|^\beta \log_e\left(\frac{1 + 2|z_1|^2}{1 + |z_1|^2}\right) + |z_2|^\beta \log_e\left(\frac{1 + 2|z_2|^2}{1 + |z_2|^2}\right)\right] - \alpha|z_1 - z_2|^2.$$

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