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MINIMAL ELEMENT THEOREMS AND EKELAND'S PRINCIPLE WITH SET RELATIONS

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Abstract. We present two existence principles for minimal points of subsets of the product space $X \times 2^Y$, where X stands for a separated uniform space and Y a topological vector space. The two principles are distinct with respect to the involved ordering structure in 2^Y.

We derive from them new variants of Ekeland's principle for set-valued maps as well as a minimal point theorem in $X \times Y$ and Ekeland's principle for vector-valued functions.

1. Introduction

Ekeland's variational principle and its equivalent formulations belong to the cornerstones of Nonlinear Functional Analysis with applications in many fields of analysis, optimization and operations research. During the last years, an increasing interest could be observed for versions involving a set-valued function, compare e.g. [3], [14], [15], [28].

A set-valued mapping from a set X into a set Y is usually understood to be a relation $F \subset X \times Y$ *not* necessarily satisfying the uniqueness property, i.e., we have not that $(x, y_1), (x, y_2) \in F$ implies $y_1 = y_2$.

In contrast to this, we understand a set-valued mapping to be a function from X to 2^Y, i.e., a relation F on $X \times 2^Y$ satisfying the uniqueness property (i.e., $(x, V_1), (x, V_2) \in F$ implies $V_1 = V_2$). This leads in a natural way to new definitions of concepts like graph, domain and minimal points of set-valued maps as well as new results in set-valued optimization theory.

Investigating an optimization problem with a set-valued objective function we need to compare its values. In this paper, we use reflexive and transitive relations on 2^Y to compare two values $F(x_1)$ and $F(x_2)$ of a map $F : X \to 2^Y$. This is not the common approach to set-valued optimization up to now. Usually, a set-valued optimization problem is reduced to a vector-valued problem by looking for minimal (efficient) points of the set $\bigcup_{x \in X} F(x)$.

We start considering subsets of $X \times 2^Y$ and looking for minimal elements of them with respect to appropriate ordering relations on 2^Y. We present existence results for such elements called *Minimal Element Theorems with Set Relations*.

We shall draw several conclusions of the minimal element theorems.

First, we derive new variants of Ekeland's variational principle for set-valued maps. Our Ekeland-type theorems are much more general than and cover most of the known results of the field as special cases.

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Secondly, we conclude a minimal point theorem in $X \times Y$. Such theorems are well-established and useful tools in vector optimization and related fields, cf. [25], [11], [10], [14] for example.

Finally, the well-known Ekeland-type principle for vector-valued functions $f : X \to Y$ (e.g. [26], [16], [11]) turns out to be a consequence of our new Ekeland-type principles for set-valued maps as well as of the minimal point theorem.

The paper is organized as follows. In the next section we introduce two ordering relations for elements of 2^Y, where Y is a linear topological space, as well as related boundedness concepts in the space 2^Y. In Section 3 we present scalarization methods for subsets of 2^Y. These methods are essentially used for the proofs of the minimal element theorems and may be of independent interest. In Section 5 our main results, two minimal element theorems with set relations, are presented. Section 6 contains the new set-valued variational principles and several conclusions.

2. Ordering Relations and Boundedness in 2^Y

Let Y be a topological vector space. We denote by 2^Y the set of all subsets of Y including the empty set $\emptyset$. As usual, the sum of two sets $V_1, V_2 \in 2^Y$ is defined by $V_1 + V_2 := \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\}$. We use the convention $\emptyset + V_1 = \emptyset$. The product of $\alpha \in \mathbb{R}$ and $V \in 2^Y \backslash \{\emptyset\}$ is defined by $\alpha V := \{\alpha v : v \in V\}$. Moreover, we define $\alpha \cdot \emptyset = \emptyset$ for $\alpha \neq 0$ and $0 \cdot \emptyset = \{0\}$.

This section is concerned with ordering relations for sets as well as with order boundedness concepts in 2^Y. Such relations have been introduced for the case $Y = \mathbb{R}$ in a paper [30] by Young from 1931. A comprehensive survey on these relations and related power structures is [2]. Kuroiwa, Tananka and Truong [20], [18], [19] started developing a new approach to set-valued optimization using the same relations. A detailed approach to ordering relations on power sets as well as links to several algebraic concepts can be found in [12].

We define two relations $\preccurlyeq$, $\curlyeqprec$ on 2^Y as follows. These relations are two out of six being natural generalizations of partial orderings on a linear space Y to relations on 2^Y, compare [20] for details.

Definition 2.1. Let $K \subseteq Y$ be a convex cone containing $0 \in Y$ and $V_1, V_2 \in 2^Y$. We define:

$$V_1 \preccurlyeq_K V_2 \Longleftrightarrow V_2 \subseteq V_1 + K;$$
$$V_1 \curlyeqprec_K V_2 \Longleftrightarrow V_1 \subseteq V_2 - K.$$

If there is no risk of confusion the relations are simply denoted by $\preccurlyeq$ and $\curlyeqprec$.

Note that Luc [23], Chapters 2.5 and 5.1 implicitly used these relations describing the constraints for a set-valued optimization problem. This can be seen observing that $V \preccurlyeq_K \{0\}$ iff $V \cap (-K) \neq \emptyset$ and $V \curlyeqprec_K \{0\}$ iff $V \subseteq -K$.

Both relations can also be expressed by the ordering $\leq_K$ in Y, which is defined by $y_1 \leq_K y_2$ iff $y_2 - y_1 \in K$:

$$V_1 \preccurlyeq V_2 \Longleftrightarrow \forall v_2 \in V_2 \; \exists v_1 \in V_1 : \; v_1 \leq_K v_2; \tag{1}$$
$$V_1 \curlyeqprec V_2 \Longleftrightarrow \forall v_1 \in V_1 \; \exists v_2 \in V_2 : \; v_1 \leq_K v_2. \tag{2}$$

Furthermore, the following relationships are easy to show:

$$V_1 \curlyeqprec V_2 \quad \Longleftrightarrow \quad -V_2 \preccurlyeq -V_1 \quad \Longleftrightarrow \quad V_2 \preccurlyeq_{-K} V_1. \tag{3}$$

In the following, we study the properties of $\preccurlyeq$ having in mind that by (3) we are able to obtain the same properties for $\curlyeqprec$ with K replaced by $-K$ if necessary.

The relations $\preccurlyeq$ and $\curlyeqprec$ are reflexive and transitive. We have no antisymmetry but

$$(V_1 \preccurlyeq V_2,\ V_2 \preccurlyeq V_1) \quad \Longleftrightarrow \quad V_1 + K = V_2 + K. \tag{4}$$

Introducing the equivalence relation $V_1 \sim V_2$ iff $V_1 \preccurlyeq V_2$ and $V_2 \preccurlyeq V_1$ we may generate a partial ordering on the set of equivalence classes. Furthermore, it can be shown that for $\alpha_1, \alpha_2 \geq 0$ we have

$$(V_1 \preccurlyeq V_2, V_3 \preccurlyeq V_4) \quad \Longrightarrow \quad \alpha_1 V_1 + \alpha_2 V_3 \preccurlyeq \alpha_1 V_2 + \alpha_2 V_4. \tag{5}$$

Note that K has not to be pointed (K is pointed iff $K \cap -K = \{0\}$) for proving (1)-(5). Let $V \in 2^Y$ be a subset of Y. We say that $\bar{v} \in V$ is a $\leq_K$–minimal element of V if $v \in V$, $v \leq_K \bar{v}$ implies $\bar{v} \leq_K v$. The set of all $\leq_K$–minimal elements of V is denoted by $\operatorname{Min} V$. If K is a convex pointed cone then we have $\bar{v} \in \operatorname{Min} V$ iff $v \in V$, $v \leq_K \bar{v}$ implies $\bar{v} = v$. In this case,

$$(V_1 \preccurlyeq V_2,\ V_2 \preccurlyeq V_1) \quad \Longrightarrow \quad \operatorname{Min}(V_1) = \operatorname{Min}(V_2). \tag{6}$$

A subset $V \subset Y$ is said to be *lower externally stable* iff $V \subseteq \operatorname{Min} V + K$. This property is called *domination property* by several authors. Compare Luc [23] and the references therein. By direct calculation, one may find for $V_1, V_2 \in 2^Y$ being lower externally stable sets that

$$V_1 \preccurlyeq V_2 \quad \Longleftrightarrow \quad \operatorname{Min} V_1 \preccurlyeq \operatorname{Min} V_2. \tag{7}$$

A similar assertion follows for $\curlyeqprec$ replacing $\leq_K$–minimal by $\leq_K$–maximal elements and lower by upper external stability.

The following relationships can easily be verified:

$$\forall V \in 2^Y : \ \ V \preccurlyeq \emptyset,\ Y \preccurlyeq V,\ \emptyset \curlyeqprec V,\ V \curlyeqprec Y;$$

$$\begin{aligned} \emptyset \preccurlyeq V \Rightarrow V = \emptyset; \quad V \preccurlyeq Y \Rightarrow Y = V + K; \\ V \curlyeqprec \emptyset \Rightarrow V = \emptyset; \quad Y \curlyeqprec V \Rightarrow Y = V - K. \end{aligned}$$

These relationships motivate the following boundedness concepts for subsets of 2^Y.

Definition 2.2. A subset $\mathcal{V} \subseteq 2^Y$ is said to be $\preccurlyeq$*–bounded below* if there exists some topologically bounded subset $\tilde{V} \subseteq Y$ such that $\tilde{V} \preccurlyeq V$ holds for all $V \in \mathcal{V}$. The set $\tilde{V}$ is called a *lower* $\preccurlyeq$*–bound* of $\mathcal{V}$. A subset $\mathcal{V} \subseteq 2^Y$ is said to be $\curlyeqprec$*–bounded above* and $\tilde{V}$ is called an *upper* $\curlyeqprec$*–bound* of $\mathcal{V}$ if $-\mathcal{V} := \{-V :\ V \in \mathcal{V}\}$ is $\preccurlyeq$–bounded below with the lower $\preccurlyeq$–bound $-\tilde{V}$.

Definition 2.3. A subset $\mathcal{V} \subseteq 2^Y$ is said to be $\curlyeqprec$*–bounded below* if there exists some nonempty subset $\tilde{V} \subseteq Y$ such that $\tilde{V} \curlyeqprec V$ holds for all $V \in \mathcal{V}$. The set $\tilde{V}$ is called a *lower* $\curlyeqprec$*–bound* of $\mathcal{V}$. A subset $\mathcal{V} \subseteq 2^Y$ is said to be $\preccurlyeq$*–bounded above* and $\tilde{V}$ is called an *upper* $\preccurlyeq$*–bound* of $\mathcal{V}$ if $-\mathcal{V}$ is $\curlyeqprec$–bounded below with the lower $\curlyeqprec$–bound $-\tilde{V}$.

Using the complementary relations of $\preccurlyeq$ and $\curlyeqprec$, denoted by $\not\preccurlyeq$ and $\not\curlyeqprec$ we introduce further boundedness concepts.

Definition 2.4. A subset $\mathcal{V} \subseteq 2^Y$ is said to be *weakly $\preccurlyeq$–bounded below* if there exists some topologically bounded subset $\tilde{V} \subseteq Y$ such that $V \not\preccurlyeq \tilde{V}$ holds for all $V \in \mathcal{V}$. The set $\tilde{V}$ is called a *weak lower $\preccurlyeq$–bound* of $\mathcal{V}$. A subset $\mathcal{V} \subseteq 2^Y$ is said to be *weakly $\curlyeqprec$–bounded above* and $\tilde{V}$ is called a *weak upper $\curlyeqprec$–bound* of $\mathcal{V}$ if $-\mathcal{V}$ is weakly $\preccurlyeq$–bounded below with the weak lower $\preccurlyeq$–bound $-\tilde{V}$.

Definition 2.5. A subset $\mathcal{V} \subseteq 2^Y$ is said to be *weakly $\curlyeqprec$–bounded below* if there exists some nonempty subset $\tilde{V} \subseteq Y$ such that $V \not\curlyeqprec \tilde{V}$ holds for all $V \in \mathcal{V}$. The set $\tilde{V}$ is called a *weak lower $\curlyeqprec$–bound* of $\mathcal{V}$. A subset $\mathcal{V} \subseteq 2^Y$ is said to be *weakly $\preccurlyeq$–bounded above* and $\tilde{V}$ is called a *weak upper $\preccurlyeq$–bound* of $\mathcal{V}$ if $-\mathcal{V}$ is weakly $\curlyeqprec$–bounded below with the weak lower $\curlyeqprec$–bound $-\tilde{V}$.

Note that we have a kind of duality between nonempty sets and topologically bounded sets in the definition of the above boundedness concepts. This duality can be observed throughout the paper. Further, we have to take care using the symbol $\emptyset$: It denotes an element of 2^Y but of course a subset $\mathcal{V} \subseteq 2^Y$ can also be the empty set.

Remark 2.6. Let $\operatorname{cl} K \neq Y$. If $\mathcal{V} \subseteq 2^Y$ is $\preccurlyeq$–bounded below then $\mathcal{V}$ is weakly $\preccurlyeq$–bounded below. Indeed, let $\mathcal{V}$ be $\preccurlyeq$–bounded below, i.e., there exists some topologically bounded set $\tilde{V} \subseteq Y$ such that $\tilde{V} \preccurlyeq V$ for all $V \in \mathcal{V}$. We show that $\tilde{V} + y$ is a weak lower $\preccurlyeq$–bound of $\mathcal{V}$, where $y \notin \operatorname{cl} K$. Assuming the contrary, i.e., $V \preccurlyeq \tilde{V} + y$ for some $V \in \mathcal{V}$ it follows $\tilde{V} + y \subseteq V + K \subseteq \tilde{V} + K$. An induction argument yields $\tilde{V} + ny \subseteq \tilde{V} + K$ for all $n \in \mathbb{N}$. Dividing by n and letting $n \to \infty$ implies $y \in \operatorname{cl} K$, a contradiction.

Example 2.7. Let $K = \mathbb{R}^2_+$ and $\mathcal{V} = \{Y \setminus -K\}$. Then $\mathcal{V}$ is weakly $\preccurlyeq$–bounded below but $\mathcal{V}$ is not $\preccurlyeq$–bounded below.

Remark 2.8. Let $\operatorname{cl} K \neq Y$. If $\mathcal{V} \subseteq 2^Y$ is $\curlyeqprec$–bounded below then $\mathcal{V}$ is weakly $\curlyeqprec$–bounded below. Indeed, let $\mathcal{V}$ be $\curlyeqprec$–bounded below, i.e., there exists some nonempty set $\tilde{V} \subseteq Y$ such that $\tilde{V} \curlyeqprec V$ for all $V \in \mathcal{V}$. We show that $\tilde{v} - y$ is a weak lower $\curlyeqprec$–bound of $\mathcal{V}$, where $\tilde{v} \in \tilde{V}$ and $y \notin \operatorname{cl} K$. Assuming the contrary, i.e., $V \curlyeqprec \tilde{v} - y$ for some $V \in \mathcal{V}$ it follows $\tilde{v} \in \tilde{V} \subseteq V + K \subseteq \tilde{v} - y + K$. An induction argument yields $\tilde{v} \in \tilde{v} - ny + K$ for all $n \in \mathbb{N}$. Dividing by n and letting $n \to \infty$ implies $y \in \operatorname{cl} K$, a contradiction.

Example 2.9. Let $K = \mathbb{R}^2_+$ and $\mathcal{V} = \{\{y\} : y \in Y \setminus -K\}$. Then $\mathcal{V}$ is weakly $\curlyeqprec$–bounded below but $\mathcal{V}$ is not $\curlyeqprec$–bounded below.

3. Scalarization Methods on 2^Y

We present several nonlinear scalarization functionals defined on 2^Y. They are generalizations of the functionals introduced in [7] and extensively studied in [29], [8], [11] and [10]. The most important property of our functionals turns out to be the monotonicity with respect to the relations $\preccurlyeq$, $\curlyeqprec$. Let $V_1, V_2 \in 2^Y$. We call a functional $z : 2^Y \to \mathbb{R} \cup \{\pm\infty\}$ *$\preccurlyeq$–monotone* iff $V_1 \preccurlyeq V_2$ implies $z(V_1) \leq z(V_2)$. It is called *$\curlyeqprec$–monotone* iff $V_1 \curlyeqprec V_2$ implies $z(V_1) \leq z(V_2)$.

Theorem 3.1. *Let Y be a topological vector space, $K \subseteq Y$ a convex cone and $k^0 \in K \setminus -\operatorname{cl} K$. Let $\mathcal{V} \subseteq 2^Y$ be nonempty and $\preccurlyeq$–bounded, i.e., there is a topological bounded set $V' \subseteq Y$ and a nonempty set $V'' \subseteq Y$ such that*

$$\forall V \in \mathcal{V} : \; V' \preccurlyeq V \preccurlyeq V''.$$

Then, the functional $z^l : 2^Y \to \mathbb{R} \cup \{\pm\infty\}$, defined by

$$z^l(V) := \inf\{t \in \mathbb{R} : \, tk^0 + V'' \subseteq V + \operatorname{cl} K\},$$

has the following properties:

(i) *z^l is bounded on $\mathcal{V}$;*
(ii) *$V \in \mathcal{V}$, $\alpha \in \mathbb{R}$ implies $z^l(V + \alpha k^0) = z^l(V) + \alpha$;*
(iii) *z^l is $\preccurlyeq$–monotone.*

Proof. Since V'' is an upper $\preccurlyeq$–bound, we have $V'' \neq \emptyset$ and $V'' \subseteq V + K$. Hence $V \neq \emptyset$ and $z^l(V) \leq 0$ for all $V \in \mathcal{V}$.

Assume that z^l is not bounded below. Then for each $n \in \mathbb{N}$, we can find some $t_n < -n$ and some $V_n \in \mathcal{V}$ such that $-nk^0 + V'' = (-n - t_n)k^0 + t_n k^0 + V'' \subseteq K \setminus -\operatorname{cl} K + V_n + \operatorname{cl} K \subseteq V_n + \operatorname{cl} K$. Since $\mathcal{V}$ is $\preccurlyeq$–bounded below by $V' \subseteq Y$ we have $-nk^0 + V'' \subseteq V_n + \operatorname{cl} K \subseteq V' + \operatorname{cl} K$ for all $n \in \mathbb{N}$. Hence $-nk^0 + v_0 \in V' + \operatorname{cl} K$ for arbitrary $v_0 \in V''$. Dividing by n and letting $n \to \infty$ we get $k^0 \in -\operatorname{cl} K$ since V' is bounded. This contradicts the assumption $k^0 \in K \setminus -\operatorname{cl} K$.

Assertion (ii) is obvious. To show (iii) let $V_1 \preccurlyeq V_2$. Then $V_2 + \operatorname{cl} K \subseteq V_1 + K + \operatorname{cl} K \subseteq V_1 + \operatorname{cl} K$. This implies $z^l(V_1) \leq z^l(V_2)$ by definition of z^l. $\square$

An analogous result for the relation $\curlyeqprec$ is an immediate conclusion.

Corollary 3.2. *Let Y, K, k^0 be as in Theorem 3.1. Let $\mathcal{V} \subseteq 2^Y$ be nonempty and $\curlyeqprec$–bounded, i.e., there is a nonempty set $W' \subseteq Y$ and a topologically bounded set $W'' \subseteq Y$ such that*

$$\forall V \in \mathcal{V} : \; W' \curlyeqprec V \curlyeqprec W''.$$

Then, the functional $z^u : 2^Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$z^u(V) := -\inf\{t \in \mathbb{R} : \, t\left(-k^0\right) + W' \subseteq V - \operatorname{cl} K\}$$

is bounded on $\mathcal{V}$, satisfies $z^u(V + \alpha k^0) = z^u(V) + \alpha$ for all $V \in \mathcal{V}$, $\alpha \in \mathbb{R}$ and is $\curlyeqprec$–monotone.

Proof. Note that $\mathcal{V}$ is $\curlyeqprec$–bounded iff it is $\preccurlyeq_{-K}$–bounded with upper bound W'. Taking into account that $-z^u(V)$ coincides with $z^l(V)$ replacing V'' by W', K by $-K$ and k^0 by $-k^0$ we may apply Theorem 3.1 to obtain the assertions of the corollary. $\square$

Let Y be a topological vector space and $K \subseteq Y$ a convex cone. We use the following assumption for weakening the boundedness condition.

(C)	There exists a proper closed convex cone $C \subseteq Y$ with nonempty interior satisfying $K \subseteq C$.

Remark 3.3. If Y is a locally convex space and $K \subseteq Y$ a convex cone such that $K \setminus -\mathrm{cl}\, K \neq \emptyset$ then assumption (C) is satisfied and there exists an element $k^0 \in K \cap \mathrm{int}\, C$. Indeed, if $k^0 \in K \setminus -\mathrm{cl}\, K$ then we have $\left\{-k^0\right\} \cap \mathrm{cl}\, K = \emptyset$ and we can apply a classical separation theorem to the convex compact set $\left\{-k^0\right\}$ and the closed convex set $\mathrm{cl}\, K$. We obtain the existence of a continuous linear functional $y^* \in Y^*$ such that $y^*(-k^0) < 0 \leq y^*(k)$ for all $k \in \mathrm{cl}\, K$. The desired cone C can be defined by $C := \{y \in Y : \, y^*(y) \geq 0\}$.

Theorem 3.4. *Let Y be a topological vector space, $K \subseteq Y$ a convex cone satisfying assumption (C). Let $k^0 \in K \cap \mathrm{int}\, C$ and let $\mathcal{V} \subseteq 2^Y$ be nonempty, $\preccurlyeq_C$–bounded above and weakly $\preccurlyeq_C$–bounded below. Then, the functional $c^l : 2^Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by*

$$c^l(V) := \inf\left\{t \in \mathbb{R} : \, V \preccurlyeq_C \left\{tk^0\right\}\right\}$$

has the following properties:

(i) *c^l is bounded on $\mathcal{V}$;*
(ii) *$V \in \mathcal{V}$, $\alpha \in \mathbb{R}$ implies $c^l(V + \alpha k^0) = c^l(V) + \alpha$;*
(iii) *c^l is $\preccurlyeq$–monotone;*
(iv) *If $\mathcal{V}$ consists of compact sets $V \subseteq Y$ and $K \setminus \{0\} \subseteq \mathrm{int}\, C$ then*

$$\Big(V_1 \preccurlyeq V_2 \, , \; V_1 \cap V_2 = \emptyset\Big) \implies c^l(V_1) < c^l(V_2).$$

Proof. Let V'' be an upper $\preccurlyeq_C$–bound of $\mathcal{V}$, i.e., $V'' \neq \emptyset$ and $V'' \subseteq V + C$ for all $V \in \mathcal{V}$. Let $v'' \in V''$ be given. Because of $k^0 \in \mathrm{int}\, C$ there exists a neighborhood U of zero such that $U \subseteq -k^0 + \mathrm{int}\, C$. Choosing some $\sigma > 0$ such that $-v'' \in \sigma U$ we obtain $v'' \in -\sigma U \subseteq \sigma(k^0 - \mathrm{int}\, C) \subseteq \sigma k^0 - C$. Hence $\sigma k^0 \in V'' + C \subseteq V + C$ for all $V \in \mathcal{V}$. This means $c^l(V) \leq \sigma$ for all $V \in \mathcal{V}$, i.e., c^l is bounded above.

Assume that c^l is not bounded below. Then, for all $n \in \mathbb{N}$, we can find $t_n < -n$ and $V_n \in \mathcal{V}$ such that $V_n \preccurlyeq_C \left\{t_n k^0\right\}$. Hence $-nk^0 = (-n - t_n)k^0 + t_n k^0 \in C + V_n + C \subseteq V_n + C$. Thus

$$\forall n \in \mathbb{N}, \; \exists v_n \in V_n : \; -nk^0 \in v_n + C. \tag{8}$$

Since $\mathcal{V}$ is supposed to be weakly $\preccurlyeq$–bounded below there exists some topologically bounded set $V' \subseteq Y$ such that $V' \not\subseteq V + C$ for all $V \in \mathcal{V}$. Hence for each $n \in \mathbb{N}$ there exists $v'_n \in V'$ such that $v'_n \notin v_n + C$. It follows

$$-k^0 - v'_n/n \overset{(8)}{\in} (v_n - v'_n)/n + C \subseteq (Y \setminus -\mathrm{int}\, C) + C \subseteq Y \setminus -\mathrm{int}\, C.$$

Letting $n \to \infty$ we get $k^0 \notin \mathrm{int}\, C$ which contradicts the assumption $k^0 \in K \cap \mathrm{int}\, C$. Hence c^l is bounded on $\mathcal{V}$.

Assertions (ii) and (iii) are obvious. Let us prove (iv). By definition of the infimum, for each $n \in \mathbb{N}$ there exists $v_n \in V_2$ such that $\left(c^l(V_2) + 1/n\right) k^0 \in v_n + C$. Since V_2 is compact, we can find a subnet of the sequence $\{v_n\}_{n\in\mathbb{N}}$ converging to some $\bar{v} \in V_2$. Hence $c^l(V_2)k^0 \in \bar{v} + C$. Let $V_1 \preccurlyeq V_2$ and $V_1 \cap V_2 = \emptyset$. We have $\bar{v} \in V_2 \subseteq V_1 + K \setminus \{0\} \subseteq V_1 + \mathrm{int}\, C$. Hence there is some $\delta > 0$ such that $\bar{v} - \delta k^0 \in V_1 + \mathrm{int}\, C$. It follows $c^l(V_2)k^0 \in \bar{v} + C \subseteq \delta k^0 + V_1 + C$. Applying assertion (ii) we obtain $c^l(V_1) + \delta = c^l(V_1 + \delta k^0) \leq c^l(V_2)$. $\square$

The analogous result using the relation $\preccurlyeq$ can not be deduced from Theorem 3.4 by a construction similar to that of Corollary 3.2. Instead, we define a new scalarization functional. Compare the remark following the proof of Theorem 3.5.

Theorem 3.5. *Let Y be a topological vector space, $K \subseteq Y$ a convex cone satisfying assumption (C). Let $k^0 \in K \cap \operatorname{int} C$ and let $\mathcal{V} \subseteq 2^Y$ be nonempty, $\preccurlyeq_C$–bounded above and weakly $\preccurlyeq_C$–bounded below. Then, the functional $c^u : 2^Y \to \mathbb{R} \cup \{\pm\infty\}$*

$$c^u(V) := \inf \left\{ t \in \mathbb{R} : \ V \preccurlyeq_C \left\{ tk^0 \right\} \right\}$$

has the following properties:

(i) *c^u is bounded on $\mathcal{V}$;*
(ii) *$V \in \mathcal{V}$, $\alpha \in \mathbb{R}$ implies $c^u(V + \alpha k^0) = c^u(V) + \alpha$;*
(iii) *c^u is $\preccurlyeq$–monotone;*
(iv) *If $\mathcal{V}$ consists of compact sets $V \subseteq Y$ and $K \setminus \{0\} \subseteq \operatorname{int} C$ then*

$$\Big(V_1 \preccurlyeq V_2 \ , \ V_1 \cap V_2 = \emptyset \Big) \implies c^u(V_1) < c^u(V_2).$$

Proof. Let V_0 be an upper $\preccurlyeq$–bound of $\mathcal{V}$, i.e., $W'' \subseteq Y$ is topologically bounded and $V \subseteq W'' - C$ for all $V \in \mathcal{V}$. Let U be a neighborhood of zero such that $U \subseteq -k^0 + \operatorname{int} C$. Choosing $\sigma > 0$ such that $-W'' \subseteq \sigma U$ we obtain $V \subseteq W'' - C \subseteq -\sigma U - C \subseteq \sigma(k^0 - \operatorname{int} C) - C \subseteq \sigma k^0 - C$. Hence $c^u(V) \le \sigma$ for all $V \in \mathcal{V}$, i.e., c^u is bounded above on $\mathcal{V}$.

Assuming that c^u is not bounded below, for all $n \in \mathbb{N}$ we can find some $V_n \in \mathcal{V}$ such that $V_n \subseteq -nk^0 - C$. Since $\mathcal{V}$ is weakly $\preccurlyeq$–bounded below there exists some nonempty set $W' \subseteq Y$ such that $V \not\subseteq W' - C$, hence $V \not\subseteq w' - C$ where $w' \in W'$ is arbitrarily chosen. Hence for all $n \in \mathbb{N}$ there exists $v_n \in V_n$ such that $v_n - w' \notin -C$ and $-v_n - nk^0 \in C$. We obtain

$$-k^0 - w'/n = -k^0 - v_n/n + (v_n - w')/n \subseteq C + (Y \setminus -\operatorname{int} C) \subseteq Y \setminus -\operatorname{int} C.$$

Letting $n \to \infty$ we get $k^0 \notin \operatorname{int} C$, which contradicts the assumption $k^0 \in K \cap \operatorname{int} C$.

Assertions (ii) and (iii) are obvious. It remains to prove (iv). By the definition of the infimum there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subseteq V_1$ such that $v_n \notin (c^u(V_1) - 1/n)k^0 - C$. Since V_1 is compact, there is a subnet of $\{v_n\}_{n\in\mathbb{N}} \subseteq V_1$ converging to some $\bar{v} \in V_1$. Hence

$$\bar{v} \notin c^u(V_1)k^0 - \operatorname{int} C. \tag{9}$$

Let $V_1 \preccurlyeq V_2$ and $V_1 \cap V_2 = \emptyset$. We have $\bar{v} \in V_2 - K \setminus \{0\} \subseteq V_2 - \operatorname{int} C$. Choose $v \in V_2$ such that $\bar{v} \in v - \operatorname{int} C$. Then there exists some $\delta > 0$ such that

$$\bar{v} \in v - \delta k^0 - \operatorname{int} C. \tag{10}$$

Assuming that $v \in (c^u(V_1) + \delta)k^0 - C$ we obtain

$$\bar{v} \overset{(10)}{\in} v - \delta k^0 - \operatorname{int} C \subseteq c^u(V_1)k^0 - C - \operatorname{int} C \subseteq c^u(V_1)k^0 - \operatorname{int} C.$$

This contradicts (9). Hence we have $v \notin (c^u(V_1) + \delta)k^0 - C$. Since $v \in V_2$ it follows $c^u(V_1) + \delta \le c^u(V_2)$, i.e., $c^u(V_1) < c^u(V_2)$. □

Let us discuss why the construction of Corollary 3.2 fails in the present setting. This is due to the fact that the boundedness assumptions are not longer symmetric: In Theorem 3.5 we supposed $\mathcal{V}$ to be $\preccurlyeq_C$–bounded above and weakly $\preccurlyeq_C$–bounded below. This is true if and only if $-\mathcal{V}$ is $\preccurlyeq_C$–bounded below and weakly $\preccurlyeq_C$–bounded above. However, the following example shows that the weak $\preccurlyeq_C$–boundedness from above does not imply the boundedness of the functional c^l of Theorem 3.4.

Example 3.6. Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $k^0 = (1,1)$ and consider the set $\mathcal{V} = \{\{(-1,1)\}, \{(-1,2)\}, \{(-1,3)\}, ...\} \subseteq 2^Y$ consisting of singletons. Then $\{(0,0)\}$ is a weak upper $\preccurlyeq_C$–bound and $\{(-1,1)\}$ is a lower $\preccurlyeq_C$–bound of $\mathcal{V}$. But $c^l(\{(-1,n)\}) = n$ for all $n \in \mathbb{N}$, i.e., c^l is not bounded above on $\mathcal{V}$.

4. Basic Tools

For the convenience of the reader we present two basic tools for the proof of minimal element theorems wit set relations.

4.1. **The Brézis-Browder Principle.** The first tool is a very general existence principle for minimal elements in quasi-ordered sets due to Brézis and Browder [1], 1976.

Theorem 4.1. *Let $(W, \preceq)$ be a quasi-ordered set (i.e., $\preceq$ is a reflexive and transitive relation on W) and let $\phi : W \to \mathbb{R}$ be a function satisfying*

(A1) *ϕ is bounded below;*
(A2) *$w_1 \preceq w_2$ implies $\phi(w_1) \leq \phi(w_2)$;*
(A3) *For every $\preceq$–decreasing sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq W$ there exists some $w \in W$ such that $w \preceq w_n$ for all $n \in \mathbb{N}$.*

Then, for every $w_0 \in W$ there exists some $\bar{w} \in W$ such that

(i) *$\bar{w} \preceq w_0$;*
(ii) *$\hat{w} \preceq \bar{w}$ implies $\phi(\hat{w}) = \phi(\bar{w})$.*

Proof. See [1, Corollary 1]. □

4.2. **Uniform Spaces.** Our results involve a uniform space X (cf. [17]). Examples for uniform spaces not being necessarily metrizable are topological vector spaces and K–metric spaces (see [24] or [22]). If the reader is only interested in results for metric spaces the following considerations can be skipped. Then, one has to replace the *families of quasi-metrics*, in the following denoted by $\{q_\lambda\}_{\lambda\in\Lambda}$ or by q_Λ, by the metric.

In [14] we presented a characterization of uniform spaces via *families of quasi-metrics* introduced by Fang [5]. We shall give a short summary of these results.

Definition 4.2. Let X be a nonempty set and let $(\Lambda, \prec)$ be a directed set. A system $\{q_\lambda\}_{\lambda\in\Lambda}$ of functions $q_\lambda : X \times X \to [0, \infty)$ satisfying

(Q1) $\Big(\lambda \in \Lambda,\, x \in X\Big) \implies q_\lambda(x, x) = 0$;
(Q2) $\Big(\lambda \in \Lambda,\, x, y \in X\Big) \implies q_\lambda(x, y) = q_\lambda(y, x)$;
(Q3) $\forall \lambda \in \Lambda$, $\exists \mu \in \Lambda$ with $\lambda \prec \mu$: $x, y, z \in X \implies q_\lambda(x, y) \leq q_\mu(x, z) + q_\mu(z, y)$;

(Q4) $\Big(x, y \in X,\ \lambda, \mu \in \Lambda,\ \lambda \prec \mu\Big) \implies q_\lambda(x, y) \le q_\mu(x, y)$

is called a *family of quasi-metrics.* If, in addition, the condition

(Q5) $\Big(\forall \lambda \in \Lambda :\ q_\lambda(x, y) = 0\Big) \implies x = y$

is satisfied, the family of quasi-metrics is said to be *separating.*

Theorem 4.3. *A topological space (X, τ) is a (separated) uniform space iff its topology τ can be generated by a (separating) family of quasi-metrics.*

Proof. See [14]. □

Convention 4.4. For an easy dealing with uniform spaces we introduce the following notation. Let $\{q_\lambda\}_{\lambda\in\Lambda}$ the family of quasi-metrics which generates the topology of the uniform space X. We write q_Λ iff an assertion holds for all $\lambda \in \Lambda$. If X is a metric space, then q_Λ is its metric.

5. Minimal Element Theorems with Set Relations

This section contains the main results of the paper. We present two minimal element theorems with respect to the set ordering relations introduced in Section 2. Let us consider a subset $\mathcal{A}$ of $X \times 2^Y$, where X is a separated uniform space and Y is a topological vector space. We introduce the following notation:

$$\mathcal{V}(\mathcal{A}) := \{V \in 2^Y : \exists x \in X : (x, V) \in \mathcal{A}\}.$$

5.1. **Minimal Element Theorem I.** Using the relation $\preccurlyeq$ we introduce the following ordering relation on $X \times 2^Y$:

$$(x_1, V_1) \preccurlyeq_{k^0} (x_2, V_2) \quad \iff \quad V_1 + k^0 q_\Lambda(x_1, x_2) \preccurlyeq V_2.$$

According to Convention 4.4, the last inequality has to be read as

$$\forall \lambda \in \Lambda :\ V_1 + k^0 q_\lambda(x_1, x_2) \preccurlyeq V_2.$$

The relation $\preccurlyeq_{k^0}$ is a reflexive and transitive relation on $X \times 2^Y$. We present our *Minimal Element Theorem* involving $\preccurlyeq_{k^0}$.

Theorem 5.1. *Let X be a separated uniform space, Y a topological vector space, $K \subseteq Y$ a convex cone and $k^0 \in K \setminus -\operatorname{cl} K$. Let $\mathcal{A}$ be a nonempty subset of $X \times 2^Y$ such that for some $(x_0, V_0) \in \mathcal{A}$ and for $\mathcal{A}_0 := \{(x, V) \in \mathcal{A} :\ (x, V) \preccurlyeq_{k^0} (x_0, V_0)\}$ the following conditions are satisfied:*

- (M1) *$\mathcal{V}(\mathcal{A}_0)$ is $\preccurlyeq$–bounded above, i.e., V_0 is nonempty;*
- (M2) *$\mathcal{V}(\mathcal{A}_0)$ is $\preccurlyeq$–bounded below;*
- (M3) *For every $\preccurlyeq_{k^0}$–decreasing sequence $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{A}_0$ there exists some $(x, V) \in \mathcal{A}_0$ such that $(x, V) \preccurlyeq_{k^0} (x_n, V_n)$ for all $n \in \mathbb{N}$.*

Then, there exists $(\bar{x}, \bar{V}) \in \mathcal{A}$ such that

- (i) *$(\bar{x}, \bar{V}) \preccurlyeq_{k^0} (x_0, V_0)$;*
- (ii) *$\Big((\hat{x}, \hat{V}) \in \mathcal{A},\ (\hat{x}, \hat{V}) \preccurlyeq_{k^0} (\bar{x}, \bar{V})\Big) \implies \hat{x} = \bar{x}$.*

Under the additional assumption (C) and if $k^0 \in K \cap \operatorname{int} C$, (M2) can be relaxed to

(M2′) *$\mathcal{V}(\mathcal{A}_0)$ is weakly $\preccurlyeq_C$–bounded below.*

If, in addition, $k^0 \in K \setminus \{0\} \subseteq \operatorname{int} C$ and if for each $(x, V) \in \mathcal{A}_0$, V is compact, then (ii) can be strengthened to

$$\text{(ii}'\text{)} \quad \Big((\hat{x}, \hat{V}) \in \mathcal{A},\ (\hat{x}, \hat{V}) \preccurlyeq_{k^0} (\bar{x}, \bar{V}) \Big) \implies \Big(\hat{x} = \bar{x} \quad \textit{and} \quad \hat{V} \cap \bar{V} \neq \emptyset \Big).$$

Proof. We shall apply the Brézis-Browder principle to the quasi-ordered set $(\mathcal{A}_0, \preccurlyeq_{k^0})$ and the functional

$$\phi : \mathcal{A}_0 \to \mathbb{R}, \quad \phi(x, V) := z^l(V),$$

where $z^l : \mathcal{V}(\mathcal{A}_0) \to \mathbb{R}$ is the scalarization functional of Theorem 3.1. In the definition of z^l the upper $\preccurlyeq$–bound V'' has to be replaced by V_0.

We have to check the assumptions of Theorem 4.1. By (M1), (M2) and Theorem 3.1 (i), ϕ is well-defined and bounded. Theorem 3.1 (ii) and (iii) yield

$$(x_1, V_1) \preccurlyeq_{k^0} (x_2, V_2) \implies \phi(x_1, V_1) + q_\Lambda(x_1, x_2) \leq \phi(x_2, V_2). \tag{11}$$

Hence, ϕ is $\preccurlyeq_{k^0}$–monotone on $\mathcal{A}_0$, i.e., assumption (A2) of Theorem 4.1 is satisfied. Of course, (M3) implies assumption (A3).

Theorem 4.1 yields the existence of an element $(\bar{x}, \bar{V}) \in \mathcal{A}_0$ (i.e., (i) holds) such that

$$\Big((\hat{x}, \hat{V}) \in \mathcal{A}_0,\ (\hat{x}, \hat{V}) \preccurlyeq_{k^0} (\bar{x}, \bar{V}) \Big) \implies \phi(\hat{x}, \hat{V}) = \phi(\bar{x}, \bar{V}). \tag{12}$$

Let $(\hat{x}, \hat{V}) \in \mathcal{A}$ such that $(\hat{x}, \hat{V}) \preccurlyeq_{k^0} (\bar{x}, \bar{V})$. The transitivity of $\preccurlyeq_{k^0}$ yields $(\hat{x}, \hat{V}) \in \mathcal{A}_0$. Applying (12) and (11) we obtain $q_\Lambda(\hat{x}, \bar{x}) = 0$. Since X is separated, we have $\hat{x} = \bar{x}$, i.e., (ii) holds.

To see that (M2) can be replaced by (M2′) we proceed as above but using the functional c^l of Theorem 3.4 instead of z^l in the Definition of ϕ. To prove (ii') assume that $\hat{V} \cap \bar{V} = \emptyset$. Then, (iv) of Theorem 3.4 yields $\phi(\hat{x}, \hat{V}) < \phi(\bar{x}, \bar{V})$. This contradicts (12). $\square$

5.2. **Minimal Element Theorem II.** Using the $\curlyeqprec$–relation we present a second minimal element theorem. We introduce the following ordering relation on $X \times 2^Y$:

$$(x_1, V_1) \curlyeqprec_{k^0} (x_2, V_2) \iff V_1 + k^0 q_\Lambda(x_1, x_2) \curlyeqprec V_2.$$

This relation is also reflexive and transitive.

Theorem 5.2. *Let X be a separated uniform space, Y a topological vector space, $K \subseteq Y$ a convex cone and $k^0 \in K \setminus -\operatorname{cl} K$. Let $\mathcal{A}$ be a nonempty subset of $X \times 2^Y$ such that for some $(x_0, V_0) \in \mathcal{A}$ and for $\mathcal{A}_0 := \{(x, V) \in \mathcal{A} : (x, V) \curlyeqprec_{k^0} (x_0, V_0)\}$ the following conditions are satisfied:*

- (M1) *$\mathcal{V}(\mathcal{A}_0)$ is $\curlyeqprec$–bounded above;*
- (M2) *$\mathcal{V}(\mathcal{A}_0)$ is $\curlyeqprec$–bounded below;*
- (M3) *For every $\curlyeqprec_{k^0}$–decreasing sequence $\{(x_n, V_n)\}_{n \in \mathbb{N}} \subseteq \mathcal{A}_0$ there exists some $(x, V) \in \mathcal{A}_0$ such that $(x, V) \curlyeqprec_{k^0} (x_n, V_n)$ for all $n \in \mathbb{N}$.*

Then, there exists $(\bar{x}, \bar{V}) \in \mathcal{A}$ such that

- (i) *$(\bar{x}, \bar{V}) \curlyeqprec_{k^0} (x_0, V_0)$;*
- (ii) *$(\hat{x}, \hat{V}) \in \mathcal{A}$, $(\hat{x}, \hat{V}) \curlyeqprec_{k^0} (\bar{x}, \bar{V})$ implies $\hat{x} = \bar{x}$.*

Under the additional assumption (C) and if $k^0 \in K \cap \operatorname{int} C$, (M1) and (M2) can be relaxed to

(M1′) $\{V_0\}$ *is* $\curlyeqprec_C$*–bounded above;*
(M2′) $\mathcal{V}(\mathcal{A}_0)$ *is weakly* $\curlyeqprec_C$*–bounded below.*

If, in addition, $k^0 \in K \setminus \{0\} \subseteq \operatorname{int} C$ *and if for each* $(x, V) \in \mathcal{A}_0$*,* V *is compact, then (ii) can be strengthened to*

(ii′) $(\hat{x}, \hat{V}) \in \mathcal{A}$*,* $(\hat{x}, \hat{V}) \curlyeqprec_{k^0} (\bar{x}, \bar{V})$ *implies* $\hat{x} = \bar{x}$ *and* $\hat{V} \cap \bar{V} \neq \emptyset$*.*

Proof. We shall apply the Brézis-Browder principle to the quasi-ordered set $(\mathcal{A}_0, \curlyeqprec_{k^0})$ and the functional

$$\phi : \mathcal{A}_0 \to \mathbb{R},\ \phi(x, V) := z_u(V)$$

where $z_u : \mathcal{V}(\mathcal{A}_0) \to \mathbb{R}$ is the scalarization functional of Corollary 3.2. In the definition of z_u, the set W' has to be a lower $\preccurlyeq$–bound of $\mathcal{V}(\mathcal{A}_0)$ which exists according to (M2).

We have to check the assumptions of Theorem 4.1. This can be done using (M1), (M2), (M3) and Corollary 3.2. Theorem 4.1 yields the existence of the desired element $(\bar{x}, \bar{V}) \in \mathcal{A}_0$ in the same way as in the proof of Theorem 5.1.

To see that (M1) and (M2) can be replaced by (M1′) and (M2′) proceed as above but using the functional c^u of Theorem 3.5 instead of z^u in the definition of ϕ.

To prove (ii′) assume that $\hat{V} \cap \bar{V} = \emptyset$. Theorem 3.5 (iv) yields $\phi(\hat{x}, \hat{V}) < \phi(\bar{x}, \bar{V})$. But we must have $\phi(\hat{x}, \hat{V}) = \phi(\bar{x}, \bar{V})$, a contradiction. □

Note that Theorem 5.2 can be transformed into a *Maximal Element Theorem* with respect to the reflexive and transitive relation $\preccurlyeq_{-k^0}$ on $X \times 2^Y$ defined by

$$(x_1, V_1) \preccurlyeq_{-k^0} (x_2, V_2) \quad \Longleftrightarrow \quad V_1 - k^0 q_\Lambda(x_1, x_2) \preccurlyeq_{-K} V_2$$

observing that a sequence $\{(x_n, V_n)\}_{n \in \mathbb{N}} \subseteq \mathcal{A}_0$ is $\curlyeqprec_{k^0}$–decreasing if and only if it is $\preccurlyeq_{-k^0}$–increasing. One can see that it is not possible to obtain Theorem 5.2 from Theorem 5.1 by replacing $\curlyeqprec_{k^0}$ by $\preccurlyeq_{-k^0}$. We face the alternative either to state minimal element theorems for both the relations $\preccurlyeq_{k^0}$, $\curlyeqprec_{k^0}$ or to state a minimal as well as a maximal element theorem involving one of the relations.

6. Ekeland's Principle with Set Relations

In this section, we present several conclusions of Theorem 5.1 and 5.2. Recently, Truong [27] proved a variant of Ekeland's variational principle involving only the $\preccurlyeq$–relation. A similar, but more general variant can be obtained from Theorem 5.1. A new variational principle will be derived from Theorem 5.2. Another variant of Ekeland's principle for set-valued maps with point relations (cf. [3], [14]) as well as a minimal point theorem in $X \times Y$ turn out to be consequences of Theorem 5.1 as well as of Theorem 5.2.

Let X be a set, Y a linear topological space, $F : X \to 2^Y$ a set-valued map and K a convex cone in Y. In contrast to known definitions we call the set

$$\operatorname{graph} F := \left\{(x, V) \in X \times 2^Y : V = F(x)\right\}$$

the *graph* of F and the *image* of a subset $M \subseteq X$ (see also [27]) is defined by

$$F(M) := \{F(x) : x \in M\}.$$

Note that $\operatorname{graph} F$ is a subset of $X \times 2^Y$, not of $X \times Y$ and the image $F(M)$ is a subset of 2^Y, not of Y!

Moreover, we shall introduce the concept of the *domain* of a set-valued map F in a suitable way for each of the relations $\preccurlyeq$, $\curlyeqprec$. We define

$$\begin{aligned} &\preccurlyeq\text{-dom}\, F := \{x \in X : \; F(x) \preccurlyeq V \text{ for some nonempty } V \subseteq Y\}\,, \\ &\curlyeqprec\text{-dom}\, F := \{x \in X : \; F(x) \curlyeqprec V \text{ for some topologically bounded } V \subseteq Y\}\,. \end{aligned}$$

Clearly, $x \in \preccurlyeq\text{-dom}\, F$ means that the set $\{F(x)\} \subseteq 2^Y$ consisting of just one element is $\preccurlyeq$–bounded above, which is equivalent to $F(x) \neq \emptyset$. Similarly, $x \in \curlyeqprec\text{-dom}\, F$ means that the set $\{F(x)\} \subseteq 2^Y$, consisting of just one element, is $\curlyeqprec$–bounded above.

6.1. **Ekeland's Principle with Set Relations I.** First, we state a variational principle involving the ordering relation $\preccurlyeq$.

Theorem 6.1. *Let X be a separated uniform space, Y a topological vector space, $K \subseteq Y$ a convex cone and $k^0 \in K \setminus -\operatorname{cl} K$. Let $F : X \to 2^Y$ be a set-valued mapping, $x_0 \in \preccurlyeq\text{-dom}\, F$, $S(x_0) := \left\{x \in X : F(x) + k^0 q_\Lambda(x, x_0) \preccurlyeq F(x_0)\right\}$ and $\mathcal{A}_0 := \{(x, V) \in \operatorname{graph} F : x \in S(x_0)\}$ such that the following conditions are satisfied:*

- (E1) *$F(S(x_0))$ is $\preccurlyeq$–bounded below;*
- (E2) *For every $\preccurlyeq_{k^0}$–decreasing sequence $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{A}_0$ there exists some $(x, V) \in \mathcal{A}_0$ such that $(x, V) \preccurlyeq_{k^0} (x_n, V_n)$ for all $n \in \mathbb{N}$.*

Then, there exists $\bar{x} \in \preccurlyeq\text{-dom}\, F$ such that

- (i) *$F(\bar{x}) + k^0 q_\Lambda(\bar{x}, x_0) \preccurlyeq F(x_0)$;*
- (ii) *$\forall x \neq \bar{x}$, $\exists \lambda \in \Lambda$: $F(x) + k^0 q_\lambda(x, \bar{x}) \not\preccurlyeq F(\bar{x})$.*

Under the additional assumption (C) and if $k^0 \in K \cap \operatorname{int} C$, (E1) can be relaxed to

- (E1′) *$F(S(x_0))$ is weakly $\preccurlyeq_C$–bounded below.*

Proof. Set $\mathcal{A} := \operatorname{graph} F$ and apply Theorem 5.1. It only remains to note that $x_0 \in \preccurlyeq\text{-dom}\, F$ implies condition (M1) of Theorem 5.1. $\square$

Condition (ii) tells us that there does not exist an $x \in X \setminus \{\bar{x}\}$ such that $F_\lambda(x) \preccurlyeq F_\lambda(\bar{x}) = F(\bar{x})$ where

$$F_\lambda : X \to 2^Y, \quad F_\lambda(x) := F(x) + k^0 q_\lambda(x, \bar{x}).$$

This means, $\bar{x}$ is an s-minimizer in the sense of [28] of F_λ. Of course, $F_\lambda(\bar{x})$ is also a minimal element of $\{F_\lambda(x) : \; x \in X\}$ with respect to $\preccurlyeq$.

The assumptions of Theorem 6.1 may look somewhat artificial. We give a sufficient condition for (E2).

Theorem 6.2. *Let X, Y, K, k^0, F, x_0, $S(x_0)$, $\mathcal{A}_0$ be as in Theorem 6.1 and let (E1) be satisfied. Then (E2) is in force if the following condition is satisfied:*

- (E2′) *For every $x \in X$ the set $S(x) := \left\{x' \in X : F(x') + k^0 q_\Lambda(x', x) \preccurlyeq F(x)\right\}$ is sequentially complete in X.*

Proof. Let $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{A}_0$ be a $\preccurlyeq_{k^0}$–decreasing sequence, i.e., $V_n = F(x_n)$ and

$$F(x_{n+1}) + k^0 q_\Lambda(x_{n+1}, x_n) \preccurlyeq F(x_n) \tag{13}$$

for all $n \in \mathbb{N}$. By the transitivity of $\preccurlyeq_{k^0}$ we get

$$F(x_m) + k^0 q_\Lambda(x_m, x_n) \preccurlyeq F(x_n) \tag{14}$$

for all $m \geq n, m \in \mathbb{N}$. Applying the functional $\phi : \mathcal{A}_0 \to \mathbb{R}$ from the proof of Theorem 5.1 to relation (13) we obtain

$$\phi(x_{n+1}, F(x_{n+1})) + q_\Lambda(x_{n+1}, x_n) \leq \phi(x_n, F(x_n)).$$

The sequence $\{\phi(x_n, F(x_n))\}_{n\in\mathbb{N}}$ is nonincreasing and bounded below by (E1) and the corresponding properties of ϕ, hence convergent and all the more a Cauchy sequence. Applying ϕ to (14) we may conclude that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy as well. Since $S(x_0)$ is sequentially complete, $\{x_n\}_{n\in\mathbb{N}}$ converges to some $x \in S(x_0)$. Moreover, (14) implies that $x_m \in S(x_n)$ for all $m \geq n, m \in \mathbb{N}$. Since by (E2') $S(x_n)$ is sequentially complete we have $x \in S(x_n)$ for all $n \in \mathbb{N}$. This means $(x, F(x)) \preccurlyeq_{k^0} (x_n, F(x_n))$ as desired. □

Remark 6.3. We indicate a special situation where (E2') is satisfied. Let X be a sequentially complete separated uniform space. Ferro [6] introduced the concept of lower $\preccurlyeq$–semicontinuity (D-lower semicontinuity in [6]) and Truong [27] proved an Ekeland-type theorem using this continuity property. A set-valued map $F : X \to 2^Y$ is said to be sequentially lower $\preccurlyeq$–semicontinuous iff for every $V \in 2^Y$ the set $\{x' \in X : F(x') \preccurlyeq V\}$ is a sequentially closed subset of X. We claim that (E2') is satisfied if F is sequentially lower $\preccurlyeq$–semicontinuous with $\preccurlyeq$–closed values (for each $x \in X$, $F(x) + K$ is a closed set). Indeed, let $\{x_n\}_{n\in\mathbb{N}} \subseteq S(x)$ be a sequence such that $x_n \to x'$. Fix $\lambda \in \Lambda$. Then there exists $\mu \in \Lambda$ such that $q_\lambda(x', x) \leq q_\mu(x', x_n) + q_\mu(x_n, x)$. Since $x_n \to x'$ we can find for each $\varepsilon > 0$ a number $n_\varepsilon \in \mathbb{N}$ such that $q_\mu(x', x_n) \leq \varepsilon$ for all $n \in \mathbb{N}$, $n > n_\varepsilon$. Hence $q_\mu(x_n, x) \geq q_\lambda(x', x) - \varepsilon$. Since $F(x_n) + k^0 q_\mu(x_n, x) \preccurlyeq F(x)$, for $n > n_\varepsilon$ holds $F(x_n) + k^0 (q_\lambda(x', x) - \varepsilon) \preccurlyeq F(x)$. This implies $F(x_n) \preccurlyeq F(x) - k^0 (q_\lambda(x', x) - \varepsilon)$ for $n > n_\varepsilon$. Since F is lower $\preccurlyeq$–semicontinuous we have $F(x') \preccurlyeq F(x) - k^0 (q_\lambda(x', x) - \varepsilon)$, hence

$$F(x) \subseteq F(x') + k^0 \big(q_\lambda(x', x) - \varepsilon\big) + K. \tag{15}$$

Take $y \in F(x)$. Then, from (15) it follows that $y + \varepsilon k^0 \in F(x') + k^0 q_\lambda(x', x) + K$. Letting $\varepsilon \to 0$ we obtain $y \in F(x') + k^0 q_\lambda(x', x) + K$ since the latter set is closed. Hence $x' \in S(x)$.

Remark 6.4. In [27] the concept of K–boundedness (cf. [23]) is defined as follows: A subset A of a linear topological space Y is said to be K–bounded if there is a topologically bounded set $M \subseteq Y$ such that $A \subseteq M + K$. It can easily be seen that the boundedness condition in [27] coincides with condition (E1) of Theorem 6.1.

In [27], Y is supposed to be a locally convex space, K is a closed pointed convex cone and $k^0 \in K \setminus \{0\}$. Therefore we have $k^0 \in K \setminus -\operatorname{cl} K$. Remark 3.3 yields that assumption (C) is satisfied and $k^0 \in K \cap \operatorname{int} C$. Therefore, these assumptions have not to be proposed additionally. Hence, our boundedness condition (E1') is in fact weaker than the boundedness condition in [27] (cf. Remark 2.6 and Example

2.7). This extends the area of applicability of Ekeland's principle as the following example shows.

Example 6.5. Let $X = [0, \infty)$, $Y = \mathbb{R}^2$ with Euclidean norm, $B_r(y)$ be the closed ball of radius $r \geq 0$ centered at $y \in Y$, $K = C = \mathbb{R}^2_+$ and

$$F : X \to 2^Y, \quad F(x) := B_x(0) \setminus -\operatorname{int} K.$$

One may check that F is not $\preccurlyeq$–bounded, but $(-1, -1)^T$ is a weak lower $\preccurlyeq$–bound. Hence the results of [27] are not applicable. Moreover, there does not exist a $\preccurlyeq$–minimal value of F, but it is $\preccurlyeq$–lower semicontinuous. Thus, we can apply Theorem 6.2 in combination with Remark 6.3.

Theorem 6.2 and Remark 6.3 – 6.6 show that Theorem 5.1 of [27] is a very special case of Theorem 6.1 concerning the properties of F: We use (E2) instead of K-lower semicontinuity and also weaker boundedness assumptions, cf. Remark 6.4. Of course, we deal with larger classes of spaces X (uniform instead of metric spaces) and Y (topological vector spaces instead of locally convex spaces) as well as of cones in Y (not necessarily closed and pointed).

Remark 6.6. If $x_0 \in X$ is an εk^0–minimal point of F in the sense of [27], i.e., $F(x) \not\preccurlyeq F(x_0) - \varepsilon k^0$ then relation (i) of Theorem of 6.1 can be split into the two relations (i$_1$) $F(\bar{x}) \preccurlyeq F(x_0)$; (i$_2$) $q_\Lambda(\bar{x}, x_0) \leq \varepsilon$. Indeed, while (i$_1$) is immediate, (i) is equivalent to $F(x_0) \subseteq F(\bar{x}) + k^0 q_\Lambda(\bar{x}, x_0) + K$. If $q_\mu(\bar{x}, x_0) > \varepsilon$ for some $\mu \in \Lambda$, we have

$$F(x_0) \subseteq F(\bar{x}) + k^0 \left(q_\mu(\bar{x}, x_0) - \varepsilon\right) + \varepsilon k^0 + K \subseteq F(\bar{x}) + \varepsilon k^0 + K,$$

which contradicts the εk^0–minimality of x_0.

The following lemma tells us that there is always a $\preccurlyeq$–k^0–minimal solution if F is $\preccurlyeq$–bounded below.

Lemma 6.7. *Let X, Y, K and k_0 as in Theorem 6.1, $F : X \to 2^Y$. If F is $\preccurlyeq$–bounded below, then there exists $x_0 \in X$ such that*

$$\forall x \in X : \ F(x) \not\preccurlyeq F(x_0) - k^0.$$

Proof. Assume the contrary, namely, for all $x_0 \in X$ there exists some $x \in X$ such that $F(x) \preccurlyeq F(x_0) - k^0$. By induction we can construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $F(x_n) \preccurlyeq F(x_0) - nk^0$ for all $n \in \mathbb{N}$. Since $F(x_0) - nk^0 \preccurlyeq F(x_0)$ and by assumption we conclude that $\mathcal{V} := \{F(x_n) : \ n \in \mathbb{N}\}$ is $\preccurlyeq$–bounded. Applying the functional z^l of Theorem 3.1 to the above inequality we get $z^l(F(x_n)) \leq z^l(F(x_0)) - n$ for all $n \in \mathbb{N}$, i.e., z^l is not bounded below on $\mathcal{V}$. This contradicts (i) of Theorem 3.1 saying that z^l is bounded on $\mathcal{V}$. □

Note that, involving assumption (C), it is enough to suppose that F is weakly $\preccurlyeq$–bounded below (compare Theorem 6.1). The same applies for the following theorem, the classical form of Ekeland's principle, see Theorem 1 *bis* in [4].

Theorem 6.8. *Let the assumptions of Theorem 6.1 be satisfied. Then there exists $\bar{x} \in X$ such that*

$$\forall x \in X : F(x) \not\preccurlyeq F(\bar{x}) - k^0 \tag{16}$$

$$\forall x \in X, x \neq \bar{x},\ \exists \lambda \in \Lambda : F(x) + k^0 q_\lambda(x, \bar{x}) \not\preccurlyeq F(\bar{x}). \tag{17}$$

Proof. According to Lemma 6.7 there exists $x_0 \in X$ such that

$$\forall x \in X :\ F(x) \not\preccurlyeq F(x_0) - k^0. \tag{18}$$

Applying Theorem 6.1 and taking into account Remark 6.6 we obtain $\bar{x} \in X$ satisfying (17) as well as $F(\bar{x}) \preccurlyeq F(x_0)$. If $x' \in X$, $F(x') \preccurlyeq F(\bar{x}) - k^0$ we obtain by transitivity $F(x') \preccurlyeq F(x_0) - k^0$ contradicting (18). Hence (16) is true. □

6.2. **Ekeland's Principle with Set Relations II.** In this section, we shall prove an analogous variational principle for the relation $\preccurlyeq$.

Theorem 6.9. *Let X be a separated uniform space, Y a topological vector space, $K \subseteq Y$ a convex cone and $k^0 \in K \setminus -\operatorname{cl} K$. Let $F : X \to 2^Y$ be a set-valued mapping, $x_0 \in \preccurlyeq\text{-}\operatorname{dom} F$, $S(x_0) := \left\{x \in X : F(x) + k^0 q_\Lambda(x, x_0) \preccurlyeq F(x_0)\right\}$ and $\mathcal{A}_0 := \{(x, V) \in \operatorname{graph} F : x \in S(x_0)\}$ such that the following conditions are satisfied:*

- (E1) *$F(S(x_0))$ is $\preccurlyeq$–bounded below;*
- (E2) *For every $\preccurlyeq_{k^0}$–decreasing sequence $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{A}_0$ there exists some $(x, V) \in \mathcal{A}_0$ such that $(x, V) \preccurlyeq_{k^0} (x_n, V_n)$ for all $n \in \mathbb{N}$.*

Then, there exists $\bar{x} \in \preccurlyeq\text{-}\operatorname{dom} F$ such that

- (i) *$F(\bar{x}) + k^0 q_\Lambda(\bar{x}, x_0) \preccurlyeq F(x_0)$;*
- (ii) *$\forall x \neq \bar{x} : \exists \lambda \in \Lambda :\ F(x) + k^0 q_\lambda(x, \bar{x}) \not\preccurlyeq F(\bar{x})$.*

Under the additional assumption (C) and if $k^0 \in K \cap \operatorname{int} C$, then we can even allow $x_0 \in \preccurlyeq_C\text{-}\operatorname{dom} F$ and (E1) can be replaced by

- (E1′) *$F(S(x_0))$ is weakly $\preccurlyeq_C$–bounded below.*

Proof. Set $A := \operatorname{graph} F$ and apply Theorem 5.2 noting that $x_0 \in \preccurlyeq\text{-}\operatorname{dom} F$ and $x_0 \in \preccurlyeq_C\text{-}\operatorname{dom} F$ imply (M1) and (M1′) of Theorem 5.2, respectively. □

Theorem 6.10. *Let X, Y, K, k^0, F, $S(x_0)$, $\mathcal{A}_0$ be as in Theorem 6.9 and let (E1) be satisfied. Then (E2) is in force if the following condition is satisfied:*

- (E2′) *For every $x \in X$ the set $S(x) := \left\{x' \in X :\ F(x') + k^0 q_\Lambda(x', x) \preccurlyeq F(x)\right\}$ is sequentially complete in X.*

Proof. Follow the lines of the proof of Theorem 6.2. □

Remark 6.11. A map $F : X \to 2^Y$ is called sequentially lower $\preccurlyeq$–semicontinuous iff for every $V \in 2^Y$ the set $\{x' \in X :\ F(x') \preccurlyeq V\}$ is a sequentially closed subset of X. Following arguments similar to those in Remark 6.3, we can prove that (E2') of Remark 6.10 is satisfied if F is sequentially lower $\preccurlyeq$–semicontinuous with $\preccurlyeq$–closed values (for each $x \in X$, $F(x) - K$ is a closed set).

Remark 6.12. Considerations similar to Remark 6.6, Lemma 6.7 and Theorem 6.8 can be done.

6.3. **Ekeland's Principle with Point Relations and a Minimal Point Theorem.** In [14] we proved a minimal point theorem and its equivalence to a variant of Ekeland's principle for set-valued maps as well as many conclusions of them (e.g. results of [3], [11], [10], [16], [21]). In this subsection we show that these theorems are corollaries of Theorem 5.1 as well as of Theorem 5.2. We are concerned with elements $w = (w_X, w_Y) = (x, y)$ of the product space $W = X \times Y$, X a separated uniform space, Y a topological vector space. We introduce the ordering relation $\preceq_{k^0}$ on W using an element $k^0 \in K \setminus -\operatorname{cl} K$:

$$(x_1, y_1) \preceq_{k^0} (x_2, y_2) \quad \Longleftrightarrow \quad y_1 + k^0 q_\Lambda (x_1, x_2) \leq_K y_2. \tag{19}$$

If K is a convex cone, $\preceq_{k^0}$ is a reflexive and transitive relation. If K additionally is pointed, the relation is also antisymmetric. See e.g. [11], [14]. Identifying an element $(x, y) \in X \times Y$ with $(x, \{y\}) \in X \times 2^Y$, we can easily see that $\preceq_{k^0}$ coincides with the above defined ordering $\preccurlyeq_{k^0}$ as well as with $\curlyeqprec_{k^0}$.

The following corollary is a variant of Ekeland's variational principle for set-valued maps involving the ordering relation $\preceq_{k^0}$ applied to elements of the set

$$\operatorname{gr} F := \{(x, y) \in X \times Y : y \in F(x)\}$$

usually denoting the graph of F. Note the difference to the definition at the beginning of Section 6.

Corollary 6.13. *Let X be a separated uniform space, Y a topological vector space, $K \subseteq Y$ a convex cone and $k_0 \in K \setminus -\operatorname{cl} K$. For the set-valued mapping $F : X \to 2^Y$, let $w_0 = (x_0, y_0) \in \operatorname{gr} F$ be given such that for the set $A_0 := \{w \in \operatorname{gr} F : w \preceq_{k^0} w_0\}$ the following assumptions are satisfied:*

(E1) *The set $(A_0)_Y := \{y \in Y : \exists x \in X : w = (x, y) \in A_0\}$ is $\leq_K$–bounded below;*
(E2) *For every $\preceq_{k^0}$–decreasing sequence $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq A_0$ there exists some point $(x, y) \in A_0$ such that $(x, y) \preceq_{k^0} (x_n, y_n)$ for all $n \in \mathbb{N}$.*

Then, there exists some point $(\bar{x}, \bar{y}) \in \operatorname{gr} F$ such that

(i) *$\bar{y} + k^0 q_\Lambda(\bar{x}, x_0) \leq_K y_0$;*
(ii) *If $(x, y) \in \operatorname{gr} F$ and $x \neq \bar{x}$, then there is $\lambda \in \Lambda$ such that $y + k^0 q_\lambda(x, \bar{x}) \not\leq_K \bar{y}$.*

Under the additional assumption (C) and if $k^0 \in K \cap \operatorname{int} C$, (E1) can be relaxed to

(E1′) *There exists some $\tilde{y} \in Y$ such that $(A_0)_Y \cap (\tilde{y} - \operatorname{int} C) = \emptyset$;*

and if $k^0 \in K \setminus \{0\} \subseteq \operatorname{int} C$ then $\bar{y}$ is $\leq_K$–minimal in $F(\bar{x})$.

Proof. Choose $\mathcal{A} = \{(x, \{y\}) : (x, y) \in \operatorname{gr} F\}$ and apply Theorem 5.1 or 5.2. □

The following corollary is a minimal point theorem in $X \times Y$. Compare Theorem 9 in [14].

Corollary 6.14. *Let X be a separated uniform space, Y a topological vector space, $K \subseteq Y$ a convex cone and $k^0 \in K \setminus -\operatorname{cl} K$. Let $A \subseteq W = X \times Y$ such that for some $w_0 \in A$ and for $A_0 := \{w \in A : w \preceq_{k^0} w_0\}$ the following conditions are satisfied:*

(M2) *The set $(A_0)_Y := \{y \in Y : \exists x \in X : (x, y) \in A_0\}$ is $\leq_K$–bounded below;*
(M3) *For every $\preceq_{k^0}$–decreasing sequence $\{w_n\}_{n \in \mathbb{N}} \subseteq A_0$ there exists some $w \in A_0$ such that $w \preceq_{k^0} w_n$ for all $n \in \mathbb{N}$.*

Then, there exists $\bar{w} \in A$ such that

(i) $\bar{w} \preceq_{k^0} w_0$;

(ii) $\Big(\hat{w} \in A,\ \hat{w} \preceq_{k^0} \bar{w}\Big) \implies \hat{w}_X = \bar{w}_X$.

Under the additional assumption (C) and if $k^0 \in K \cap \operatorname{int} C$, *(M2) can be replaced by*

(M2′) *There exists some* $\tilde{y} \in Y$ *such that* $(A_0)_Y \cap (\tilde{y} - \operatorname{int} C) = \emptyset$.

If, in addition, $k^0 \in K \setminus \{0\} \subseteq \operatorname{int} C$ *then (ii) can be strengthened to*

(ii′) $\Big(\hat{w} \in A,\ \hat{w} \preceq_{k^0} \bar{w}\Big) \implies \hat{w} = \bar{w}$, *(i.e.,* $\bar{w}$ *is* $\preceq_{k^0}$*–minimal in* A*).*

Proof. Choose $\mathcal{A} = \{(x, \{y\}) : (x, y) \in A\}$ and apply Theorem 5.1 or 5.2. □

The equivalence of Corollary 6.13 and Corollary 6.14 as well as a fixed point theorem of Kirk-Caristi type has been established in [14]. Because of this equivalence the minimal point theorem in $X \times Y$ can be understood as Ekeland's principle for set-valued maps with respect to an ordering relation for elements of $\operatorname{gr} F \subseteq X \times Y$.

6.4. **Ekeland's Principle for Single-valued Maps.** Finally, we present a conclusion for the case of a single-valued map $f : X \to Y$. As above, identifying an element $y \in Y$ with $\{y\} \in 2^Y$ the ordering relations $\preccurlyeq$ and $\curlyeqprec$ coincide, hence the following corollary may be deduced from Theorem 6.1 as well as from Theorem 6.9. It is also possible to derive it from Corollary 6.13. As proposed in [11], [10], we extend the space Y by an element ∞ such that $y \leq_K \infty$ for all $y \in Y$. As usual, the *domain* of f is said to be the set $\operatorname{dom} f := \{x \in X : f(x) \neq \infty\}$.

Corollary 6.15. *Let* X *be a separated uniform space,* Y *a topological vector space,* $K \subseteq Y$ *a convex cone and* $k_0 \in K \setminus -\operatorname{cl} K$. *Let* $f : X \to Y \cup \{\infty\}$ *be a proper function satisfying the conditions*

(E1) f *is* $\leq_K$*–bounded below;*

(E2) *For every* $x \in \operatorname{dom} f$ *the set* $S(x) := \big\{x' \in X : f(x') + k^0 q_\Lambda(x', x) \leq_K f(x)\big\}$ *is sequentially complete.*

Then, for each $x_0 \in \operatorname{dom} f$ *there exists* $\bar{x} \in X$ *such that*

(i) $f(\bar{x}) + k^0 q_\Lambda(\bar{x}, x_0) \leq_K f(x_0)$;

(ii) *For all* $x \in X \setminus \{\bar{x}\}$ *there is* $\lambda \in \Lambda$ *such that* $f(x) + k^0 q_\lambda(x, \bar{x}) \not\leq_K f(\bar{x})$.

Under the additional assumption (C) and if $k^0 \in K \cap \operatorname{int} C$, *(E1) can be replaced by*

(E1′) *There exists some* $\tilde{y} \in Y$ *such that* $f(S(x_0)) \cap (\tilde{y} - \operatorname{int} C) = \emptyset$.

Proof. To apply Corollary 6.13 or Theorem 6.1 (and Remark 6.2) we may define a set-valued function $F : X \to 2^Y$ by setting $F(x) := \{f(x)\}$ if $f(x) \neq \infty$ and $F(x) := \emptyset$ else. A third variant of a proof can be given by setting $F(x) := \{f(x)\}$ if $f(x) \neq \infty$ and $F(x) := Y$ else and applying Theorem 6.9 (and Remark 6.10). □

Taking into account Remark 6.6 and 6.12, respectively, we see that Corollary 6.15 is a generalization of Theorem 4 of [16] as well as Theorem 4 of [21]. It also covers Corollary 2 of [11] as well as Corollary 2 of [10]. This shows that our minimal set theorems are powerful generalizations of the minimal point theorems of [25], [10], [11], [14].

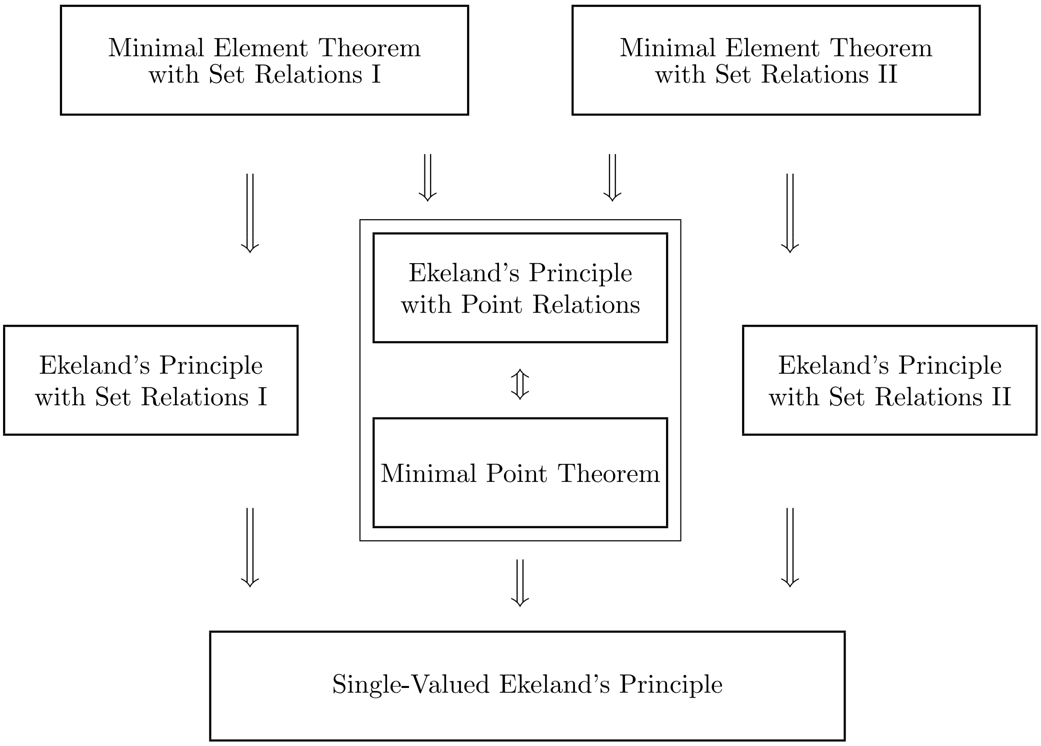

FIGURE 1. Relationships between the main results

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