Journal of Nonlinear and Convex Analysis
Volume 7, Number 1, 2006, 39–50

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

INCREASING-ALONG-RAYS PROPERTY FOR VECTOR FUNCTIONS

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Abstract. In this paper we extend to the vector case the notion of increasing-along-rays function. The proposed definition is given by means of a nonlinear scalarization through the so-called oriented distance function from a point to a set.

We prove that the considered class of functions enjoys properties similar to those holding in the scalar case, with regard to optimization problems, relations with (generalized) convex functions and characterization in terms of Minty type variational inequalities.

1. Introduction

The notion of increasing-along-rays (IAR) scalar function arises mainly in the study of abstract convexity (see e.g. [22]) and can be viewed as a generalization of the concept of quasiconvex function. Properties of IAR scalar functions have been investigated in [6, 7]. Here, several properties of this class of functions with regard to optimization problems have been pointed out and furthermore it has been shown that IAR functions can be characterized by means of a generalized Minty variational inequality [21]. In this paper we extend to the vector case the notion of IAR function. In Section 2 we briefly recall the notion of scalar IAR functions and its basic properties. Since the proposed definition of increasing-along-rays vector function is given using a nonlinear scalarization, namely the so-called oriented distance function from a point to a set, introduced in [17], Section 3 presents some basic facts on this concept and the main relations between a vector minimization problem and its scalarized counterpart. In Section 4 the notion of vector IAR function is presented and some basic properties with respect to vector optimization are pointed out. Sections 5,6 and 7 are finally devoted to the investigation of the newly defined class of functions. We underline that the introduced notion depends on the norm in the image space. This fact was not relevant in the scalar case and we regard it as a special feature of vector setting. Being interested in conditions on f which make this property independent of the norm in the image space, we prove that this is the case when vector convex functions are considered. Vector quasiconvex functions do not enjoy the norm-independence property, but we prove that these functions enjoy the increasing-along-rays property for a suitable choice of the norm in the image space. Further, we show that vector increasing-along-rays functions can be characterized in terms of existence of a solution of a (scalar) generalized Minty variational inequality.

Key words and phrases. generalized convexity, increasing-along-rays property, star-shaped set, Minty variational inequality.

2. Scalar increasing-along-rays functions

Throughout the paper we consider finite dimensional normed spaces X and Y. We refer to the norm in the considered spaces by $\|\cdot\|$, since it will be clear from the context to which of the normed spaces it applies. Further, K will denote a nonempty subset of X.

In this section we recall the notion of scalar-valued IAR function and some of its basic properties which further we extend to a general space Y.

Definition 2.1. Let the subset $K \subseteq X$ be star-shaped at x^0. A function φ defined on K is called increasing-along-rays at x^0 (for short, $\varphi \in IAR(K, x^0)$), if its restriction on the segment $R_{x^0,x} \cap K$, with $R_{x^0,x} = \{x^0 + \alpha x | \alpha \geq 0\}$, is increasing, for each $x \in K$. (A function g of one real variable is increasing on the interval I if $t_2 \geq t_1$, $t_1, t_2 \in I$ implies $g(t_2) \geq g(t_1)$.)

When $K = X$, the function is increasing along the whole ray $R_{x^0,x}$. We refer to such special case as $IAR\left(x^0\right)$. It is clear that when $X = \mathbb{R}$ and K is an interval, $\varphi \in IAR(K, x^0)$ if and only if it is quasiconvex with a global minimum over K at x^0. However the following example shows that when $n \geq 2$ and K is a convex set, the class of functions $\varphi \in IAR(K, x^0)$ is broader then the class of quasiconvex functions with a global minimum at x^0.

Example 2.1. Let $\varphi(x_1, x_2) = x_1^2 x_2^2$, and $K = \mathbb{R}^2$. Then, for $x^0 = \left(0\,,\,0\right)$ it is easily seen that $\varphi \in IAR(K, x^0)$, but φ is not quasiconvex.

We consider the following problem:

$$P(\varphi, K) \qquad\qquad \min \varphi(x),\ x \in K \subseteq X.$$

A point $x^0 \in K$ is a (global) solution of $P(\varphi, K)$ when $\varphi(x) - \varphi(x^0) \geq 0,\ \forall x \in K$. The solution is strict if $\varphi(x) - \varphi(x^0) > 0$, $\forall x \in K \backslash \{x^0\}$. We denote by $\operatorname{argmin}(\varphi, K)$ the set of solutions of $P(\varphi, K)$. Local solutions of $P(\varphi, K)$ have a clear definition and we omit it.

The properties which are stated in the following results motivate some of the interest for the class $IAR(K, x^0)$ and are the core of the problems we present in the following sections.

Proposition 2.1 ([6])**.** *Let the subset $K \subseteq X$ be star-shaped at x^0 and $\varphi \in IAR(K, x^0)$. Then:*

i) *x^0 is a solution of $P(\varphi, K)$;*
ii) *no point $x \in K$, $x \neq x^0$, can be a strict local solution of $P(\varphi, K)$.*
iii) $\operatorname{argmin}(\varphi, K)$ *is star-shaped at x^0.*

Proposition 2.2 ([25])**.** *Let the subset $K \subseteq X$ be star-shaped at x^0 and φ be a function defined on K. Then $\varphi \in IAR(K, x^0)$ if and only if for each $c \in \mathbb{R}$ with $c \geq \varphi(x^0)$, the set $\operatorname{lev}_{\leq c}\varphi = \{x \in K : \varphi(x) \leq c\}$ is star-shaped at x^0.*

IAR functions can be, as well, characterized through some generalized variational inequalities of Minty type (see e.g. [6, 7]). The following notions are classical and are presented for the sake of completeness.

Definition 2.2. Let the subset $K \subseteq X$ be star-shaped at x^0 and let φ be a function defined on an open set containing K. The function φ is said to be radially lower semicontinuous over K along rays starting at x^0, if for each $x \in K$, the restriction of φ on the interval $R_{x^0,x} \cap K$ is lower semicontinuous.

We write $\varphi \in RLSC(K, x^0)$ to denote that φ satisfies the previous definition. For any real function φ defined on an open set containing K, the lower Dini directional derivative at the point $x \in K$ in the direction $u \in X$ is defined as an element of $\bar{\mathbb{R}} := [-\infty, +\infty]$ by

$$\varphi'_-(x, u) = \liminf_{t \to +0} \frac{\varphi(x + tu) - \varphi(x)}{t}.$$

The problem of finding $x^0 \in K$ such that K is star-shaped at x^0 and x^0 satisfies the inequalities

$$MVI(\varphi'_-, K) \qquad \varphi'_-(y, x^0 - y) \leq 0, \forall y \in K$$

can be regarded as a generalized Minty variational inequality. This problem obviously reduces to the usual Minty variational inequality problem of differential type (see e.g. [21]) when φ is differentiable on an open set containing K.

Theorem 2.1 ([6])**.** *Let the subset $K \subseteq X$ be star-shaped set at x^0.*

i) *If x^0 solves $MVI(\varphi'_-, K)$ and $\varphi \in RLSC(K, x^0)$, then $\varphi \in IAR(K, x^0)$.*
ii) *Conversely, if $\varphi \in IAR(K, x^0)$, then x^0 is a solution of $MVI(\varphi'_-, K)$.*

We recall furthermore that the importance of IAR functions in scalar optimization is stressed by the fact that they also enjoy several well-posedness properties (see e.g. [6]).

3. Oriented distance function and scalar characterizations of vector optimality concepts

Let f be a function from X to Y and let C be a closed convex pointed cone in Y with $\operatorname{int} C \neq \emptyset$. We consider the vector optimization problem

$$VP(f, K) \qquad \min_C \ f(x), \ x \in K \subseteq X.$$

Usually, the solutions of problem $VP(f, K)$ are called points of efficiency, but here we prefer to call them minimizers. We say that the point $x^0 \in K$ is e-minimizer (respectively w-minimizer) for $VP(f, K)$ when $f(x) - f(x^0) \notin -C\backslash\{0\}$ ($f(x) - f(x^0) \notin -\operatorname{int} C$), for every $x \in K$. Further, a point $x^0 \in K$ is an ideal minimizer when $f(x) - f(x^0) \in C$. Recall that ideal minimizers are not likely to happen for $VP(f, K)$.

Given a set $A \subset Y$ and a point $y \in Y$, the distance from y to A is given by the function (depending on the norm chosen on Y) $d_A(y) = \inf_{a \in A} \|y - a\|$. In [17] the author proposes a generalization of the distance notion, known as oriented distance. The oriented distance from y to A is given by the function $\Delta_A(y) = d_A(y) - d_{Y \backslash A}(y)$. The main properties of Δ_A are summarized in the next proposition.

Proposition 3.1. i) *Δ_A is 1-Lipschitzian;*
ii) *$\Delta_A(y) < 0$ for every $y \in \operatorname{int} A$, $\Delta_A(y) = 0$ for every $y \in \partial A$ and $\Delta_A(y) > 0$ for every $y \in \operatorname{int}(Y \setminus A)$;*

iii) *if A is closed, then it holds $A = \{y : \Delta_A(y) \leq 0\}$;*
iv) *if A is convex, then Δ_A is convex;*
v) *if A is a cone, then Δ_A is positively homogeneous;*
vi) *if A is a closed convex cone, then Δ_A is nonincreasing with respect to the ordering relation induced on Y, i.e. the following is true: if $y_1, y_2 \in Y$ then*

$$y_1 - y_2 \in A \quad \Longrightarrow \quad \Delta_A(y_1) \leq \Delta_A(y_2)$$

If A has nonempty interior, then

$$y_1 - y_2 \in \operatorname{int} A \quad \Longrightarrow \quad \Delta_A(y_1) < \Delta_A(y_2)$$

When $A = C$ is a closed convex cone, the following characterization holds (see e.g. [14])

$$\Delta_{-C}(y) = \max\{\langle \xi, y\rangle, \xi \in C' \cap S\},$$

where $C' = \{\xi \in Y : \langle \xi, c\rangle \geq 0,\ \forall c \in C\}$ denotes the positive polar of the cone of C and $S = \{\xi \in Y : \|\xi\| = 1\}$ is the unit sphere in Y. In the sequel we also denote by $B = \{y \in Y : \|y\| \leq 1\}$ the unit ball in Y.
Let us observe the following generalization of l^p norms. Let $y \in \mathbb{R}^n$ and C be a polyhedral cone generated by n linearly independent vectors. Hence, also C' is a polyhedral cone generated by n linearly independent vectors $\xi^1, \xi^2, \ldots, \xi^n$. In this case we can define on $\mathbb{R}^n$, the norms

$$\|y\| = \left(\sum_{i=i}^{n} |\langle \xi^i, y\rangle|^p\right)^{\frac{1}{p}}, \quad \text{for} \quad 1 \leq p < +\infty$$

and

$$\|y\| = \max\{|\langle \xi^i, y\rangle|, i = 1, \ldots, n\}, \quad \text{for} \quad p = +\infty.$$

We refer to these norms as l^p_C norms.

Recently, function Δ_{-C} has been used (see e.g. [1, 5, 15, 16, 20, 25]) to scalarize the vector optimization problem $VP(f, K)$. The scalar problem is

$$P(\varphi_{x^0}, K) \qquad\qquad \min\ \varphi_{x^0}(x),\ x \in K,$$

where $x^0 \in K$ and $\varphi_{x^0}(x) = \Delta_{-C}(f(x) - f(x^0))$. Relations among solutions of $P(\varphi_{x^0}, K)$ and those of problem $VP(f, K)$ are investigated in [15, 26]. For the reader's convenience, we quote here the characterization of the w-minimizers.

Theorem 3.1 ([15, 26])**.** *The point $x^0 \in K$ is a w-minimizer for problem $VP(f, K)$ if and only if it is a solution of problem $P(\varphi_{x^0}, K)$.*

Function Δ_{-C} has been used in [12, 13] to obtain first and second-order optimality conditions in Lagrangian form for a constrained vector optimization problem. In these papers the authors give necessary and sufficient optimality conditions by means of Dini-type derivatives, for functions of class $C^{0,1}$ (i.e. locally Lipschitz) and $C^{1,1}$ (i.e. with locally Lipschitz gradient). Further applications of function Δ_{-C} can be found in [20], where the well-posedness of problem $VP(f, K)$ is related with that of the scalar problem $P(\varphi_{x^0}, K)$.

4. Vector increasing-along-rays functions

From now on, if not otherwise specified, K denotes a subset of X, star-shaped at x^0. We recall that in [18], the following definition of cone monotonic function is given.

Definition 4.1 ([18])**.** Assume that the space X is partially ordered by a closed convex pointed cone D. A function $f : X \to Y$ is said to be increasing at $x^0 \in X$, when

$$X \cap (x^0 - D) \subseteq \{x \in X : f(x) \in f(x^0) - C\}$$

If we try to rephrase this definition in a radial context, we get that, if K is star-shaped at x^0, then f is increasing on K along the rays starting at x^0 when , $\forall x \in K$ and $\forall t_1, t_2$, with $t_2 \geq t_1 \geq 0$, it holds $f(x^0 + t_1(x - x^0)) \in f(x^0 + t_2(x - x^0)) - C$. Anyway, this definition reveals to be too strong for our purposes, since, in such a case it is easily seen that x^0 is an ideal minimizer for f over K.

This consideration leads us to introduce the following notion of vector increasing along rays (VIAR) function.

Definition 4.2. Let $K \subseteq X$ be star-shaped at x^0. A function $f : K \to Y$ is said to be increasing-along-rays starting at x^0 (for short $f \in VIAR(K, x^0)$), when function $\varphi_{x^0}(x) \in IAR(K, x^0)$.

The previous definition has a clear geometrical meaning and reduces to the notion of IAR function when $f : X \to \mathbb{R}$. The VIAR property is a monotonicity (along rays) property, defined through the oriented distance function and not through the order induced on Y by the cone C. The oriented distance function clearly depends on the norm considered on the space Y and hence one would expect that the VIAR property depends also on it. This is the case, as the following simple examples show.

Example 4.1.
- i) Consider the function $f : \mathbb{R} \to \mathbb{R}^2$, defined as $f(x) = (x, g(x))$, where $g(x) = 2x$ if $x \in [0, 1]$ and $g(x) = -\frac{1}{4}x + \frac{9}{4}$ if $x \in (1, +\infty)$ and let $C = \mathbb{R}^2_+$, $K = \mathbb{R}_+$ and $x_0 = 0$. Then it is easy to show that function $f \in VIAR(K, x_0)$ if $\mathbb{R}^2$ is endowed with the Euclidean norm l^2, but $f \notin VIAR(K, x_0)$ if $\mathbb{R}^2$ is endowed with the norm l^∞.
- ii) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^3$, defined as $f(x_1, x_2) = (x_1,\ g(x_1),\ 0)$, where $g(x)$ is defined in the previous point i). Let $C \subset \mathbb{R}^3$ be the polyhedral cone generated by the linearly independent vectors $\xi^1 = (-1/30,\ 1,\ 0)$, $\xi^2 = (1,\ 1/30,\ 0)$ and $\xi^3 = (0,\ 0,\ 1)$ (hence it is easily seen that $C' = C$). Let $K = \mathbb{R}^2_+$ and $x^0 = 0$. Then $f \in VIAR\left(K, x^0\right)$ if $\mathbb{R}^3$ is endowed with the Euclidean norm l^2, while $f \notin VIAR\left(K, x^0\right)$ if $\mathbb{R}^3$ is endowed by the l^∞_C norm.

Investigating the VIAR property, we are also interested in conditions on f, which, like in the scalar case, make this property independent on the (equivalent) norms that can be introduced on Y. For C-convex functions such independence is shown in Theorem 5.1 below. The next propositions give some basic properties of VIAR functions which should be compared with those in Section 2.

Proposition 4.1. *Let the subset $K \subseteq X$ be star-shaped at x^0. If $f \in VIAR(K, x^0)$, then*

$$f(x^0 + t_2(x - x^0)) - f(x^0 + t_1(x - x^0)) \notin -\operatorname{int} C, \tag{1}$$

$\forall t_2 \geq t_1 > 0$ *and* $\forall x \in K$.

Proof. Omitted as immediate. □

The reversal of Proposition 4.1 does not hold, as the next example shows.

Example 4.2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ endowed with the l^∞ norm, $K = \mathbb{R}_+$, $C = \mathbb{R}^2_+$ and $x_0 = 0$. Consider the function $f : K \to \mathbb{R}^2$, defined as $f(x) = (2x - x^2, x^2 - 2x)$. Then f fulfills condition (1), but $f \notin VIAR(\mathbb{R}_+, x_0)$.

Proposition 4.2. *Let the subset $K \subseteq X$ be star-shaped at x^0. Then $f \in VIAR(K, x^0)$ if and only if for every $x \in K$ and $\varepsilon > 0$ such that $f(x) \in f(x^0) - C + \varepsilon B$, it holds $f(x^0 + t(x - x^0)) \in f(x^0) - C + \varepsilon B$, for every $t \in [0, 1]$.*

Proof. The proof is immediate, observing that

$$\{x \in K : f(x) \in f(x^0) - C + \varepsilon B\} = \{x \in K : \varphi_{x^0}(x) \leq \varepsilon\} = \operatorname{lev}_{\leq \varepsilon} \varphi_{x^0}$$

and recalling Proposition 2.2. □

Proposition 4.3. *Let the subset $K \subseteq X$ be star-shaped at x^0. Then:*

i) *x^0 is a w-minimizer of f over K.*
ii) *The set $f^{-1}(f(x^0))$ is star-shaped at x^0.*

Proof.
i) Since $\varphi_{x^0} \in IAR(K, x^0)$, then x^0 is a minimizer of φ_{x^0} over K and hence a w-minimizer of f.
ii) The set $f^{-1}(f(x^0))$ is the set of global minimizers of φ_{x^0} and hence the result follows from Proposition 1 in [6]. □

Remark 4.1. We wish to observe, similarly to the scalar case, that if $f \in VIAR(K, x^0)$, then f is C-quasiconnected [2, 20]. It follows that, similarly to the scalar case, $VIAR$ functions enjoy some well-posedness properties. For more details on this topic we refer to [20].

5. Classes of VIAR functions

In the scalar case, it is known that quasiconvex functions are in the class $IAR(K, x^0)$, where x^0 is a minimizer for f over the convex set K. We are going to see that in the vector case some differences arise. Namely, let x^0 be w-minimizer for a function $f : X \to Y$; we will see that if f is C-convex, then $f \in VIAR(K, x^0)$, whatever the norm we choose on the space Y. On the contrary, if f is C-quasiconvex, then it possesses the $VIAR$ property only for suitable choices of the norm in Y.

Definition 5.1 ([18]). Let K be a convex subset of X.

i) The function $f : X \to Y$ is C-convex on K if for every $x^1, x^2 \in K$ and for every $t \in [0, 1]$ it holds

$$f((1 - t)x^1 + tx^2) - (1 - t)f(x^1) - tf(x^2) \in -C$$

ii) The function $f : X \to Y$ is C-quasiconvex on K if for every $y \in Y$, the (level) set
$$\{x \in K : f(x) \in y - C\}$$
is convex.

Remark 5.1. We wish to recall that a function f is C-convex if and only if the scalar function $\langle \xi, f \rangle$ is convex for every $\xi \in C'$. The same result does not hold for C-quasiconvex functions [18]. Anyway, we remind that when $Y = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$, then f is C-quasiconvex if and only if every component of f is quasiconvex. The case when $Y = \mathbb{R}^n$ and C is a polyhedral cone generated by n linearly independent vectors has been treated in [4]. Since also C' is generated by n linearly independent vectors $\xi^1, \dots, \xi^n$, it is known that f is C-quasiconvex if and only if $\langle \xi^i, f \rangle$ is quasiconvex for every $i = 1, \dots, n$. Finally in [3] the case of general (i.e. non polyhedral) cones is also developed. For a complete and updated reference on the topic we refer to [19].

Theorem 5.1. *Let K be a convex subset of X and let f be C-convex on K. If x^0 is a w-minimizer of f over K, then $f \in VIAR(K, x^0)$, whatever the norm chosen in Y.*

Proof. f is C-convex if and only if the scalar function $\langle \xi, f(x) \rangle$, is convex for every $\xi \in C'$. Hence, whatever the norm chosen in Y, function $\varphi_{x^0}(x)$ is the maximum of convex functions, hence is convex. □

Now we turn our attention to C-quasiconvex functions. We claim that if f is a C-quasiconvex function and x^0 is w-minimizer of f over K, then we can always choose a norm on Y, such that $f \in VIAR(K, x^0)$. To prove it we need some preliminary concepts and results. We recall that a convex set $A \subseteq C$ is a base for the cone C when $0 \notin A$ and for every $k \in C$, $k \neq 0$, there are unique elements $a \in A$ and $t > 0$, such that $k = ta$.

Lemma 5.1 ([18]). *Let $k \in \operatorname{int} C$, $\alpha > 0$ and consider the hyperplane $H_\alpha = \{y \in Y : \langle k, y \rangle = \alpha\}$. Then the set $G_\alpha = H_\alpha \cap C'$ is a compact base for C'.*

Given the set G_1, let $\tilde{B}_1 = \operatorname{conv}\{G_1 \cup (-G_1)\}$ (here $\operatorname{conv} A$ denotes the convex hull of the set A). Since $\tilde{B}_1$ is a balanced, convex, absorbing and bounded set, with $0 \in \operatorname{int} \tilde{B}_1$, the Minkowsky functional $\gamma_{\tilde{B}_1}(y) := \inf \left\{\lambda, \lambda > 0,\ y \in \lambda \tilde{B}_1\right\}$ is a norm on Y (see e.g. [23]).

We denote the norm defined by function $\gamma_{\tilde{B}_1}$ as $\|\cdot\|_{C,k}$, to stress the dependence on both the ordering cone and the given $k \in \operatorname{int} C$.

Theorem 5.2. *Let K be a convex set, let f be C-quasiconvex on K and let x^0 be w-minimizer for f over K. Then, whatever $k \in \operatorname{int} C$, if Y is endowed with the norm $\|\cdot\|_{C,k}$, then $f \in VIAR(K, x^0)$.*

Proof. Recall that, since Y is endowed with the norm $\|\cdot\|_{C,k}$, we have $C' \cap S = \{\xi \in C' : \langle \xi, k \rangle = 1\}$. For $\varepsilon > 0$, we have
$$\{x \in K : \varphi_{x^0}(x) \leq \varepsilon\} = \{x \in K : \max_{\xi \in C' \cap S} \langle \xi, f(x) - f(x^0) \rangle \leq \varepsilon\}$$

$$= \{x \in K : \max_{\xi \in C' \cap S} \langle \xi, f(x) - f(x^0) \rangle \leq \varepsilon \max_{\xi \in C' \cap S} \langle \xi, k \rangle \}$$

$$= \{x \in K : \max_{\xi \in C' \cap S} \langle \xi, f(x) - f(x^0) - \varepsilon k \rangle \leq 0\} = \{x \in K : f(x) \in f(x^0) + \varepsilon k - C\}.$$

Since f is C-quasiconvex on K, this last set is convex for every $\varepsilon > 0$ and so the level set of φ_{x^0}, $\{x \in K : \varphi_{x^0}(x) \leq \varepsilon\}$ is convex too. It follows that φ_{x^0} is quasiconvex with x^0 as minimizer over K and hence is in the class $IAR(K, x^0)$. □

The next example shows that also for C-quasiconvex functions, the $VIAR$ property depends on the norm chosen on the space Y.

Example 5.1. Let $K \subset \mathbb{R} = [0, +\infty)$ and consider the function $f : K \to \mathbb{R}^2$, $f = (f_1, f_2)$, defined as $f_1(x) = -\frac{1}{2}x$, if $x \in [0,1]$, $f_1(x) = -\frac{1}{2}x^3$, if $x \in (1, +\infty)$ and $f_2(x) = x$. Further, let $C = \mathbb{R}^2_+$, $x_0 = 0$. If we fix $k = (1,1) \in \operatorname{int} C$, we obtain that the norm $\| \cdot \|_{C,k}$ coincides with the l^1 norm on $\mathbb{R}^2$, so that $\varphi_{x_0}(x) = \max\{f_1(x), f_2(x)\}$ (see e.g. [14]). Clearly, x_0 is a w-minimizer and by Theorem 5.2, if $\mathbb{R}^2$ is endowed with the l^1 norm, $f \in VIAR(K, x^0)$, which it is also easily seen directly.
Assume now that $\mathbb{R}^2$ is endowed with a different norm, constructed as follows. Consider the set $A = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 = -x_1 + 3, x_1 \in [1,2]\}$ and let $\tilde{A} = \operatorname{conv}(A \cup (-A))$. The Minkowsky functional of the set $\tilde{A}$ defines a norm on $\mathbb{R}^2$ and direct calculations show that when this norm is considered, $\varphi_{x_0} \notin IAR(K, x^0)$ and hence $f \notin VIAR(K, x^0)$.

When $Y = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$ Theorem 5.2 holds for the most common l^p norms.

Proposition 5.1. *Let $Y = \mathbb{R}^n$, $C = \mathbb{R}^n_+$ and let f be a C-quasiconvex function. If x^0 is w-minimizer for f over K (K convex) and Y is endowed with a l^p norm ($1 \leq p \leq +\infty$), then $f \in VIAR(K, x^0)$.*

Proof. From the representation $\Delta_{-C}(y) = \max\{\langle \xi, y \rangle, \xi \in C' \cap S\}$, with easy calculations we obtain thet if $\mathbb{R}^n$ is endowed with a l^p norm ($1 \leq p \leq +\infty$), then $\Delta_{-\mathbb{R}^n_+}(y) = \max\{\sum_{i \in I_+(y)} \xi_i y_i, \xi \in \mathbb{R}^n_+ \cap S\}$, where $I_+(y) = \{i = 1, \ldots, n : y_i > 0\}$. Now, without loss of generality assume that $f(x^0) = 0$, take any $x \in K$, consider the ray $R_{x^0,x}$ and observe that the restriction of every function f_i (a component of f) on this ray is quasiconvex. To show that $f \in VIAR(K, x^0)$, consider any two numbers $t_1, t_2 > 0$, with $t_2 > t_1$. If $f_i(x + t_1(x - x^0)) > 0$, then it cannot be $f_i(x^0 + t_2(x - x^0)) < f_i(x + t_1(x - x^0))$. In fact, if this happens, the level set, $\{t : f_i(x^0 + t(x - x^0)) \leq \max\{0, f_i(x^0 + t_2(x - x^0))\}\}$ is not convex, since t_1 does not belong to this set. From these considerations, it follows readily that the function $\varphi_{x^0}(x) \in IAR(K, x^0)$, which completes the proof. □

The proof of the next result follows along the lines of Proposition 5.1 and we omit it.

Proposition 5.2. *Let $Y = \mathbb{R}^n$ and let C' be a polyhedral cone generated by n linearly independent vectors $\xi^1, \ldots, \xi^n$. If f is C-quasiconvex on the convex set $K \subseteq X$, x^0 is a w-minimizer for f over K and Y is endowed with a l^p_C norm, then $f \in VIAR(K, x^0)$.*

6. VIAR functions and variational inequalities

In the scalar case, a function $f \in IAR(K, x^0)$ is characterized by Theorem 2.1 by means of a Minty variational inequality problem. The study of vector optimization problems by means of Minty-type variational inequalities has been first presented in [11] and has been deepened in [24]. This approach is based on a vector-valued variational inequality. Let $f : K \subseteq X \to Y$ be a function of class C^1 on an open set containing the convex set K. The vector variational inequality of Minty type is defined as the problem of finding a point $x^0 \in K$ such that

$$MVVI(f', K) \qquad f'(y)(x^0 - y) \cap C = \{0\},\ \forall y \in K.$$

If it holds

$$MVVI^w(f', K) \qquad f'(y)(x^0 - y) \cap \operatorname{int} C = \emptyset,\ \forall y \in K$$

then $x^0 \in K$ is a weak solution of $MVVI(f', K)$. However, it can be shown that if x^0 is a solution (or a weak solution) of $MVVI(f', K)$, then differently from the scalar case, f does not necessarily belong to the class $VIAR(K, x^0)$ (see e.g. Example 1 in [8]). The former gap can be filled in by relating to $VIAR$ functions suitable scalar variational inequality problems.

Definition 6.1 ([18])**.** Let f be a function defined on a set $K \subseteq X$. We say that f is C-continuous at $\bar{x}$ when for every neighborhood U of $\bar{x} \in X$, there exists a neighborhood V of $f(\bar{x}) \in Y$, such that

$$f(x) \in V + C, \quad \forall x \in U \cap K.$$

We say that f is C-continuous on K, when f is C-continuous at any point of K.

Definition 6.2 ([18])**.** Let $\{h(x,t) : t \in T\}$ be a family of scalar-valued functions on K, where T is a nonempty parameter set. We say that this family is lower equi-semicontinuous at $\bar{x} \in K$ when for every $\varepsilon > 0$, there exists a neighborhood U of $\bar{x}$, such that

$$h(x,t) \geq h(\bar{x},t) - \varepsilon, \quad \forall x \in U \cap K \text{ and } t \in T.$$

We recall the following result.

Proposition 6.1 ([18])**.** *f is C-continuous at a point $\bar{x} \in K$ if and only if the family $G = \{\langle \xi, f \rangle : \xi \in C' \cap S\}$ is lower equi-semicontinuous at that point.*

The proof of the next proposition comes immediately from the previous result.

Proposition 6.2. *Let $f : X \to Y$ be C-continuous on K and let $x^0 \in K$. Then function $\varphi_{x^0}(x)$ is lower semicontinuous on K.*

The previous definitions and results can be rephrased in a radial sense.

Definition 6.3. Let $K \subseteq X$ be star-shaped at x^0 and let f be a function defined on an open set containing K. The function f is said to be C-radially continuous in K along the rays starting at x^0 (for short, $f \in C\text{-}RC(K, x^0)$), if for every $x \in K$, the restriction of f on the interval $R_{x^0,x} \cap K$ is C-continuous.

Proposition 6.3. *Let $f \in C\text{-}RC(K, x^0)$. Then $\varphi_{x^0}(x)$ is radially lower semicontinuous in K along the rays starting at x^0 ($\varphi_{x^0} \in RLSC(K, x^0)$).*

So, we can give the following result, which characterizes $VIAR$ functions in terms of a suitable variational inequality.

Proposition 6.4. *Let $K \subseteq X$ be star-shaped at x^0. Assume that f is a function defined on an open set containing K.*

i) *Let $f \in C\text{-}RC(K, x^0)$. If x^0 solves $MVI((\varphi_{x^0})'_-, K)$, then $f \in VIAR(K, x^0)$.*
ii) *conversely, if $f \in VIAR(K, x^0)$, then x^0 solves $MVI((\varphi_{x^0})'_-, K)$.*

Proof. The proof follows recalling Proposition 6.3 and Theorem 2.1. □

Similarly to the scalar case, the assumption $f \in C\text{-}RC(K, x^0)$ appears in only one of the two opposite implications. The next example shows that this assumption cannot be dropped at all.

Example 6.1. Let $K = \mathbb{R}$, $x_0 = 0$, $C = \mathbb{R}^2_+$ and consider the function $f : \mathbb{R} \to \mathbb{R}^2$ defined as $f(x) = (g(x), g(x))$, with

$$g(x) = \begin{cases} 1, & \text{if } x \neq 2 \\ 3, & \text{if } x = 2 \end{cases}$$

and assume that $\mathbb{R}^2$ is endowed with the norm l^∞. Then $f \notin C\text{-}RC(K, x_0)$ and it holds $(\varphi_{x_0})'_-(y, x_0 - y) \leq 0$, $\forall y \in \mathbb{R}$, but $f \notin VIAR(K, x_0)$.

Corollary 6.1. *Let $x^0 \in \ker K$ and let $f \in C - RC(K, x^0)$. If x^0 solves $MVI((\varphi_{x^0})'_-, K)$, then x^0 is w-minimizer for f over K.*

We close this section with some comparisons between problem $MVI((\varphi_{x^0})'_-, K)$ and the vector variational inequality problem $MVVI(f', K)$, assuming that f is a function of class C^1 on an open set containing K. In [9] it has been observed that every solution of $MVI((\varphi_{x^0})'_-, K)$ is also a weak solution of $MVVI(f', K)$, regardless of the norm introduced in the space Y. The converse does not necessarily hold as shown by Example 2 in [9]. Anyway, Theorem 9 in [9] ensures that if f is a C-convex function on the convex set K, then any $x^0 \in K$ which is a weak solution of $MVVI(f', K)$ solves also $MVI((\varphi_{x^0})'_-, K)$ (regardless of the norm in Y). The next example shows that the coincidence of the two concepts of solution cannot be extended to C-quasiconvex functions.

Example 6.2. Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ $X = \mathbb{R}$, $K = [0, \pi + \sqrt[3]{2}]$ and consider the function $f : \mathbb{R}_+ \to \mathbb{R}^2$ defined as $f(x) = (f_1(x), f_2(x))$, where

$$f_1(x) = \begin{cases} 0, & 0 \leq x \leq \pi \\ (\pi - x)^3, & x > \pi \end{cases}$$

and

$$f_2(x) = \begin{cases} \cos x - 1, & 0 \leq x \leq \pi \\ -2, & x > \pi \end{cases}$$

Then f is of class C^1 and C-quasiconvex on K. It is easily seen that $x_0 = 0$ is a weak solution of $MVVI(f', K)$, but x_0 is not a solution of $MVI((\varphi_{x_0})'_-, K)$, whatever the norm in Y, since x_0 is not w-minimizer of f over K (recall Corollary 6.1).

It is worth mentioning also that when f is of class C^1, then φ_{x^0} is subdifferentiable (see e.g. [10]) and it can be proved that problem $MVI((\varphi_{x^0})'_-, K)$ is equivalent to the more popular Minty generalized variational inequality for the set-valued map $\partial\varphi_{x^0}(\cdot)$ (the subdifferential), which requires to find a point $x^0 \in K$ such that the inequality

$$\langle v, x^0 - y\rangle \leq 0$$

is satisfied for every $v \in \partial\varphi_{x^0}(y)$ and $y \in K$.

7. The case f is C-quasiconvex

The main reason to introduce a variational inequality is to define an alternative approach to the underlying optimization problem. The best opportunity is to have coincidence of solutions sets. In our case, if $f \in C\text{-}RC(K, x^0)$, any solution of $MVI((\varphi_{x^0})', K)$ is a w-minimizer for $VP(f, K)$. The following example shows that the converse is not true in general.

Example 7.1. Let $X = K = Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $k = (1, 1)$ and assume that Y is endowed with the norm $\|\cdot\|_{C,k}$ (which coincides with the l^1 norm). Let $f(x_1, x_2) = (x_1^2x_2^2, -x_1^2x_2^2)$. Then the set of the w-minimizers is $\mathbb{R}^2$, but any point $x^0 = (0, x_2)$ is not a solution of $MVI((\varphi_{x^0})'_-, K)$. This can be seen since $\varphi_{x^0}(x_1, x_2) = x_1^2x_2^2$.

The next theorem proves that when f is assumed to be C-quasiconvex, the coincidence of the solution sets of $MVI((\varphi_{x^0})'_-, K)$ and $VP(f, K)$ is guaranteed.

Theorem 7.1. *Assume that Y is endowed with the norm $\|\cdot\|_{C,k}$. Let $K \subseteq X$ be a convex set and $f : K \to Y$ be C-quasiconvex. Then x^0 is a w-minimizer of f over K if and only if is a solution of $MVI((\varphi_{x^0})'_-, K)$.*

Proof. Let $x^0 \in K$ be a w-minimizer. Then, Theorem 5.2 applies and hence $f \in VIAR(K, x^0)$. Proposition 6.4 completes the proof. □

Remark 7.1. If $Y = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$ (or C is a polyhedral cone generated by n linearly independent vectors), then in the previous result, the norm $\|\cdot\|_{C,k}$ can be replaced by the l^p norm (the l^p_C norm).

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Manuscript received September 18, 2004
revised May 3, 2005

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