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STRONG CONVERGENCE TO ZEROS OF ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \cap_{\lambda>0} R(I+\lambda A)$. Then, we consider a sequence $\{x_n\}$ generated by $x \in C,\ x_n = \alpha_n x + (1-\alpha_n)J_{\lambda_n}x_n$ $(\forall n \in \mathbf{N})$, where $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,\infty)$ and J_{λ_n} is the resolvent of A and prove that if $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\alpha_n/\lambda_n = 0$, $\{x_n\}$ converges strongly to some element of $A^{-1}0$. And we consider a sequence $\{x_n\}$ generated by $x_1 = x \in C,\ x_{n+1} = \alpha_n x + (1-\alpha_n)J_{\lambda_n}x_n$ $(\forall n \in \mathbf{N})$, where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ and proved that if $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n+1}| < \infty$, $\liminf_{n\to\infty}\lambda_n > 0$ and $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| < \infty$, $\{x_n\}$ converges strongly to some element of $A^{-1}0$.

1. Introduction

Throughout this paper, let E be a real Banach space with norm $\|\cdot\|$ and let $\mathbf{N}$ be the set of all positive integers. Let $A \subset E \times E$ be an m-accretive operator such that $A^{-1}0 \neq \emptyset$. An m-accretive operator is equivalent to a maximal monotone operator in a Hilbert space. Let $x \in E$ and $\{\lambda_n\} \subset (0,\infty)$. At first, Rockafellar [21] considered the proximal point algorithm, i.e. $x_1 = x,\ \ x_{n+1} = J_{\lambda_n}x_n$ $(\forall n \in \mathbf{N})$ where J_{λ_n} is the resolvent of A and proved weak convergence to an element of $A^{-1}0$ in a Hilbert space. But the strong convergence of the proximal point algorithm failed; see Gűler [7]. So, Kamimura and Takahashi [10] considered a sequence $\{x_n\}$ generated by Halpern type iteration [8], that is,

$$x_1 = x, \quad x_{n+1} = \alpha_n x + (1-\alpha_n)J_{\lambda_n}x_n \ (\forall n \in \mathbf{N}) \tag{1}$$

where $\{\alpha_n\} \subset [0,1]$ and they proved that $\{x_n\}$ converges strongly to an element of $A^{-1}0$ if $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}\lambda_n = \infty$. Then, Kamimura and Takahashi [11, 12] extended this result to a Banach space, (see also [27]). And Solodov and Svaiter [25], Bauschke and Combettes [2] and the author and Takahashi [14] considered a sequence generated by Haugazeau's hybrid method [9] and proved strong convergence to an element of $A^{-1}0$ in a Hilbert space, (see also [15, 17]). Then, Kamimura and Takahashi [13] and Ohsawa and Takahashi [19] extended Solodov and Svaiter's result to a Banach space, separately. And author, K. Shimoji and W. Takahashi [18] considered a sequence $\{x_n\}$ generated by Browder type [3], that is,

$$x_n = \alpha_n x + (1-\alpha_n)J_{\lambda_n}x_n \quad (\forall n \in \mathbf{N}) \tag{2}$$

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where $\{\alpha_n\} \subset (0,1)$ and proved strong convergence to an element of $A^{-1}0$ in a Hilbert space when $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \alpha_n/\lambda_n = 0$.

In this paper, we extend the result [18] to a Banach space in section 3. Next, we prove strong convergence to an element of $A^{-1}0$ by (1) under $\liminf_{n\to\infty} \lambda_n > 0$ in section 4.

2. Preliminaries and Lemmas

We write $x_n \to x$ to indicate that a sequence $\{x_n\}$ converges strongly to x. Let C be a subset of E and let $T : C \longrightarrow E$. T is called Lipschitzian if there exists a nonnegative number k such that $\|Tx - Ty\| \leq k\|x - y\|$ for all $x, y \in C$. T is said to be a contraction if T is Lipschitzian with $k < 1$. T is called nonexpansive if T is Lipschitzian with $k = 1$, that is, $\|Tx - Ty\| \leq \|x - y\|$ holds for each $x, y \in C$. We denote by $F(T)$ the set of all fixed points of T. We define the modulus of convexity of E δ_E as follows: δ_E is a function of $[0,2]$ into $[0,1]$ such that $\delta_E(\varepsilon) = \inf\{1 - \|x+y\|/2 : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}$ for every $\varepsilon \in [0,2]$. E is called uniformly convex if $\delta_E(\varepsilon) > 0$ for each $\varepsilon > 0$. E is called strictly convex if $\|x + y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space E, we have that if $\|x\| = \|y\| = \|\lambda x + (1-\lambda)y\|$ for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. It is known that a uniformly convex Banach space is strictly convex. Let $G = \{g : [0,\infty) \longrightarrow [0,\infty) : g(0) = 0, g :$ continuous, strictly increasing, convex$\}$. Xu [29] proved the following theorem: Let E be a uniformly convex Banach space. Then, for every bounded subset B of E, there exists $g_B \in G$ such that

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g_B(\|x - y\|) \tag{3}$$

for all $x, y \in B$ and $0 \leq \lambda \leq 1$. E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{4}$$

exists for every $x, y \in S(E)$, where $S(E) = \{x \in E : \|x\| = 1\}$. And the norm of E is said to be uniformly Gâteaux differentiable if for each $y \in S(E)$, (4) is attained uniformly for $x \in S(E)$. It is known that the duality mapping $J : E \longrightarrow 2^{E^*}$ is single valued and norm to weak* uniformly continuous on bounded subsets of E if E has a uniformly Gâteaux differentiable norm. Let μ be a continuous, linear functional on l^∞. We call μ a Banach limit [1] when μ satisfies $\|\mu\| = \mu(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $\{a_n\} \in l^\infty$. We know that $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for every $\{a_n\} \in l^\infty$. We have the following from [28]; see also [5].

Lemma 2.1. *Let C be a convex subset of E whose norm is uniformly Gâteaux differentiable and let $z \in C$. Let $\{x_n\} \subset E$ be a bounded sequence and let μ be a Banach limit. Then, $\mu_n\|x_n - z\|^2 = \min_{y\in C} \mu_n\|x_n - y\|^2$ if and only if $\mu_n(y - z, J(x_n - z)) \leq 0$ for all $y \in C$.*

Let C be a convex subset of E and let K be a nonempty subset of C. Let P be a retraction of C onto K, that is, $Px = x$ for every $x \in K$. P is said to be sunny if $P(Px + t(x - Px)) = Px$ whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. We know the following [4, 20].

Lemma 2.2. *Let C be a convex subset of a smooth Banach space and let K be a nonempty subset of C. Let P be a retraction of C onto K. Then, P is sunny and nonexpansive if and only if $(x - Px, J(y - Px)) \leq 0$ for every $x \in C$ and $y \in K$. Hence, there is at most one sunny, nonexpansive retraction of C onto K.*

An operator $A \subset E \times E$ is called accretive if for (x_1, y_1), $(x_2, y_2) \in A$, there exists $j \in J(x_1 - x_2)$ such that $(y_1 - y_2, j) \geq 0$, where J is the duality mapping of E. An accretive operator A is said to satisfy the range condition if $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$, where $D(A)$ is the domain of A, $R(I + \lambda A)$ is the range of $I + \lambda A$ and $\overline{D(A)}$ is the closure of $D(A)$. And an accretive operator A is said to be m-accretive if $R(I + \lambda A) = E$ for every $\lambda > 0$. If A is accretive, then we can define, for each $r > 0$, a mapping $J_r : R(I + rA) \longrightarrow D(A)$ by $J_r = (I + rA)^{-1}$. J_r is called the resolvent of A. We know that J_r is nonexpansive and $A^{-1}0 = F(J_r)$ for every $r > 0$. We also define the Yosida approximations A_r by $A_r = (I - J_r)/r$; see [26, 27] for more details. We have the following result for the resolvents [16], see also [26, 27].

Lemma 2.3. *Let $A \subset E \times E$ be an accretive operator which satisfies the range condition. Then, $\frac{1}{\lambda}\|(I - J_\lambda)J_r x\| \leq \frac{1}{r}\|(I - J_r)x\|$ holds for every $r, \lambda > 0$ and $x \in R(I + rA)$.*

And we have the following [6], see also [26, 27].

Lemma 2.4. *Let $A \subset E \times E$ be an accretive operator. Then, for each $r, \lambda > 0$ and $x \in R(I + rA) \cap R(I + \lambda A)$, $\|J_\lambda x - J_r x\| \leq \frac{|\lambda - r|}{\lambda}\|x - J_\lambda x\|$ holds.*

3. Browder Type

Using an idea of [23] (see also [24]), we get the following.

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \cap_{\lambda > 0} R(I + \lambda A)$. Let $\{x_n\}$ be a sequence generated by (2), where $x \in C$, $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$. If $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{\alpha_n}{\lambda_n} = 0$, $\{x_n\}$ converges strongly to $z \in A^{-1}0$. Further if $Px := \lim_{n\to\infty} x_n$ $(\forall x \in C)$, P is a sunny nonexpansive retraction of C onto $A^{-1}0$.*

Proof. Let $T_n y = \alpha_n x + (1 - \alpha_n)J_{\lambda_n} y$ for every $n \in \mathbf{N}$ and $y \in C$. We have $T_n : C \longrightarrow C$ and T_n is a contraction for all $n \in \mathbf{N}$ since J_{λ_n} is nonexpansive and $0 < \alpha_n < 1$. So, for each $n \in \mathbf{N}$, there exists a unique element $x_n \in C$ such that $x_n = \alpha_n x + (1 - \alpha_n)J_{\lambda_n} x_n$. Let $z_0 \in A^{-1}0$. We get

$$\begin{aligned}\|x_n - z_0\| &= \|\alpha_n(x - z_0) + (1 - \alpha_n)(J_{\lambda_n} x_n - z_0)\| \\ &\leq \alpha_n\|x - z_0\| + (1 - \alpha_n)\|J_{\lambda_n} x_n - z_0\| \\ &\leq \alpha_n\|x - z_0\| + (1 - \alpha_n)\|x_n - z_0\|\end{aligned}$$

for every $n \in \mathbf{N}$. So, we obtain $\|x_n - z_0\| \leq \|x - z_0\|$ for all $n \in \mathbf{N}$ which implies $\{x_n\}$ is bounded. Further, we have

$$\|x_n - J_{\lambda_n} x_n\| = \alpha_n\|x - J_{\lambda_n} x_n\| \leq \alpha_n(\|x - z_0\| + \|J_{\lambda_n} x_n - z_0\|) \leq 2\alpha_n\|x - z_0\|$$

for each $n \in \mathbf{N}$. As $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \alpha_n/\lambda_n = 0$, we get

$$\lim_{n\to\infty} \|x_n - J_{\lambda_n} x_n\| = \lim_{n\to\infty} \frac{1}{\lambda_n}\|x_n - J_{\lambda_n} x_n\| = 0. \tag{5}$$

Let $r > 0$. We obtain

$$\begin{aligned}\|x_n - J_r x_n\| &\le \|x_n - J_{\lambda_n} x_n\| + \|J_{\lambda_n} x_n - J_r J_{\lambda_n} x_n\| + \|J_r J_{\lambda_n} x_n - J_r x_n\| \\ &\le 2\|x_n - J_{\lambda_n} x_n\| + \frac{r}{\lambda_n}\|x_n - J_{\lambda_n} x_n\|\end{aligned}$$

for every $n \in \mathbf{N}$ by Lemma 2.3. Therefore, we have

$$\lim_{n\to\infty} \|x_n - J_r x_n\| = 0 \tag{6}$$

for all $r > 0$ from (5). Since A_{λ_n} is accretive, we get

$$\begin{aligned}\alpha_n(x - z_0, J(x_n - z_0)) &= \alpha_n(x_n - z_0, J(x_n - z_0)) \\ &\quad + (1-\alpha_n)((x_n - J_{\lambda_n} x_n) - (z_0 - J_{\lambda_n} z_0), J(x_n - z_0)) \\ &\ge \alpha_n \|x_n - z_0\|^2\end{aligned}$$

for every $n \in \mathbf{N}$ and $z_0 \in A^{-1}0$. So, we obtain

$$\|x_n - z_0\|^2 \le (x - z_0, J(x_n - z_0)) \tag{7}$$

for all $n \in \mathbf{N}$. And we have

$$\begin{aligned}(x_n - x, J(x_n - z_0)) &= \frac{1-\alpha_n}{\alpha_n}(J_{\lambda_n} x_n - x_n, J(x_n - z_0)) \\ &= \frac{1-\alpha_n}{\alpha_n}\{(J_{\lambda_n} x_n - z_0, J(x_n - z_0)) - (x_n - z_0, J(x_n - z_0))\} \\ &= \frac{1-\alpha_n}{\alpha_n}\{(J_{\lambda_n} x_n - z_0, J(x_n - z_0)) - \|x_n - z_0\|^2\} \le 0\end{aligned} \tag{8}$$

for each $n \in \mathbf{N}$ and $z_0 \in A^{-1}0$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ and let μ be a Banach limit. Let g be a real valued function on C defined by $g(y) = \mu_i \|x_{n_i} - y\|^2$ for every $y \in C$. By [23, Proposition 2], we get g is continuous and convex, and g satisfies $\lim_{\|y\|\to\infty} g(y) = \infty$. So, there exists $x_0 \in C$ such that $g(x_0) = \inf_{y\in C} g(y)$. Let $y_1, y_2 \in C$ with $y_1 \ne y_2$ such that $g(y_1) = g(y_2) = \inf_{y\in C} g(y)$ and let B be a bounded subset of E containig $\{x_{n_i} - y_1\}$ and $\{x_{n_i} - y_2\}$. There exists $g_B \in G$ such that

$$\begin{aligned}\Big\|x_{n_i} - \frac{y_1+y_2}{2}\Big\|^2 &= \Big\|\frac{1}{2}(x_{n_i} - y_1) + \frac{1}{2}(x_{n_i} - y_2)\Big\|^2 \\ &\le \frac{1}{2}\|x_{n_i} - y_1\|^2 + \frac{1}{2}\|x_{n_i} - y_2\|^2 - \frac{1}{4} g_B(\|y_1 - y_2\|)\end{aligned}$$

which implies

$$g\Big(\frac{y_1+y_2}{2}\Big) \le \frac{1}{2}g(y_1) + \frac{1}{2}g(y_2) - \frac{1}{4}g_B(\|y_1 - y_2\|) < \inf_{y\in C} g(y).$$

This is a contradiction. So, we obtain $y_1 = y_2$. Therefore, there exists a unique element y_0 of C such that $g(y_0) = \inf_{y\in C} g(y)$. We suppose $y_0 \notin A^{-1}0$. Let $r > 0$

and let B be a bounded subset of E containing $\{x_{n_i} - y_0\}$ and $\{x_{n_i} - J_r y_0\}$. We have

$$\begin{aligned}\Big\|x_{n_i} - \frac{J_r y_0 + y_0}{2}\Big\|^2 &\le \frac{1}{2}\|x_{n_i} - y_0\|^2 + \frac{1}{2}\|x_{n_i} - J_r y_0\|^2 - \frac{1}{4} g_B(\|y_0 - J_r y_0\|)\\ &\le \frac{1}{2}\|x_{n_i} - y_0\|^2 + \frac{1}{2}\{\|x_{n_i} - J_r x_{n_i}\| + \|J_r x_{n_i} - J_r y_0\|\}^2 - \frac{1}{4} g_B(\|y_0 - J_r y_0\|)\\ &\le \frac{1}{2}\|x_{n_i} - y_0\|^2 + \frac{1}{2}\{\|x_{n_i} - J_r x_{n_i}\| + \|x_{n_i} - y_0\|\}^2 - \frac{1}{4} g_B(\|y_0 - J_r y_0\|)\\ &= \frac{1}{2}\|x_{n_i} - y_0\|^2 + \frac{1}{2}\{\|x_{n_i} - J_r x_{n_i}\|^2 + 2\|x_{n_i} - J_r x_{n_i}\| \cdot \|x_{n_i} - y_0\|\\ &\qquad + \|x_{n_i} - y_0\|^2\} - \frac{1}{4} g_B(\|y_0 - J_r y_0\|)\end{aligned}$$

for some $g_B \in G$ which implies

$$g\Big(\frac{J_r y_0 + y_0}{2}\Big) \le \frac{1}{2} g(y_0) + \frac{1}{2} g(y_0) - \frac{1}{4} g_B(\|y_0 - J_r y_0\|) < \inf_{y \in C} g(y)$$

by (6). This is a contradiction. So, we get $y_0 \in A^{-1}0$. It follows from (7) and Lemma 2.1 that $\mu_i \|x_{n_i} - y_0\|^2 \le \mu_i (x - y_0, J(x_{n_i} - y_0)) \le 0$. There exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that

$$\lim_{j\to\infty} \|x_{n_{i_j}} - y_0\|^2 = 0$$

because

$$\lim_{j\to\infty} \|x_{n_{i_j}} - y_0\|^2 = \liminf_{i\to\infty} \|x_{n_i} - y_0\|^2 \le \mu_i \|x_{n_i} - y_0\|^2 \le 0.$$

On the other hand, let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be sebsequences of $\{x_n\}$ such that $x_{n_i} \to z_1 \in A^{-1}0$ and $x_{n_j} \to z_2 \in A^{-1}0$. By (8), we obtain $(x_{n_i} - x, J(x_{n_i} - z_2)) \le 0$ for all $i \in \mathbf{N}$ and $(x_{n_j} - x, J(x_{n_j} - z_1)) \le 0$ for each $j \in \mathbf{N}$. Since

$$\begin{aligned}&|(x_{n_i} - x, J(x_{n_i} - z_2)) - (z_1 - x, J(z_1 - z_2))|\\ &\quad \le |(x_{n_i} - x, J(x_{n_i} - z_2)) - (z_1 - x, J(x_{n_i} - z_2))|\\ &\qquad + |(z_1 - x, J(x_{n_i} - z_2)) - (z_1 - x, J(z_1 - z_2))|\\ &\quad \le \|x_{n_i} - z_1\| \cdot \|x_{n_i} - z_2\| + |(z_1 - x, J(x_{n_i} - z_2)) - (z_1 - x, J(z_1 - z_2))|\end{aligned}$$

for every $i \in \mathbf{N}$ and J is norm to weak* uniformly continuous on bounded subsets of E, we have $(z_1 - x, J(z_1 - z_2)) \le 0$. Similarly, $(z_2 - x, J(z_2 - z_1)) \le 0$. So, we get $\|z_1 - z_2\|^2 = (z_1 - z_2, J(z_1 - z_2)) \le 0$, that is, $z_1 = z_2$. Therefore, $\{x_n\}$ converges strongly to some element of $A^{-1}0$. Hence, we can define a mapping P of C onto $A^{-1}0$ by $Px = \lim_{n\to\infty} x_n$ because x is an arbitrary point of C. By the argument above, we obtain $(Px - x, J(Px - z_0)) \le 0$ for all $x \in C$ and $z_0 \in A^{-1}0$. So, P is a sunny nonexpansive retraction from Lemma 2.2. □

The following generalizes the result of [18, Theorem 4.2].

Theorem 3.2. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let $A \subset E \times E$ be an m-accretive operator such that $A^{-1}0 \ne \emptyset$. Let $\{x_n\}$ be a sequence generated by (2), where $x \in E$, $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$. If $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{\alpha_n}{\lambda_n} = 0$, $\{x_n\}$ converges strongly to*

$z \in A^{-1}0$. Further if $Px := \lim_{n\to\infty} x_n$ ($\forall x \in E$), P is a sunny nonexpansive retraction of E onto $A^{-1}0$.

4. Halpern Type Iteration

Using the method employed in [22], we get the following.

Theorem 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \cap_{\lambda>0} R(I+\lambda A)$. Let $\{x_n\}$ be a sequence generated by (1), where $x \in C$, $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$. If $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, $\liminf_{n\to\infty} \lambda_n > 0$ and $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$, $\{x_n\}$ converges strongly to $z \in A^{-1}0$. Further, if $Px := \lim_{n\to\infty} x_n$ ($\forall x \in C$), P is a sunny nonexpansive retraction of C onto $A^{-1}0$.*

Proof. Let $z_0 \in A^{-1}0$. We have $\|x_n - z_0\| \leq \|x - z_0\|$ for every $n \in \mathbf{N}$. In fact, suppose that $\|x_n - z_0\| \leq \|x - z_0\|$ for some $n \in \mathbf{N}$. We get

$$\begin{aligned}\|x_{n+1} - z_0\| &= \|\alpha_n(x - z_0) + (1-\alpha_n)(J_{\lambda_n}x_n - z_0)\| \\ &\leq \alpha_n\|x - z_0\| + (1-\alpha_n)\|x_n - z_0\| \leq \|x - z_0\|.\end{aligned}$$

So, $\{x_n\}$ is bounded. From Lemma 2.4, we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(\alpha_n - \alpha_{n-1})x + (1-\alpha_n)J_{\lambda_n}x_n - (1-\alpha_{n-1})J_{\lambda_{n-1}}x_{n-1}\| \\ &= \|(\alpha_n - \alpha_{n-1})x + (1-\alpha_n)(J_{\lambda_n}x_n - J_{\lambda_{n-1}}x_{n-1}) + (\alpha_{n-1} - \alpha_n)J_{\lambda_{n-1}}x_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}| \cdot \|x - J_{\lambda_{n-1}}x_{n-1}\| \\ &\quad + (1-\alpha_n)\{\|J_{\lambda_n}x_n - J_{\lambda_n}x_{n-1}\| + \|J_{\lambda_n}x_{n-1} - J_{\lambda_{n-1}}x_{n-1}\|\} \\ &\leq |\alpha_n - \alpha_{n-1}| \cdot \|x - J_{\lambda_{n-1}}x_{n-1}\| \\ &\quad + (1-\alpha_n)\Big\{\|x_n - x_{n-1}\| + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n}\|x_{n-1} - J_{\lambda_n}x_{n-1}\|\Big\} \\ &\leq (|\alpha_n - \alpha_{n-1}| + |\lambda_n - \lambda_{n-1}|) \cdot M_0 + (1-\alpha_n)\|x_n - x_{n-1}\|\end{aligned}$$

for every $n = 2, 3, \cdots$, where $M_0 = \sup_{n=2,3,\cdots} \{\|x - J_{\lambda_{n-1}}x_{n-1}\| + \|x_{n-1} - J_{\lambda_n}x_{n-1}\|/\lambda_n\}$.
Let $m, n \in \mathbf{N}$. We have

$$\begin{aligned}&\|x_{n+m+1} - x_{n+m}\| \\ &\quad\leq (|\alpha_{n+m} - \alpha_{n+m-1}| + |\lambda_{n+m} - \lambda_{n+m-1}|)M_0 + (1-\alpha_{n+m})\|x_{n+m} - x_{n+m-1}\| \\ &\quad\leq (|\alpha_{n+m} - \alpha_{n+m-1}| + |\lambda_{n+m} - \lambda_{n+m-1}|)M_0 \\ &\qquad + (1-\alpha_{n+m})\{(|\alpha_{n+m-1} - \alpha_{n+m-2}| + |\lambda_{n+m-1} - \lambda_{n+m-2}|)M_0 \\ &\qquad + (1-\alpha_{n+m-1})\|x_{n+m-1} - x_{n+m-2}\|\} \\ &\quad\leq \{(|\alpha_{n+m} - \alpha_{n+m-1}| + |\lambda_{n+m} - \lambda_{n+m-1}|) + (|\alpha_{n+m-1} - \alpha_{n+m-2}| \\ &\qquad + |\lambda_{n+m-1} - \lambda_{n+m-2}|)\}M_0 + (1-\alpha_{n+m})(1-\alpha_{n+m-1})\|x_{n+m-1} - x_{n+m-2}\| \\ &\quad\leq \cdots \\ &\quad\leq M_0 \cdot \Big\{\sum_{k=m}^{n+m-1} (|\alpha_{k+1} - \alpha_k| + |\lambda_{k+1} - \lambda_k|)\Big\} + \Big\{\prod_{k=m}^{n+m-1} (1-\alpha_{k+1})\Big\}\|x_{m+1} - x_m\|.\end{aligned}$$

Hence, we get

$$\limsup_{n\to\infty} \|x_{n+1} - x_n\| = \limsup_{n\to\infty} \|x_{n+m+1} - x_{n+m}\| \leq M_0 \cdot \Big\{ \sum_{k=m}^{\infty} (|\alpha_{k+1} - \alpha_k| + |\lambda_{k+1} - \lambda_k|) \Big\}$$

for each $m \in \mathbf{N}$. By $\sum_{k=1}^{\infty}(|\alpha_{k+1} - \alpha_k| + |\lambda_{k+1} - \lambda_k|) < \infty$, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. So, we obtain

$$\lim_{n\to\infty} \|x_n - J_{\lambda_n} x_n\| = 0 \tag{9}$$

since $\|x_n - J_{\lambda_n} x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - J_{\lambda_n} x_n\| \leq \|x_{n+1} - x_n\| + \alpha_n \|x - J_{\lambda_n} x_n\|$ and $\lim_{n\to\infty} \alpha_n = 0$. By Lemma 2.3, we have

$$\begin{aligned} \|x_n - J_{\lambda_m} x_n\| &\leq \|x_n - J_{\lambda_n} x_n\| + \|J_{\lambda_n} x_n - J_{\lambda_m} J_{\lambda_n} x_n\| + \|J_{\lambda_m} J_{\lambda_n} x_n - J_{\lambda_m} x_n\| \\ &\leq 2\|x_n - J_{\lambda_n} x_n\| + \frac{\lambda_m}{\lambda_n} \|x_n - J_{\lambda_n} x_n\| \end{aligned}$$

for all $m, n \in \mathbf{N}$. Hence, from (9) and $\liminf_{n\to\infty} \lambda_n > 0$, we get

$$\lim_{n\to\infty} \|x_n - J_{\lambda_m} x_n\| = 0 \tag{10}$$

for every $m \in \mathbf{N}$. Let $\{\beta_m\} \subset (0, 1)$ with $\lim_{m\to\infty} \beta_m = 0$ and let $\{y_m\}$ be a sequence of C such that $y_m = \beta_m x + (1 - \beta_m) J_{\lambda_m} y_m$ for every $m \in \mathbf{N}$. By Theorem 3.1, $\lim_{m\to\infty} y_m = z \in A^{-1}0$. Let μ be a Banach limit. It follows from (10) and

$$\|x_n - J_{\lambda_m} y_m\|^2 \leq \|x_n - J_{\lambda_m} x_n\|^2 + \|x_n - y_m\|^2 + 2\|x_n - J_{\lambda_m} x_n\| \cdot \|x_n - y_m\|$$

for each $n \in \mathbf{N}$ that

$$\mu_n \|x_n - J_{\lambda_m} y_m\|^2 \leq \mu_n \|x_n - y_m\|^2 \tag{11}$$

for all $m \in \mathbf{N}$. Since

$$(1 - \beta_m)(x_n - J_{\lambda_m} y_m) = (x_n - y_m) - \beta_m (x_n - x),$$

we obtain

$$\begin{aligned} (1 - \beta_m)^2 \|x_n - J_{\lambda_m} y_m\|^2 &\geq \|x_n - y_m\|^2 - 2\beta_m (x_n - x, J(x_n - y_m)) \\ &= (1 - 2\beta_m)\|x_n - y_m\|^2 + 2\beta_m (x - y_m, J(x_n - y_m)) \end{aligned}$$

for every $m, n \in \mathbf{N}$. Hence, we have

$$\begin{aligned} &(1 - \beta_m)^2 \mu_n \|x_n - J_{\lambda_m} y_m\|^2 \\ &\geq (1 - 2\beta_m)\mu_n \|x_n - y_m\|^2 + 2\beta_m \mu_n (x - y_m, J(x_n - y_m)) \end{aligned}$$

for all $m \in \mathbf{N}$. By (11),

$$\begin{aligned} &(1 - \beta_m)^2 \mu_n \|x_n - y_m\|^2 \\ &\geq (1 - 2\beta_m)\mu_n \|x_n - y_m\|^2 + 2\beta_m \mu_n (x - y_m, J(x_n - y_m)), \end{aligned}$$

that is,

$$\frac{\beta_m}{2}\mu_n\|x_n - y_m\|^2 \geq \mu_n(x - y_m, J(x_n - y_m)) \tag{12}$$

for each $m \in \mathbf{N}$. Let $\varepsilon > 0$. As J is norm to weak* uniformly continuous on bounded subsets of E and $y_m \to z$, there exists $m_1 \in \mathbf{N}$ such that for every $m \geq m_1$,

$$|(x - z, J(x_n - z)) - (x - z, J(x_n - y_m))| < \frac{\varepsilon}{3}$$
$$|(x - z, J(x_n - y_m)) - (x - y_m, J(x_n - y_m))| < \frac{\varepsilon}{3}$$

for all $n \in \mathbf{N}$. And from (12) and $\beta_m \to 0$, there exists $m_2 \in \mathbf{N}$ such that

$$\mu_n(x - y_m, J(x_n - y_m)) < \frac{\varepsilon}{3}$$

for each $m \geq m_2$. Hence, there exists $m_0 \in \mathbf{N}$ such that for every $m \geq m_0$,

$$\begin{aligned}\mu_n(x - z, J(x_n - z)) &= \{\mu_n(x - z, J(x_n - z)) - \mu_n(x - z, J(x_n - y_m))\}\\ &\quad + \{\mu_n(x - z, J(x_n - y_m)) - \mu_n(x - y_m, J(x_n - y_m))\}\\ &\quad + \mu_n(x - y_m, J(x_n - y_m))\\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Since ε is arbitrary,

$$\mu_n(x - z, J(x_n - z)) \leq 0.$$

Further, by $\|x_{n+1} - x_n\| \to 0$, we get

$$|(x - z, J(x_n - z)) - (x - z, J(x_{n+1} - z))| \to 0.$$

Therefore, we obtain

$$\limsup_{n\to\infty}(x - z, J(x_n - z)) \leq 0 \tag{13}$$

by [22, Proposition 2]. From

$$(1 - \alpha_n)(J_{\lambda_n}x_n - z) = (x_{n+1} - z) - \alpha_n(x - z),$$

we have

$$(1 - \alpha_n)^2\|J_{\lambda_n}x_n - z\|^2 \geq \|x_{n+1} - z\|^2 - 2\alpha_n(x - z, J(x_{n+1} - z))$$

for all $n \in \mathbf{N}$. Let $\varepsilon > 0$. By (13), there exists $n_0 \in \mathbf{N}$ such that

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1 - \alpha_n)^2\|J_{\lambda_n}x_n - z\|^2 + 2\alpha_n(x - z, J(x_{n+1} - z))\\ &\leq (1 - \alpha_n)\|x_n - z\|^2 + \{1 - (1 - \alpha_n)\}\varepsilon\end{aligned}$$

for every $n \geq n_0$. Hence,

$$\begin{aligned}&\|x_{n+1} - z\|^2\\ &\quad\leq (1 - \alpha_n)\{(1 - \alpha_{n-1})\|x_{n-1} - z\|^2 + (1 - (1 - \alpha_{n-1}))\varepsilon\} + \{1 - (1 - \alpha_n)\}\varepsilon\\ &\quad= (1 - \alpha_n)(1 - \alpha_{n-1})\|x_{n-1} - z\|^2 + \{1 - (1 - \alpha_n)(1 - \alpha_{n-1})\}\varepsilon\\ &\quad\leq \cdots\end{aligned}$$

$$\leq (1-\alpha_n)(1-\alpha_{n-1})\cdots(1-\alpha_{n_0})\|x_{n_0}-z\|^2$$
$$+\{1-(1-\alpha_n)(1-\alpha_{n-1})\cdots(1-\alpha_{n_0})\}\varepsilon$$

for each $n \geq n_0$. Therefore, $\limsup_{n\to\infty} \|x_{n+1}-z\|^2 \leq \varepsilon$. Since ε is arbitrary, we get $x_n \to z \in A^{-1}0$. Hence, we can define a mapping P of C onto $A^{-1}0$ by $Px = \lim_{n\to\infty} x_n$. From Theorem 3.1, P is a sunny nonexpansive retraction of C onto $A^{-1}0$. □

We get the following result.

Theorem 4.2. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let $A \subset E \times E$ be an m-accretive operator such that $A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by (1), where $x \in E$, $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$. If $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, $\liminf_{n\to\infty} \lambda_n > 0$ and $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$, $\{x_n\}$ converges strongly to $z \in A^{-1}0$. Further, if $Px := \lim_{n\to\infty} x_n$ ($\forall x \in E$), P is a sunny nonexpansive retraction of E onto $A^{-1}0$.*

5. Application

Let $\beta_i \in (0,1)$ $(i = 1,2,\cdots,r)$ such that $\sum_{i=1}^{r} \beta_i = 1$ and let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \cdots, T_r$ be nonexpansive mappings of C into itself with $\cap_{i=1}^{r} F(T_i) \neq \emptyset$ and let $T = \sum_{i=1}^{r} \beta_i T_i$. Then, T is nonexpansive of C into itself and $F(T) = \cap_{i=1}^{r} F(T_i)$. In fact, $\cap_{i=1}^{r} F(T_i) \subset F(T)$ is trivial. Let $z \in F(T)$ and $u \in \cap_{i=1}^{r} F(T_i)$. We get

$$\begin{aligned}
\|z-u\| &= \|\beta_1(T_1z-u)+\beta_2(T_2z-u)+\cdots+\beta_r(T_rz-u)\| \\
&\leq \beta_1\|T_1z-u\|+\beta_2\|T_2z-u\|+\cdots+\beta_r\|T_rz-u\| \\
&\leq \beta_1\|z-u\|+\beta_2\|z-u\|+\cdots+\beta_r\|z-u\| = \|z-u\|
\end{aligned}$$

which implies $\|T_1z-u\| = \|T_2z-u\| = \cdots = \|T_rz-u\| = \|z-u\|$. Since E is strictly convex, $T_1z = T_2z = \cdots = T_rz = z$. So, let $A = I - T$. We know $A \subset E \times E$ is an accretive operator such that $C = D(A) \subset \cap_{\lambda>0} R(I+\lambda A)$ and $A^{-1}0 = F(T)$. Further, for $\lambda > 0$, $x \in R(I+\lambda A)$ and $y \in D(A)$, we have $y = J_\lambda x \Longleftrightarrow y = \frac{1}{1+\lambda}x + \frac{\lambda}{1+\lambda}Ty$. So, we obtain the following by Theorem 4.1.

Theorem 5.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let $\beta_i \in (0,1)$ $(i = 1,2,\cdots,r)$ such that $\sum_{i=1}^{r} \beta_i = 1$. Let $T_1, T_2, \cdots, T_r$ be nonexpansive mappings of C into itself such that $\cap_{i=1}^{r} F(T_i) \neq \emptyset$ and let $T = \sum_{i=1}^{r} \beta_i T_i$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$, $y_n = \frac{1}{1+\lambda_n}x_n + \frac{\lambda_n}{1+\lambda_n}Ty_n$, $x_{n+1} = \alpha_n x + (1-\alpha_n)y_n$ ($\forall n \in \boldsymbol{N}$), where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$. If $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, $\liminf_{n\to\infty} \lambda_n > 0$ and $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$, $\{x_n\}$ converges strongly to $z \in \cap_{i=1}^{r} F(T_i)$. Further, if $Px := \lim_{n\to\infty} x_n$ ($\forall x \in C$), P is a sunny nonexpansive retraction of C onto $\cap_{i=1}^{r} F(T_i)$.*

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