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# NEW APPROXIMATION SCHEMES FOR NONEXPANSIVE NONSELF-MAPPINGS IN A BANACH SPACE

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ABSTRACT. In this paper, weak and strong convergence theorems of a new three-step iteration with errors are established for nonexpansive nonself-mappings in Banach spaces. The results obtained in this paper extend and improve the several recent results in this area.

## 1. INTRODUCTION

Fixed-point iteration processes for approximating fixed point of nonexpansive mapping in Banach spaces have been studied by various authors (see [3, 4, 6, 10, 11, 12, 16, 17, 18]) using the Mann iteration process (see [6]) or the Ishikawa iteration process (see [4, 16, 18]). In 2000, Noor [8] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. In 1998, Takahashi and Kim [15] proved strong convergence of approximants to fixed points of nonexpansive nonself-mappings in reflexive Banach spaces with uniformly Gâteaux differentiable norm. In the same year, Jung and Kim [5] proved the existence of a fixed point for nonexpansive nonself-mapping in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm.

In [16], Tan and Xu introduced a modified Ishikawa process to approximate fixed points of nonexpansive self-mappings defined on nonempty closed convex bounded subsets of a uniformly convex Banach space. Suantai [14] defined a new three-step iterations which is an extension of Noor iterations and gave some weak and strong convergence theorems of such iterations for asymptotically nonexpansive mappings in uniformly Banach spaces. Recently, Shahzad [13] extended Tan and Xu's results([16],Theorem 1, p.305) to the case of nonexpansive nonself-mapping in a uniformly convex Banach space. Inspired and motivated by research going on in this area, we define and study a new three-step iteration with errors for nonexpansive nonself-mapping. This scheme can be viewed as an extension for the two-step iterative schemes of Shahzad [13]. The scheme is defined as follows.

Let $X$ be a normed space, $C$ be a nonempty convex subset of $X$, $P : X \to C$ be the nonexpansive retraction of $X$ onto $C$, and $T : C \to X$ be a given mapping. Then for a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= P(a_nTx_n + (1 - a_n - \gamma_n)x_n + \gamma_nu_n) \\ y_n &= P(b_nTz_n + c_nTx_n + (1 - b_n - c_n - \mu_n)x_n + \mu_nv_n) \qquad (1.1) \\ x_{n+1} &= P(\alpha_nTy_n + \beta_nTz_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_nw_n), \quad n \geq 1, \end{aligned}$$

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where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}, \{\lambda_n\}$ are appropriate sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}$ *and* $\{w_n\}$ are bounded sequences in $C$.

If $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then (1.1) reduces to the iteration scheme defined by Shahzad [13]

$$\begin{aligned} y_n &= P(b_n T x_n + (1 - b_n)x_n) \\ x_{n+1} &= P(\alpha_n T y_n + (1 - \alpha_n)x_n), \quad n \geq 1, \end{aligned} \tag{1.2}$$

where $\{b_n\}, \{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

If $T : C \to C$, then the iterative scheme (1.1) reduces to the three-step iterations with errors

$$\begin{aligned} z_n &= a_n T x_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n \\ y_n &= b_n T z_n + c_n T x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n \\ x_{n+1} &= \alpha_n T y_n + \beta_n T z_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n, \quad n \geq 1, \end{aligned} \tag{1.3}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}, \{\lambda_n\}$ are appropriate sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}$ *and* $\{w_n\}$ are bounded sequences in $C$.

The purpose of this paper is to establish weak and strong convergence results of the iterative scheme (1.1) for completely continuous nonexpansive nonself-mappings in a uniformly convex Banach space. Our results extend and improve the corresponding ones announced by Shahzad [13], Tan and Xu [16] and others.

Now, we recall the well known concepts and results.

Recall that a Banach space $X$ is said to satisfy *Opial's condition* [9] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

In the sequel, the following lemmas are needed to prove our main results.

**Lemma 1.1** ([16], Lemma 1 )**.** *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n = 1, 2, ...,$$

*If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then*

(1) $\lim_{n\to\infty} a_n$ *exists .*

(2) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

**Lemma 1.2** ([7], Lemma 1.4 )**.** *Let $X$ be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty), g(0) = 0$ such that*

$$\|\alpha x + \beta y + \mu z + \lambda w\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \mu\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g(\|x - y\|),$$

*for all $x, y, z, w \in B_r$, and all $\alpha, \beta, \mu, \lambda \in [0, 1]$ with $\alpha + \beta + \mu + \lambda = 1$.*

**Lemma 1.3** (Browder [1])**.** *Let $X$ be a uniformly convex Banach space, $C$ a nonempty closed convex subset of $X$ and $T : C \to X$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed point of $T$.*

**Lemma 1.4** ([14], Lemma 2.7 )**.** *Let $X$ be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in $X$ . Let $u, v \in X$ be such that* $\lim_{n\to\infty} \|x_n - u\|$ *and* $\lim_{n\to\infty} \|x_n - v\|$ *exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to $u$ and $v$, respectively, then $u = v$.*

## 2. Main Results

In this section, we prove weak and strong convergence theorems of the new three-step iterative scheme (1.1) for nonexpansive nonself-mapping in a uniformly convex Banach space. In order to prove our main results, the following lemma is needed.

**Lemma 2.1.** *Let $X$ be a uniformly convex Banach space, and let $C$ be a nonempty closed convex nonexpansive retract of $X$ with $P$ as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}, \{\gamma_n\}, \{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0, 1]$ such that $a_n + \gamma_n$, $b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0, 1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, and let $\{u_n\}, \{v_n\}$ and $\{w_n\}$ be bounded sequences in $C$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.1).*

(i) *If $q$ is a fixed point of $T$, then* $\lim_{n\to\infty} \|x_n - q\|$ *exists.*

(ii) *If* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, *then* $\lim_{n\to\infty} \|Ty_n - x_n\| = 0$.

(iii) *If either* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ *or* $0 < \liminf_{n\to\infty} \alpha_n$ *and* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, *then* $\lim_{n\to\infty} \|Tz_n - x_n\| = 0$.

(iv) *If the following conditions*

  (a) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ *and*

  (b) *either* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$ *or* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$

*are satisfied, then* $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.

*Proof.* Let $q \in F(T)$, by boundedness of the sequence $\{u_n\}, \{v_n\}$ and $\{w_n\}$, we can put

$$M = \max\{\sup_{n\geq 1} \|u_n - q\|,\ \sup_{n\geq 1} \|v_n - q\|,\ \sup_{n\geq 1} \|w_n - q\|\}.$$

(i) For each $n \geq 1$, we have

(2.1)
$$\begin{aligned}
\|x_{n+1} - q\| &= \|P(\alpha_n Ty_n + \beta_n Tz_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n) - P(q)\| \\
&\leq \|\alpha_n Ty_n + \beta_n Tz_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n - q\| \\
&\leq \alpha_n\|Ty_n - q\| + \beta_n\|Tz_n - q\| \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\| + \lambda_n\|w_n - q\| \\
&\leq \alpha_n\|y_n - q\| + \beta_n\|z_n - q\| + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\| + M\lambda_n,
\end{aligned}$$

$$\begin{aligned}
\|z_n - q\| &= \|P(a_n Tx_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n) - P(q)\| \\
&\leq a_n\|Tx_n - q\| + (1 - a_n - \gamma_n)\|x_n - q\| + \gamma_n\|u_n - q\|
\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\le a_n\|x_n-q\|+(1-a_n-\gamma_n)\|x_n-q\|+M\gamma_n\\&\le \|x_n-q\|+M\gamma_n\end{aligned}$$

and

$$\begin{aligned}\|y_n-q\|&=\|P(b_nTz_n+c_nTx_n+(1-b_n-c_n-\mu_n)x_n+\mu_nv_n)-P(q)\|\\&\le b_n\|Tz_n-q\|+c_n\|Tx_n-q\|\\&\quad+(1-b_n-c_n-\mu_n)\|x_n-q\|+\mu_n\|v_n-q\|\\&\le b_n\|z_n-q\|+c_n\|x_n-q\|+(1-b_n-c_n-\mu_n)\|x_n-q\|+M\mu_n\\&\le b_n\|z_n-q\|+(1-b_n)\|x_n-q\|+M\mu_n.\end{aligned}$$

From (2.2) we get

$$\begin{aligned}\|y_n-q\|&\le b_n(\|x_n-q\|+M\gamma_n)+(1-b_n)\|x_n-q\|+M\mu_n\\&=\|x_n-q\|+\epsilon^n_{(1)},\end{aligned}\tag{2.3}$$

where $\epsilon^n_{(1)}=Mb_n\gamma_n+M\mu_n$. Since $\sum_{n=1}^\infty\gamma_n<\infty$ and $\sum_{n=1}^\infty\mu_n<\infty$, we have $\sum_{n=1}^\infty\epsilon^n_{(1)}<\infty$.
From (2.1), (2.2) and (2.3) we get

$$\begin{aligned}\|x_{n+1}-q\|&\le\alpha_n(\|x_n-q\|+\epsilon^n_{(1)})+\beta_n(\|x_n-q\|+M\gamma_n)\\&\quad+(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|+M\lambda_n\\&=\alpha_n\|x_n-q\|+\alpha_n\epsilon^n_{(1)}+\beta_n\|x_n-q\|+M\beta_n\gamma_n\\&\quad+(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|+M\lambda_n\\&\le\|x_n-q\|+\epsilon^n_{(2)},\end{aligned}\tag{2.4}$$

where $\epsilon^n_{(2)}=\alpha_n\epsilon^n_{(1)}+M\beta_n\gamma_n+M\lambda_n$. Since $\sum_{n=1}^\infty\epsilon^n_{(2)}<\infty$ we obtain from (2.4) and Lemma 1.1 that $\lim_{n\to\infty}\|x_n-q\|$ exists.
(ii) By (i) we have that $\lim_{n\to\infty}\|x_n-q\|$ exists for any $q\in F(T)$. It follows from (2.2) and (2.3) that $\{x_n-q\},\{Tx_n-q\},\{z_n-q\},\{Tz_n-q\},\{y_n-q\}$ and $\{Ty_n-q\}$ are bounded sequences. This allows us to put

$$\begin{aligned}K=\max\{M,\ &\sup_{n\ge1}\|x_n-q\|,\ \sup_{n\ge1}\|Tx_n-q\|,\ \sup_{n\ge1}\|z_n-q\|,\\&\sup_{n\ge1}\|Tz_n-q\|,\ \sup_{n\ge1}\|y_n-q\|,\ \sup_{n\ge1}\|Ty_n-q\|\}.\end{aligned}$$

Since $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n)<1$, It follows from (2.2) and (2.3) that

$$\|z_n-q\|^2\le\|x_n-q\|^2+\epsilon^n_{(3)}\tag{2.5}$$

$$\|y_n-q\|^2\le\|x_n-q\|^2+\epsilon^n_{(4)},\tag{2.6}$$

where $\epsilon^n_{(3)}=M^2\gamma_n^2+2MK\gamma_n$ and $\epsilon^n_{(4)}=(\epsilon^n_{(1)})^2+2K\epsilon^n_{(1)}$. Since $\sum_{n=1}^\infty\epsilon^n_{(3)}<\infty$ and $\sum_{n=1}^\infty\epsilon^n_{(4)}<\infty$, by Lemma 1.2, there is a continuous strictly increasing convex function $g:[0,\infty)\to[0,\infty)$, $g(0)=0$ such that

$$\|\lambda x+\beta y+\gamma z+\mu w\|^2\le\lambda\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2+\mu\|w\|^2-\lambda\beta g(\|x-y\|)\tag{2.7}$$

for all $x, y, z, w \in B_K$ and all $\lambda, \beta, \gamma, \mu \in [0,1]$ with $\lambda + \beta + \gamma = 1$. By (2.5), (2.6) and (2.7), we have

$$
\begin{aligned}
(2.8)\quad \|x_{n+1} - q\|^2 &= \|P(\alpha_n T y_n + \beta_n T z_n \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n) - P(q)\|^2 \\
&\le \|\alpha_n(Ty_n - q) + \beta_n(Tz_n - q) \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)(x_n - q) + \lambda_n(w_n - q)\|^2 \\
&\le \alpha_n\|Ty_n - q\|^2 + \beta_n\|Tz_n - q\|^2 \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + \lambda_n\|w_n - q\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|Ty_n - x_n\|) \\
&\le \alpha_n\|y_n - q\|^2 + \beta_n\|z_n - q\|^2 + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 \\
&\quad + K^2\lambda_n - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|Ty_n - x_n\|) \\
&\le \alpha_n(\|x_n - q\|^2 + \epsilon^n_{(4)}) + \beta_n(\|x_n - q\|^2 + \epsilon^n_{(3)}) \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + K^2\lambda_n \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|Ty_n - x_n\|) \\
&= \alpha_n\|x_n - q\|^2 + \alpha_n\epsilon^n_{(4)} + \beta_n\|x_n - q\|^2 + \beta_n\epsilon^n_{(3)} \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + K^2\lambda_n \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|Ty_n - x_n\|) \\
&\le \|x_n - q\|^2 + \epsilon^n_{(5)} - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|Ty_n - x_n\|),
\end{aligned}
$$

where $\epsilon^n_{(5)} = \alpha_n\epsilon^n_{(4)} + \beta_n\epsilon^n_{(3)} + K^2\lambda_n$. It is worth to note here that $\sum_{n=1}^{\infty}\epsilon^n_{(5)} < \infty$ since $\sum_{n=1}^{\infty}\epsilon^n_{(4)} < \infty, \sum_{n=1}^{\infty}\epsilon^n_{(3)} < \infty$, and $\sum_{n=1}^{\infty}\lambda_n < \infty$. Since $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, there exists $n_0 \in \mathbb{N}$ and $\delta_1, \delta_2 \in (0,1)$ such that $0 < \delta_1 < \alpha_n$ and $\alpha_n + \beta_n + \lambda_n < \delta_2 < 1$ for all $n \ge n_0$. Hence, by (2.8), we have

$$
\begin{aligned}
(2.9)\quad \delta_1(1 - \delta_2)\sum_{n=n_0}^{m} g(\|Ty_n - x_n\|) &< \sum_{n=n_0}^{m}(\|x_n - q\|^2 - \|x_{n+1} - q\|^2) + \sum_{n=n_0}^{m}\epsilon^n_{(5)} \\
&= \|x_{n_0} - q\|^2 + \sum_{n=n_0}^{m}\epsilon^n_{(5)}.
\end{aligned}
$$

Since $\sum_{n=n_0}^{\infty}\epsilon^n_{(5)} < \infty$, by letting $m \to \infty$ in (2.9) we get $\sum_{n=n_0}^{\infty} g(\|Ty_n - x_n\|) < \infty$, and therefore $\lim_{n\to\infty} g(\|Ty_n - x_n\|) = 0$. Since $g$ is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty}\|Ty_n - x_n\| = 0$.

(iii) First, we assume that $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$. By (2.7), we have

$$
\begin{aligned}
(2.10)\quad \|x_{n+1} - q\|^2 &\le \alpha_n\|y_n - q\|^2 + \beta_n\|z_n - q\|^2 \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + K^2\lambda_n \\
&\quad - \beta_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|Tz_n - x_n\|) \\
&\le \alpha_n(\|x_n - q\|^2 + \epsilon^n_{(4)}) + \beta_n(\|x_n - q\|^2 + \epsilon^n_{(3)})
\end{aligned}
$$

$$
\begin{aligned}
&+ (1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2 + K^2\lambda_n \\
&- \beta_n(1-\alpha_n-\beta_n-\lambda_n)g(\|Tz_n-x_n\|) \\
&= \alpha_n\|x_n-q\|^2 + \alpha_n\epsilon^n_{(4)} + \beta_n\|x_n-q\|^2 + \beta_n\epsilon^n_{(3)} \\
&+ (1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2 + K^2\lambda_n \\
&- \beta_n(1-\alpha_n-\beta_n-\lambda_n)g(\|Tz_n-x_n\|) \\
&\le \|x_n-q\|^2 + \epsilon^n_{(5)} - \beta_n(1-\alpha_n-\beta_n-\lambda_n)g(\|Tz_n-x_n\|),
\end{aligned}
$$

where $\epsilon^n_{(5)} = \alpha_n\epsilon^n_{(4)} + \beta_n\epsilon^n_{(3)} + K^2\lambda_n$. Since $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, there exists $n_0 \in \mathbb{N}$ and $\delta_1, \delta_2 \in (0,1)$ such that $0 < \delta_1 < \beta_n$ and $\alpha_n + \beta_n + \lambda_n < \delta_2 < 1$ for all $n \ge n_0$. Hence, by (2.10), we have $\epsilon^n_{(5)} = \alpha_n\epsilon^n_{(4)} + \beta_n\epsilon^n_{(3)} + K^2\lambda_n$.

$$
\begin{aligned}
(2.11)\quad \delta_1(1-\delta_2)\sum_{n=n_0}^{m} g(\|Tz_n-x_n\|) &< \sum_{n=n_0}^{m}(\|x_n-q\|^2 - \|x_{n+1}-q\|^2) + \sum_{n=n_0}^{m}\epsilon^n_{(5)} \\
&= \|x_{n_0}-q\|^2 + \sum_{n=n_0}^{m}\epsilon^n_{(5)}.
\end{aligned}
$$

Since $\sum_{n=n_0}^{\infty}\epsilon^n_{(5)} < \infty$, by letting $m\to\infty$ in (2.11) we get $\sum_{n=n_0}^{\infty} g(\|Tz_n-x_n\|) < \infty$, and therefore $\lim_{n\to\infty} g(\|Tz_n-x_n\|) = 0$. Since $g$ is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|Tz_n-x_n\| = 0$.

Next, we assume that $0 < \liminf_{n\to\infty}\alpha_n$ and $\liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$. By (2.5) and (2.7), we have

$$
\begin{aligned}
(2.12)\qquad \|y_n-q\|^2 &= \|P(b_nTz_n + c_nTx_n \\
&\quad + (1-b_n-c_n-\mu_n)x_n + \mu_nv_n) - P(q)\|^2 \\
&\le \|b_n(Tz_n-q) + c_n(Tx_n-q) \\
&\quad + (1-b_n-c_n-\mu_n)(x_n-q) + \mu_n(v_n-q)\|^2 \\
&\le b_n\|Tz_n-q\|^2 + c_n\|Tx_n-q\|^2 \\
&\quad + (1-b_n-c_n-\mu_n)\|x_n-q\|^2 + \mu_n\|v_n-q\|^2 \\
&\quad - b_n(1-b_n-c_n-\mu_n)g(\|Tz_n-x_n\|) \\
&\le b_n\|z_n-q\|^2 + c_n\|x_n-q\|^2 \\
&\quad + (1-b_n-c_n-\mu_n)\|x_n-q\|^2 + \mu_nK^2 \\
&\quad - b_n(1-b_n-c_n-\mu_n)g(\|Tz_n-x_n\|) \\
&\le b_n(\|x_n-q\|^2 + \epsilon^n_{(3)}) + c_n\|x_n-q\|^2 \\
&\quad + (1-b_n-c_n-\mu_n)\|x_n-q\|^2 + \mu_nK^2 \\
&\quad - b_n(1-b_n-c_n-\mu_n)g(\|Tz_n-x_n\|) \\
&\le \|x_n-q\|^2 + \epsilon^n_{(6)} - b_n(1-b_n-c_n-\mu_n)g(\|Tz_n-x_n\|),
\end{aligned}
$$

where $\epsilon^n_{(6)} = b_n\epsilon^n_{(3)} + \mu_nK^2$.

By (2.5), (2.7) and (2.12), we also have

(2.13)
$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|P(\alpha_n Ty_n+\beta_n Tz_n \\ &\quad +(1-\alpha_n-\beta_n-\lambda_n)x_n+\lambda_n w_n)-P(q)\|^2 \\ &\le \|\alpha_n(Ty_n-q)+\beta_n(Tz_n-q) \\ &\quad +(1-\alpha_n-\beta_n-\lambda_n)(x_n-q)+\lambda_n(w_n-q)\|^2 \\ &\le \alpha_n\|y_n-q\|^2+\beta_n\|z_n-q\|^2+(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+K^2\lambda_n \\ &= \alpha_n(\|x_n-q\|^2+\epsilon^n_{(6)}-b_n(1-b_n-c_n-\mu_n)g(\|Tz_n-x_n\|)) \\ &\quad +\beta_n(\|x_n-q\|^2+\epsilon^n_{(3)})+(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+K^2\lambda_n \\ &= \alpha_n\|x_n-q\|^2+\alpha_n\epsilon^n_{(6)}-\alpha_n b_n(1-b_n-c_n-\mu_n)g(\|Tz_n-x_n\|) \\ &\quad +\beta_n\|x_n-q\|^2+\beta_n\epsilon^n_{(3)}+(1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2+K^2\lambda_n \\ &\le \|x_n-q\|^2+\epsilon^n_{(7)}-\alpha_n b_n(1-b_n-c_n-\mu_n)g(\|Tz_n-x_n\|),\end{aligned}$$

where $\epsilon^n_{(7)}=\alpha_n\epsilon^n_{(6)}+\beta_n\epsilon^n_{(3)}+K^2\lambda_n$.

It is worth to note here that $\sum_{n=1}^{\infty}\epsilon^n_{(7)}<\infty$ since $\sum_{n=1}^{\infty}\epsilon^n_{(6)}<\infty, \sum_{n=1}^{\infty}\epsilon^n_{(3)}<\infty$, and $\sum_{n=1}^{\infty}\lambda_n<\infty$.

By our assumption $0<\liminf_{n\to\infty}\alpha_n$ and $0<\liminf_{n\to\infty}b_n\le\limsup_{n\to\infty}(b_n+c_n+\mu_n)<1$, there exists $n_0\in\mathbb{N}$ and $\delta_1,\delta_2\in(0,1)$ such that $0<\delta_1<\alpha_n$, $0<\delta_1<b_n$ and $b_n+c_n+\mu_n<\delta_2<1$ for all $n\ge n_0$. Hence, by (2.13), we have

$$\begin{aligned}\text{(2.14)}\quad \delta_1^2(1-\delta_2)\sum_{n=n_0}^{m}g(\|Tz_n-x_n\|) &< \sum_{n=n_0}^{m}(\|x_n-q\|^2-\|x_{n+1}-q\|^2)+\sum_{n=n_0}^{m}\epsilon^n_{(7)} \\ &= \|x_{n_0}-q\|^2+\sum_{n=n_0}^{m}\epsilon^n_{(7)}.\end{aligned}$$

Since $\sum_{n=n_0}^{\infty}\epsilon^n_{(7)}<\infty$, by letting $m\to\infty$ in (2.14) we get $\sum_{n=n_0}^{\infty}g(\|Tz_n-x_n\|)<\infty$, and therefore $\lim_{n\to\infty}g(\|Tz_n-x_n\|)=0$. Since $g$ is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|Tz_n-x_n\|=0$.
(iv) Suppose that the conditions (1) and (2) are satisfied. Then by (ii) and (iii), we have

$$\text{(2.15)}\qquad \lim_{n\to\infty}\|Ty_n-x_n\|=0 \ \text{ and } \ \lim_{n\to\infty}\|Tz_n-x_n\|=0.$$

From $z_n=P(a_nTx_n+(1-a_n-\gamma_n)x_n+\gamma_n u_n)$ and $y_n=P(b_nTz_n+c_nTx_n+(1-b_n-c_n-\mu_n)x_n+\mu_n v_n)$, we have $\|z_n-x_n\|\le a_n\|Tx_n-x_n\|+\gamma_n\|u_n-x_n\|$ and $\|y_n-x_n\|\le b_n\|Tz_n-x_n\|+c_n\|Tx_n-x_n\|+\mu_n\|v_n-x_n\|$. It follows that

$$\begin{aligned}\|Tx_n-x_n\| &\le \|Tx_n-Tz_n\|+\|Tz_n-x_n\| \\ &\le \|x_n-z_n\|+\|Tz_n-x_n\| \\ &\le a_n\|Tx_n-x_n\|+\gamma_n\|u_n-z_n\|+\|Tz_n-x_n\|,\end{aligned}$$

which implies

$$(1-a_n)\|Tx_n - x_n\| \leq \gamma_n\|u_n - z_n\| + \|Tz_n - x_n\|.$$

If $\limsup_{n\to\infty} a_n < 1$, this together with (2.15) and $\lim_{n\to\infty}\gamma_n = 0$ imply that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.

If $\limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, there exists a positive integer $N_0$ and $\eta \in (0,1)$ such that

$$c_n \leq b_n + c_n + \mu_n < \eta \quad \forall n \geq N_0.$$

Then for $n \geq N_0$, we have

$$\begin{aligned}\|Tx_n - x_n\| &\leq \|Tx_n - Ty_n\| + \|Ty_n - x_n\| \\ &\leq \|x_n - y_n\| + \|Ty_n - x_n\| \\ &\leq b_n\|Tz_n - x_n\| + c_n\|Tx_n - x_n\| \\ &\quad + \mu_n\|v_n - x_n\| + \|Ty_n - x_n\| \\ &\leq b_n\|Tz_n - x_n\| + \eta\|Tx_n - x_n\| \\ &\quad + \mu_n\|v_n - x_n\| + \|Ty_n - x_n\|.\end{aligned}$$

Hence

$$(1-\eta)\|Tx_n - x_n\| \leq b_n\|Tz_n - x_n\| + \mu_n\|v_n - x_n\| + \|Ty_n - x_n\|.$$

This together with (2.15) and the fact that $\mu_n \to 0$ as $n \to \infty$ imply $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. □

**Theorem 2.2.** *Let $X$ be a uniformly convex Banach space, and let $C$ be a nonempty closed convex nonexpansive retract of $X$ with $P$ as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be sequences of real numbers in $[0,1]$ with $a_n + \gamma_n \in [0,1], b_n + c_n + \mu_n \in [0,1]$ and $\alpha_n + \beta_n + \lambda_n \in [0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$. If*

(i) $0 < \min\{\liminf_{n\to\infty}\alpha_n,\ \liminf_{n\to\infty}\beta_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$ *or*

(ii) $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$ *and* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$,

*then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by the iterative scheme (1.1) converge strongly to a fixed point of $T$.*

*Proof.* It follows from Lemma 2.1(i) that $\{x_n\}$ is bounded. Again by Lemma 2.1 , we have

$$\begin{aligned}&\lim_{n\to\infty}\|Ty_n - x_n\| = 0, \\ &\lim_{n\to\infty}\|Tz_n - x_n\| = 0, \\ &\lim_{n\to\infty}\|Tx_n - x_n\| = 0.\end{aligned} \tag{2.16}$$

Since $T$ is completely continuous and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Hence, by $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, it follows that $\{x_{n_k}\}$ converges. Let $\lim_{n\to\infty} x_{n_k} = q$. By continuity of $T$ and (2.16) we have that $Tq = q$, so $q$ is a fixed point of $T$. By Lemma 2.1 (i), $\lim_{n\to\infty} \|x_n - q\|$ exists. But $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$, so $\lim_{n\to\infty} \|x_n - q\| = 0$. By (2.16), we have

$$\begin{aligned}\|y_n - x_n\| &= \|P(b_nTz_n + c_nTx_n \\ &\quad + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n) - P(x_n)\| \\ &\le b_n\|Tz_n - x_n\| + c_n\|Tx_n - x_n\| + \mu_n\|v_n - x_n\| \\ &\to 0 \ \ (\text{as } n \to \infty),\end{aligned}$$

and

$$\begin{aligned}\|z_n - x_n\| &= \|P(a_nTx_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n) - P(x_n)\| \\ &\le a_n\|Tx_n - x_n\| + \gamma_n\|u_n - x_n\| \\ &\to 0 \ \ (\text{as } n \to \infty).\end{aligned}$$

It follows that $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$ . □

If $T$ is a self-mapping, then the iterative scheme (1.1) reduces to that of (1.3) and the following result is directly obtained by Theorem 2.2.

**Theorem 2.3.** *Let $X$ be a uniformly convex Banach space, and $C$ a nonempty closed convex subset of $X$. Let $T$ be a completely continuous nonexpansive self-mapping of $C$ with $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be sequences of real numbers in $[0, 1]$ with $b_n + c_n \in [0, 1]$ and $\alpha_n + \beta_n \in [0, 1]$ for all $n \ge 1$ . If*

(i) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n\} \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$ *or*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$ *and* $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$,

*then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by the iterations (1.3) converge strongly to a fixed point of $T$.*

When $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 2.2 , the following result is obtained.

**Theorem 2.4.** *Let $X$ be a uniformly convex Banach space, and let $C$ be a nonempty closed convex nonexpansive retract of $X$ with $P$ as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{\alpha_n\}$ be real sequences in $[0, 1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$, *and*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$,

*For a given $x_1 \in C$, define*

$$\begin{aligned}z_n &= P(a_nTx_n + (1 - a_n)x_n) \\ y_n &= P(b_nTz_n + (1 - b_n)x_n), \quad n \ge 1 \\ x_{n+1} &= P(\alpha_nTy_n + (1 - \alpha_n)x_n).\end{aligned}$$

*Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of $T$.*

When $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 2.2 , we obtain the following result.

**Theorem 2.5.** *Let $X$ be a uniformly convex Banach space, and let $C$ be a nonempty closed convex nonexpansive retract of $X$ with $P$ as a nonexpansive retraction. Let $T : C \to X$ be a completely continuous nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{b_n\}, \{\alpha_n\}$ be a real sequences in $[0, 1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

*For a given $x_1 \in C$, define*

$$y_n = P(b_n T z_n + (1 - b_n)x_n)$$
$$x_{n+1} = P(\alpha_n T y_n + (1 - \alpha_n)x_n), \quad n \geq 1.$$

*Then $\{x_n\}$ and $\{y_n\}$ converge strongly to a fixed point of $T$.*

In the next result, we prove weak convergence of the iterations scheme (1.1) for nonexpansive nonself-mapping in a uniformly convex Banach space satisfying Opial's condition.

**Theorem 2.6.** *Let $X$ be a uniformly convex Banach space which satisfies Opial's condition, and $C$ a nonempty closed convex nonexpansive retract of $X$ with $P$ as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\lambda_n\}$ be sequences of real numbers in $[0, 1]$ with $a_n + \gamma_n, b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0, 1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$. If*

(i) $0 < \min\{\liminf_{n\to\infty} \alpha_n, \ \liminf_{n\to\infty} \beta_n\} \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ *and* $\limsup_{n\to\infty} a_n < 1$ *or*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$ *and* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$,

*then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by the iterative scheme (1.1) converge weakly to a fixed point of $T$.*

*Proof.* It follows from Lemma 2.1 that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ and $\lim_{n\to\infty} \|Tz_n - x_n\| = 0$. Since $X$ is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 1.3, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to $u$ and $v$, respectively. From Lemma 1.3, $u, v \in F(T)$. By Lemma 2.1 (i), $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 1.4 that $u = v$. Therefore $\{x_n\}$ converges weakly to a fixed point $u$ of $T$. Since $\|y_n - x_n\| \leq b_n\|Tz_n - x_n\| + c_n\|Tx_n - x_n\| + \mu_n\|v_n - x_n\| \to 0$ (as $n \to \infty$) and $\|z_n - x_n\| \leq a_n\|Tx_n - x_n\| + \gamma_n\|u_n - x_n\| \to 0$ (as $n \to \infty$) and $x_n \to u$ weakly as $n \to \infty$, it follows that $y_n \to u$ and $z_n \to u$ weakly as $n \to \infty$. □

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