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ON THE SIZE OF THE SET OF POINTS WHERE THE METRIC PROJECTION IS DISCONTINUOUS*

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Abstract. We show that the set of points where the metric projection onto a closed set in a separable Hilbert space is single-valued but discontinuous can be covered by countably many d.c.-hypersurfaces. As a corollary, we get a similar result for the metric projection onto a Čebyšev set. This complements a result of Konyagin.

1. Introduction

Let X be real a Banach space and let $\emptyset \neq M \subset X$ be a closed set. Then we define the *distance function* as

$$\operatorname{dist}(x, M) = \inf\{\|x - m\| \colon m \in M\}$$

for any $x \in X$. The *metric projection* is a multi-valued mapping from X to $\mathcal{P}(X)$, which is defined for any $x \in X$ as

$$P_M(x) = \{m \in M \colon \|x - m\| = \operatorname{dist}(x, M)\}.$$

We will say that P_M is *continuous at* $x \in X$ provided $P_M(x) = \{m\}$ for some $m \in M$ and whenever $x_n \to x$ and $y_n \in P_M(x_n)$, then $y_n \to m$. We say that M is *Čebyšev* provided $P_M(x)$ is a singleton for every $x \in X$.

De Blasi and Myjak [DM] proved that if X is a Hilbert space, and $A \subset X$ closed, then the minimization problem $(\|\cdot\|, A, x)$ is well posed for all x outside some σ-porous set (the problem $(\|\cdot\|, A, x)$ is *well posed* provided $P_A(x) = \{y\}$, and $y_n \to y$ whenever $y_n \in A$ are such that $\|y_n - x\| \to \operatorname{dist}(x, A)$). Zajíček [Z4] showed that the distance function to a closed set in a Banach space is Fréchet differentiable outside an angle small set, provided the Banach space has a separable dual and has a uniformly Fréchet differentiable norm. Matoušková [M] showed that if X is a Banach space such that its norm is uniformly Fréchet differentiable, and its dual norm is Fréchet differentiable, $A \subset X$ is closed, then the set of points $x \in X$, where $(\|\cdot\|, A, x)$ is not well posed, can be covered by a union of a σ-cone supported set and a cone small set. It follows from [F, Corollary 3.5] that if the norm of X is Fréchet differentiable, and if the norm of X^* is Fréchet differentiable, then the set of points of continuity of P_A (for a closed $A \subset X$) in the above sense coincides with the set of points x where $(\|\cdot\|, A, x)$ is well posed. See a recent survey article [Z2] for more information about small sets in approximation theory. For various notions of smoothness of norms, see e.g. [DGZ].

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Our main result is Theorem 3, which states that the set of points, where the metric projection onto a closed set in a separable Hilbert space is single-valued but discontinuous, can be covered by countably many d.c.-hypersurfaces. It is a well known open problem whether a Čebyšev subset of the separable Hilbert space is convex – see e.g. [V, BV] for a detailed account of this problem. Konyagin [K] (see also [BV]) proved that if the set of discontinuity of the metric projection to a Čebyšev set in a Hilbert space is nonempty, then it cannot be covered by finitely many graphs of $C^{1,1}$-functions (which are d.c. by [VZ, Proposition 1.11]). As a corollary to Theorem 3, we obtain Corollary 4, which states that the set of points of discontinuity of the metric projection onto a Čebyšev set in a separable Hilbert space can be covered by countably many d.c.-hypersurfaces. This complements Konyagin's result.

In the proof of Theorem 3 we use a result of Fitzpatrick (see Theorem 1) which provides a relationship between the differentiability of the distance function and continuity of the metric projection in spaces with a sufficiently smooth norm. We also apply two results due to Zajíček: Theorem 2, which says that in spaces that allow an equivalent Hilbert norm, the distance function to a closed set is locally d.c. on the complement of that set; and [Z3, Theorem 2] which shows that the set of points of Gâteaux non-differentiability of a convex function on a separable Banach space can be covered by countably many d.c.-hypersurfaces.

2. Points of discontinuity of the metric projection

Let X be a Banach space. We say that $D \subset X$ is *a d.c.-hypersurface* provided there exists a 1-codimensional closed subspace $Y \subset X$, $0 \neq v \in X$, and continuous convex functions $f, g : Y \to \mathbb{R}$ such that $D = \{y + (f-g)(y) \cdot v : y \in Y\}$. We say that a subset $M \subset X$ *can be covered by countably many d.c.-hypersurfaces* (or "$(c-c)$-hypersurfaces" in the terminology of [Z3]) provided there exist D_j $(j \in \mathbb{N})$ such that each D_j is a d.c.-hypersurface and $M \subset \bigcup_j D_j$. For basic information about d.c. functions (i.e. those function that can be written as a difference of two continuous convex functions), see [VZ].

If $\emptyset \neq G \subset X$ is open, $f : G \to \mathbb{R}$, and $x \in G$. We say that f *is Gâteaux differentiable at* x, provided $\lim_{t\to 0} t^{-1}(f(x+tv) - f(x)) =: T(v)$ exists for each $v \in X$, and T is a continuous linear functional on X. If f is Gâteaux differentiable at x, then we will denote $Df(x) := T$.

We will need the following theorem of S. Fitzpatrick:

Theorem 1 ([F], Corollary 3.6)**.** *Suppose that M is a closed subset of a Banach space E such that the norm of E is both Fréchet differentiable and uniformly Gâteaux differentiable and the norm of E^* is Fréchet differentiable.*

The following are equivalent for a point x of $E \setminus M$:

(i) *the function $\varphi = \operatorname{dist}(\cdot, M)$ is Fréchet differentiable at x;*
(ii) *the function $\varphi = \operatorname{dist}(\cdot, M)$ is Gâteux differentiable at x, and the norm of $D\varphi(x)$ is 1;*
(iii) *the metric projection onto M is continuous at x.*

Zajíček in [Z1] proved the following:

Theorem 2 ([Z1], Theorem 5). *Let X be a Banach space such that the Fréchet derivative of the norm $\|\cdot\|$ is C-Lipschitz on S_X, $F \subset X$ is closed. Suppose that on X there exists an equivalent Hilbert norm $\|\cdot\|_h$. Then $\operatorname{dist}(\cdot, F)$ is locally d.c. on $X \setminus F$.*

Now we can prove the main theorem:

Theorem 3. *Let H be the separable Hilbert space and F be a closed nonempty subset of H. Then the set of points x where $P_F(x) = \{y\}$ for some $y \in H$, but P_F is not continuous at x, can be covered by countably many d.c. hypersurfaces.*

Proof. Let $\varphi(z) = \operatorname{dist}(z, F)$. Suppose that $P_F(x) = \{y\}$ and P_F is not continuous at x. If $x = y$, then $x \in F$ and P_F is continuous at x – a contradiction. Now suppose that $x \notin F$. It means that $\varphi(x) > 0$. By Theorem 2 and by separability of H, we can find open convex sets U_n and Lipschitz convex functions $f_n, g_n : U_n \to \mathbb{R}$, so that $H \setminus F = \bigcup_n U_n$ and $\varphi = f_n - g_n$ on U_n. It is easily seen that f_n, g_n can be extended to continuous convex functions defined on the whole space H – call the extensions again f_n, g_n, and by [Z3, Theorem 2] there exist countably many d.c.-hypersurfaces D_j so that points of Gâteaux non-differentiability of all f_n and g_n are contained in $\bigcup_j D_j$.

Without any loss of generality, we can (and do) assume that $x = 0$, $\|y\| = 1$, and that φ is Gâteaux differentiable at 0. Then

$$\lim_{t \to 0} \frac{\varphi(ty) - \varphi(0) - D\varphi(0)ty}{t} = 0$$

implies that $D\varphi(0)y = -1$ (because $\varphi(ty) = 1 - t$ for $0 \le t < 1$). This implies that $\|D\varphi(0)\| = 1$ and that is a contradiction with condition (ii) of Theorem 1. It follows that φ is not Gâteaux differentiable at x. As $x \in U_n$ for some n, it follows that at least one of the functions f_n or g_n is not Gâteaux differentiable at x (as a sum of two Gâteaux differentiable functions is again Gâteaux differentiable). So $x \in \bigcup_j D_j$. □

Theorem 3 gives the following interesting corollary for Čebyšev sets, which complements Konyagin's result.

Corollary 4. *Let M be a Čebyšev set in a separable Hilbert space. Then the set of points where P_M fails to be continuous can be covered by countably many d.c.-hypersurfaces.*

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