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LC-FUNCTIONS AND MAXIMAL MONOTONICITY

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Abstract. In this paper, we consider *LC–functions*, a class of special convex functions from the product of a reflexive Banach space and its dual into $]-\infty, \infty]$. Using Fitzpatrick functions, we will show that the theory of LC–functions is a proper extension of the theory of maximal monotone sets. Various versons of the Fenchel duality theorem lead to a number of results on maximal monotonicity, some of them new. In particular, we prove various surjectivity results, including a generalization of a known "abstract Hammerstein theorem", give sufficient conditions for a sum of maximal monotone multifunctions to be maximal monotone, and prove a generalization of the Brezis–Haraux theorem

1. Introduction

All normed spaces in the paper will be *real*. Let E be a nonzero reflexive Banach space. This paper is about *LC–functions*, a class of special convex functions from $E \times E^*$ into $]-\infty, \infty]$ (see Definition 3.2).

We will see in Theorem 3.9 that any LC–function determines a maximal monotone subset of $E \times E^*$. If A is a maximal monotone subset of $E \times E^*$ then the Fitzpatrick function of A is an LC–function, but we will show in Theorem 4.1 and Example 3.5 that not every LC–function is the Fitzpatrick functions of some maximal monotone set. So we could regard the theory of LC–functions as a proper extension of the theory of maximal monotone sets.

One can obtain a number of interesting results on LC-functions by using the Fenchel duality theorem in one of its forms, as we outline in the following remarks. In each case, we trace the development of the results, starting with the relevant form of the Fenchel duality theorem, then through the results on LC–functions, and terminating with the results on maximal monotonicity.

Rockafellar's version of the Fenchel duality theorem, a special case of which we have stated in Theorem 2.1, leads to Theorem 3.8, (the rather surprising) Corollary 3.10 and Corollary 3.13. Corollary 3.13(e) leads in turn to Rockafellar's surjectivity theorem for maximal monotone multifunctions, which we state as Theorem 6.1(b).

The Attouch–Brezis version of the Fenchel duality theorem, which we have stated as Theorem 2.2, leads to Theorem 3.14 and Corollary 3.15 for lower semicontinuous LC–functions. Theorem 3.14 leads, in turn, to Theorem 6.2, a more general new result on the surjectivity of the sum of maximal monotone multifunctions, while Corollary 3.15 leads to Theorem 6.5, a generalization of known results which have been applied to Hammerstein integral equations (see Remark 6.6).

The bivariate version of the Attouch–Brezis theorem proved in Lemma 2.4 and Theorem 2.5 leads to Corollary 3.16, which leads in turn to a sufficient condition for the sum of maximal monotone multifunctions to be maximal monotone in Theorem 4.2.

Section 5 is devoted to the *fitzpatrification* of a monotone multifunction. This is an extension of a monotone multifunction that normally fails to be monotone, but it does have a convex graph. Lemmas 5.3 and 7.1 are fundamental properties of the fitzpatrification, which lead to a new surjectivity theorem in Theorem 6.3, and a new sufficient condition for Brezis–Haraux approximation in Theorem 7.3. We show how this leads to the classical sufficient conditions in Corollary 7.5.

Lower semicontinuous LC–functions are a subset of a class of functions considered by Burachik and Svaiter in [4] and Penot and Zălinescu in [8], but we do not require LC–functions to be lower semicontinuous (see Remark 3.3).

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2. Versions of the Fenchel duality theorem

If F is a normed space and $f\colon\ F \mapsto \left]-\infty,\infty\right]$ then we write $\operatorname{dom} f := \{x \in F\colon\ f(x) \in \mathbb{R}\}$. f is said to be *proper* if $\operatorname{dom} f \neq \emptyset$.

We start off by stating a result that is an immediate consequence of Rockafellar's version of the Fenchel duality theorem (see [10, Theorem 1, pp. 82–83] for the original version and [15, Theorem 2.8.7, pp. 126–127] for more general results):

Theorem 2.1. *Let F be a nonzero normed space, $f\colon\ F \mapsto \left]-\infty,\infty\right]$ be proper and convex, $g\colon\ F \mapsto \mathbb{R}$ be convex and continuous, and $f+g \geq 0$ on F. Then there exists $x^* \in F^*$ such that $f^*(x^*) + g^*(-x^*) \leq 0$.*

Theorem 2.2 below was first proved by Attouch–Brezis $\big($this follows from [1, Corollary 2.3, pp. 131–132]$\big)$ – there is a somewhat different proof in [12, Theorem 14.2, p. 51], and a much more general result was established in [15, Theorem 2.8.6, pp. 125–126]:

Theorem 2.2. *Let K be a nonzero Banach space, $f,\ g\colon\ K \mapsto \left]-\infty,\infty\right]$ be convex and lower semicontinuous, $\bigcup_{\lambda>0} \lambda\big[\operatorname{dom} f - \operatorname{dom} g\big]$ be a closed subspace of K and $f+g \geq 0$ on K. Then there exists $z^* \in K^*$ such that $f^*(-z^*) + g^*(z^*) \leq 0$.*

Notation 2.3. If E and F are nonzero Banach spaces, we define the dual of $E \times F$ to be $F^* \times E^*$ under the pairing $\lfloor (x,y),(y^*,x^*) \rfloor := \langle x, x^* \rangle + \langle y, y^* \rangle \quad \big((x,y) \in E \times F,\ (y^*,x^*) \in F^* \times E^*\big)$. We then define the *projection maps* π_1, π_2 and the *reflection maps* ρ_1, ρ_2 on $E \times F$ by $\pi_1(x,y) := x$, $\pi_2(x,y) := y$, $\rho_1(x,y) := (-x,y)$ and $\rho_2(x,y) := (x,-y)$.

Lemma 2.4 is a stepping–stone to Theorem 2.5.

Lemma 2.4. *Let E and F be nonzero Banach spaces, $a,\ b\colon\ E \times F \mapsto \left]-\infty,\infty\right]$ be convex and lower semicontinuous, $L := \bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} a - \pi_1 \operatorname{dom} b\big]$ be a closed subspace of E and, for all $(x,u,v) \in E \times F \times F$, $a(x,u) + b(x,v) \geq 0$. Then there exists $t^* \in E^*$ such that $a^*(0,-t^*) + b^*(0,t^*) \leq 0$.*

Proof. For all $(x, u, v) \in E\times F\times F$, let $f(x, u, v) := a(x, u)$ and $g(x, u, v) := b(x, v)$. We first prove that

$$\bigcup_{\lambda>0} \lambda\big[\operatorname{dom} f - \operatorname{dom} g\big] = L \times F \times F. \tag{2.4.1}$$

To this end, let $(x, u, v) \in L \times F \times F$. Then there exist $\lambda > 0$, $(s, y) \in \operatorname{dom} a$ and $(t, z) \in \operatorname{dom} b$ such that $x = \lambda(s - t)$. Thus

$$(x, u, v) = \lambda\big[(s, y, z + v/\lambda) - (t, y - u/\lambda, z)\big] \in \lambda\big[\operatorname{dom} f - \operatorname{dom} g\big].$$

This establishes "$\supset$" in (2.4.1), and (2.4.1) now follows since the inclusion "$\subset$" is obvious. Also, for all $(x, u, v) \in E \times F \times F$, $(f + g)(x, u, v) = a(x, u) + b(x, v) \geq 0$. Since $L \times F \times F$ is a closed subspace of $E \times F \times F$, Theorem 2.2 now gives $(t^*, u^*, v^*) \in E^* \times F^* \times F^*$ such that

$$f^*(-t^*, -u^*, -v^*) + g^*(t^*, u^*, v^*) \leq 0. \tag{2.4.2}$$

So $f^*(-t^*, -u^*, -v^*) < \infty$, from which $f^*(-t^*, -u^*, -v^*) = a^*(-u^*, -t^*)$ and $v^* = 0$. Similarly, $g^*(t^*, u^*, v^*) = b^*(v^*, t^*)$ and $u^* = 0$. Thus (2.4.2) reduces to

$$a^*(0, -t^*) + b^*(0, t^*) \leq 0.$$ □

We end this section with a bivariate generalization of Theorem 2.2. Apart from some minor changes of notation, this result was first proved in [14, Theorem 4.2, pp. 9–10]. The hypothesis of Theorem 2.5 is that $h(x, \cdot)$ is the inf–convolution of $f(x, \cdot)$ and $g(x, \cdot)$, and the conclusion is that $h^*(y^*, \cdot)$ is the exact inf–convolution of $f^*(y^*, \cdot)$ and $g^*(y^*, \cdot)$. The proof given here using Lemma 2.4 is somewhat simpler than that given in [14].

Theorem 2.5. *Let E and F be nonzero Banach spaces, $f,\ g\colon\ E\times F \mapsto\,]-\infty, \infty]$ be convex and lower semicontinuous, $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big]$ be a closed subspace of E and, for all $(x, y) \in E \times F$,*

$$h(x, y) := \inf\big\{f(x, u) + g(x, v)\colon\ u,\, v \in F,\ u + v = y\big\} > -\infty.$$

Then, for all $(y^*, x^*) \in F^* \times E^* = (E \times F)^*$,

$$h^*(y^*, x^*) = \min\big\{f^*(y^*, s^*) + g^*(y^*, t^*)\colon\ s^*,\, t^* \in E^*,\ s^* + t^* = x^*\big\}.$$

Proof. h is convex and, since $\pi_1 \operatorname{dom} f \cap \pi_1 \operatorname{dom} g \neq \emptyset$, h is proper. It is easy to see that

$$h^*(y^*, x^*) \leq \inf\big\{f^*(y^*, s^*) + g^*(y^*, t^*)\colon\ s^*,\, t^* \in E^*,\ s^* + t^* = x^*\big\}.$$

So what we have to prove is that there exists $t^* \in E^*$ such that

$$f^*(y^*, x^* - t^*) + g^*(y^*, t^*) \leq h^*(y^*, x^*). \tag{2.5.1}$$

Since h is proper, $h^*(y^*, x^*) > -\infty$, so we can and will suppose that $h^*(y^*, x^*) \in \mathbb{R}$. Define $a,\ b\colon\ E \times F \mapsto\,]-\infty, \infty]$ by $a(x, u) := h^*(y^*, x^*) + f(x, u) - \langle x, x^*\rangle - \langle u, y^*\rangle$ and $b(x, v) := g(x, v) - \langle v, y^*\rangle$. Then, for all $(x, u, v) \in E\times F\times F$, the Fenchel–Young inequality implies that

$$\begin{aligned} a(x, u) + b(x, v) &= h^*(y^*, x^*) + f(x, u) - \langle x, x^*\rangle - \langle u, y^*\rangle + g(x, v) - \langle v, y^*\rangle \\ &\geq h^*(y^*, x^*) + h(x, u + v) - \langle x, x^*\rangle - \langle u + v, y^*\rangle \geq 0. \end{aligned}$$

Lemma 2.4 now gives $t^* \in E^*$ such that $a^*(0,-t^*) + b^*(0,t^*) \le 0$. By direct computation,

$$a^*(0,-t^*) = f^*(y^*, x^* - t^*) - h^*(y^*, x^*) \quad \text{and} \quad b^*(0,t^*) = g^*(y^*, t^*),$$

which implies (2.5.1). □

3. LC–functions on $E \times E^*$

From now on, E is a nonzero reflexive Banach space and E^* is its topological dual space. We norm $E \times E^*$ by $\big\|(x,x^*)\big\| := \sqrt{\|x\|^2 + \|x^*\|^2}$. Then

$$\big(E \times E^*, \|\cdot\|\big)^* = \big(E \times E^*, \|\cdot\|\big),$$

under the duality $\big\lfloor (x,x^*),(y,y^*)\big\rfloor := \langle x,y^*\rangle + \langle y,x^*\rangle$. We define $q\colon\ E \times E^* \mapsto \mathbb{R}$ by $q(v) := \frac{1}{2}\lfloor v,v\rfloor \quad (v \in E \times E^*)$. ("$q$" stands for "quadratic".) Thus $q(x,x^*) = \langle x,x^*\rangle$ and $\big|q(x,x^*)\big| \le \|x\|\|x^*\| \le \frac{1}{2}\big(\|x\|^2 + \|x^*\|^2\big) = \frac{1}{2}\big\|(x,x^*)\big\|^2$. So

$$v \in E \times E^* \quad \Longrightarrow \quad \big|q(v)\big| \le \tfrac{1}{2}\big\|v\big\|^2. \tag{3.0.1}$$

Clearly, $\ v,u \in E \times E^* \Longrightarrow \lfloor v,u\rfloor = \lfloor u,v\rfloor,\ $ and we also have the *parallelogram law*

$$v,u \in E \times E^* \quad \Longrightarrow \quad q(v) + q(u) = \tfrac{1}{2}q(v+u) + \tfrac{1}{2}q(v-u).$$

If $v = (x,x^*) \in E \times E^*$ and $u = (y,y^*) \in E \times E^*$ then

$$\begin{cases} q(v-u) = \frac{1}{2}\lfloor v-u, v-u\rfloor = \frac{1}{2}\lfloor v,v\rfloor + \frac{1}{2}\lfloor u,u\rfloor - \lfloor v,u\rfloor \\ \qquad\qquad = \langle x,x^*\rangle + \langle y,y^*\rangle - \langle x,y^*\rangle - \langle y,x^*\rangle = \langle x-y, x^*-y^*\rangle. \end{cases} \tag{3.0.2}$$

The following result appears in [4, Theorem 3.1, pp. 2381–2382] and [7, Proposition 4$\big($h$\big)\!\Longrightarrow\!$(a), pp. 860–861]. See also [5, Section 2].

Lemma 3.1. *If $f\colon\ E \times E^* \mapsto \,]-\infty,\infty]$ is proper and convex and $f \ge q$ on $E \times E^*$ then the set $M(f) := \big\{v \in E \times E^*\colon\ f(v) = q(v)\big\}$ is a monotone subset of $E \times E^*$.*

Proof. Let $v,u \in M(f)$. Then, from the parallelogram law and the convexity of f,

$$\tfrac{1}{2}q(v-u) = q(v) + q(u) - \tfrac{1}{2}q(v+u) \ge f(v) + f(u) - 2f\big(\tfrac{1}{2}(v+u)\big) \ge 0,$$

and the result follows from (3.0.2). □

Definition 3.2. We say that $h\colon\ E \times E^* \mapsto \,]-\infty,\infty]$ is an *LC–function* if h is proper and convex and

$$h^* \ge h \ge q \text{ on } E \times E^*. \tag{3.2.1}$$

"LC" stands for "larger conjugate".

Remark 3.3. There is an extensive discussion in [4] and [8] of lower semicontinuous convex functions $h\colon\ E \times E^* \mapsto \,]-\infty,\infty]$ such that $h \ge q$ and $h^* \ge q$ on $E \times E^*$. Clearly, any lower semicontinuous LC–function has this property. As we will see, the class of LC–functions has a very rich theory. For instance, it will be proved in Theorem 3.9 and Corollary 3.10 that if $f\colon\ E \times E^* \mapsto \,]-\infty,\infty]$ is an LC–function then $M(f) = M(f^*)$ and $M(f)$ is a maximal monotone subset of $E \times E^*$. Furthermore, it is not necessary to assume $\big($as was done in [4, Theorem 3.1]$\big)$ that f is lower semicontinuous. (However, it was pointed out to the author by

Constantin Zălinescu that this restriction can easily be removed using the argument of [8, Proposition 2.4].) We will indicate when results on LC–functions can be deduced from known results. By and large, the proofs in the LC–function case are somewhat simpler.

The motivation for Definition 3.4 below will become clear in Theorem 4.1.

Definition 3.4. We say that $h\colon\ E \times E^* \mapsto\ ]-\infty,\infty]$ is an *SLC–function* if h is proper and convex, $h \geq q$ on $E \times E^*$ and

$$u \in E \times E^* \quad\Longrightarrow\quad h(u) = \sup\nolimits_{v\in M(h)} \big[\lfloor u,v\rfloor - q(v)\big]. \tag{3.4.1}$$

Since

$$\sup_{v\in M(h)} \big[\lfloor u,v\rfloor - q(v)\big] = \sup_{v\in M(h)} \big[\lfloor u,v\rfloor - h(v)\big] \leq \sup_{v\in E\times E^*} \big[\lfloor u,v\rfloor - h(v)\big] =: h^*(u),$$

an SLC–function is automatically an LC–function. (3.4.1) also implies that an SLC–function is automatically lower semicontinuous. "SLC" stands for "strongly larger conjugate".

If $S\colon\ E \rightrightarrows E^*$ is a multifunction then we use the standard notation

$$G(S) := \big\{(x,x^*) \in E \times E^*\colon\ x^* \in Sx\big\}.$$

Example 3.5. Define $j\colon\ E \times E^* \mapsto \mathbb{R}$ by $j := \frac{1}{2}\|\cdot\|^2$. It is well known that $j^* = j$ on $E \times E^*$, thus (3.0.1) implies that j is an LC–function. Furthermore,

$$(x,x^*) \in M(j) \iff \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2 = \langle x,x^*\rangle \iff (x,x^*) \in G(J),$$

where $J\colon\ E \rightrightarrows E^*$ is the duality map. The above equivalence tells us that J is the subdifferential of the continuous convex function defined on E by $x \mapsto \frac{1}{2}\|x\|^2$, but we will not use this fact. Now let $z \in E$ and $\|z\| = 1$. Then, for all $v = (x,x^*) \in M(j)$, $q(v) = \langle x,x^*\rangle = \|x^*\|^2$ and so

$$\lfloor (z,0),v\rfloor - q(v) = \langle z,x^*\rangle - \langle x,x^*\rangle \leq \|x^*\| - \|x^*\|^2 \leq \tfrac{1}{4}.$$

Since $j\big((z,0)\big) = \frac{1}{2}$, j is not an SLC–function.

Lemma 3.6 on translating an LC–function by an element of $E \times E^*$ will be used explicitly in Theorem 3.8, Theorem 3.9, Theorem 3.14, and Lemma 4.4.

Lemma 3.6. *Let $h\colon\ E \times E^* \mapsto\ ]-\infty,\infty]$ be an LC–function and $u \in E \times E^*$. We define $h_u\colon\ E \times E^* \mapsto\ ]-\infty,\infty]$ by $h_u := h(\cdot + u) - \lfloor \cdot,u\rfloor - q(u)$. Then h_u is an LC–function,* $\operatorname{dom} h_u = \operatorname{dom} h - u$*, $M(h_u) = M(h) - u$ and $M({h_u}^*) = M(h^*) - u$.*

Proof. For all $v \in E \times E^*$,

$$\begin{aligned}
{h_u}^*(v) &= \sup\nolimits_{w\in E\times E^*} \big[\lfloor w,v\rfloor + \lfloor w,u\rfloor + q(u) - h(w+u)\big]\\
&= \sup\nolimits_{t\in E\times E^*} \big[\lfloor t-u,v+u\rfloor + q(u) - h(t)\big]\\
&= \sup\nolimits_{t\in E\times E^*} \big[\lfloor t,v+u\rfloor - \lfloor u,v\rfloor - h(t)\big] - q(u) = h^*(v+u) - \lfloor u,v\rfloor - q(u).
\end{aligned}$$

It follows that ${h_u}^*(v) \geq h(v+u) - \lfloor v,u\rfloor - q(u) =: h_u(v)$ and

$$h_u(v) := h(v+u) - \lfloor v,u\rfloor - q(u) \geq q(v+u) - \lfloor v,u\rfloor - q(u) = q(v).$$

Consequently, h_u is an LC–function. It is obvious that $\operatorname{dom} h_u = \operatorname{dom} h - u$. Further, since

$$v \in M(h_u) \iff h(v+u) - \lfloor v, u \rfloor - q(u) = q(v) \\ \iff h(v+u) = q(v+u) \iff v + u \in M(h)$$

and

$$v \in M({h_u}^*) \iff h^*(v+u) - \lfloor u, v \rfloor - q(u) = q(v) \\ \iff h^*(v+u) = q(v+u) \iff v + u \in M(h^*),$$

we have $M(h_u) = M(h) - u$ and $M({h_u}^*) = M(h^*) - u$, as required. □

Lemma 3.7 will be used in Corollary 3.16 and the sum theorem, Theorem 4.2. Lemma 3.7 can also be deduced from [8, Corollary 3.7].

Lemma 3.7. *Let E be a nonzero reflexive Banach space, $f,\ g\colon\ E \times E^* \mapsto\ ]{-\infty}, \infty]$ be lower semicontinuous LC–functions, $\bigcup_{\lambda > 0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big]$ be a closed subspace of E and, for all $(x, x^*) \in E \times E^*$,*

$$h(x, x^*) := \inf\big\{f(x, s^*) + g(x, t^*)\colon\ s^*,\, t^* \in E^*,\ s^* + t^* = x^*\big\}. \tag{3.7.1}$$

Then h is an LC–function, and $(x, x^*) \in M(h^*)$ if, and only if,

there exist s^*, $t^* \in E^*$ such that $(x, s^*) \in M(f^*)$, $(x, t^*) \in M(g^*)$ and $s^* + t^* = x^*$.

Proof. Since $f \ge q$ and $g \ge q$ on $E \times E^*$, (3.7.1) implies that, for all $(x, x^*) \in E \times E^*$,

$$h(x, x^*) \ge \inf\big\{\langle x, s^*\rangle + \langle x, t^*\rangle\colon\ s^*,\, t^* \in E^*,\ s^* + t^* = x^*\big\} = \langle x, x^*\rangle > -\infty,$$

and then Theorem 2.5 and the fact that $f^* \ge f$ and $g^* \ge g$ on $E \times E^*$ give

$$\begin{aligned} h^*(x, x^*) &= \min\big\{f^*(x, s^*) + g^*(x, t^*)\colon\ s^*,\, t^* \in E^*,\ s^* + t^* = x^*\big\}. \\ &\ge \inf\big\{f(x, s^*) + g(x, t^*)\colon\ s^*,\, t^* \in E^*,\ s^* + t^* = x^*\big\} = h(x, x^*). \end{aligned} \tag{3.7.2}$$

Thus h is an LC–function, and the required characterization of $M(h^*)$ is immediate from (3.7.2). □

Theorem 3.8, Theorem 3.9 and Corollary 3.10 below contain subtler property of LC–functions. To put Theorem 3.8 in context, it is not obvious at this point that if f is an LC–function then f^* is proper. The argument of Theorem 3.8 will be used in Theorem 3.9, Corollary 3.13 and Theorem 3.14. Theorem 3.9, which gives a connection between LC–functions and maximal monotonicity, and (the unexpected result) Corollary 3.10 will both be used in the later results in this section and the characterization of Fitzpatrick functions in Theorem 4.1.

Theorem 3.8. *Let E be a nonzero reflexive Banach space, $f\colon\ E \times E^* \mapsto\ ]{-\infty}, \infty]$ be an LC–function and $g\colon\ E \times E^* \mapsto \mathbb{R}$ be a continuous LC–function. Then:*

$$M(f^*) + \rho_2\, M(g^*) = E \times E^*. \tag{3.8.1}$$

Proof. Let u be an arbitrary element of $E \times E^*$. From Lemma 3.6, f_u is an LC–function and so

$$\begin{aligned} &v \in E \times E^* \\ &\qquad\Longrightarrow\quad f_u(v) + g \circ \rho_1(v) = f_u(v) + g\big(\rho_1(v)\big) \ge q(v) + q\big(\rho_1(v)\big) = 0. \end{aligned} \tag{3.8.2}$$

Thus, from Theorem 2.1, there exists $w \in E\times E^*$ such that ${f_u}^*(w)+(g\circ\rho_1)^*(-w) \le 0$. Now, by direct computation, $(g\circ\rho_1)^*(-w) = g^*\big(\rho_1(w)\big)$, thus ${f_u}^*(w) + g^*\big(\rho_1(w)\big) \le 0$. Since ${f_u}^* \ge f_u$ and $g^* \ge g$ on $E\times E^*$, it follows by combining this last inequality with (3.8.2) that ${f_u}^*(w) = q(w)$ and $g^*\big(\rho_1(w)\big) = q\big(\rho_1(w)\big)$, that is to say $w \in M({f_u}^*)$ and $\rho_1(w) \in M(g^*)$, from which $-w = \rho_2\circ\rho_1(w) \in \rho_2\, M(g^*)$. From Lemma 3.6,

$$0 = w - w \in M({f_u}^*) + \rho_2\, M(g^*) = M(f^*) - u + \rho_2\, M(g^*).$$

Since this holds for any $u \in E\times E^*$, we have proved (3.8.1). $\square$

Theorem 3.9(b) below also follows from [14, Theorem 1.4(a), p. 4] or [8, Proposition 2.1].

Theorem 3.9. *Let E be a nonzero reflexive Banach space and $f\colon\ E\times E^* \mapsto\ ]-\infty,\infty]$ be an LC–function.*

(a) *Suppose that $u \in E\times E^*$, and*

$$v \in M(f^*) \quad\Longrightarrow\quad q(v-u) \ge 0, \tag{3.9.1}$$

(i.e., u is "monotonically related" to $M(f^*)$). Then $u \in M(f^*)$.

(b) *$M(f^*)$ is a maximal monotone subset of $E\times E^*$.*

Proof. (a) The argument of Theorem 3.8 and Lemma 3.6 (with $g := j$) give $w \in M({f_u}^*) = M(f^*) - u$ such that $\rho_1(w) \in M(j^*)$. It follows from (3.9.1) that $q(w) \ge 0$. Thus $\frac{1}{2}\big\|\rho_1(w)\big\|^2 = j^*\big(\rho_1(w)\big) = q\big(\rho_1(w)\big) = -q(w) \le 0$. Consequently, $\rho_1(w) = 0$, from which $w = 0$. Thus $0 \in M(f^*) - u$, and so $u \in M(f^*)$, as required.

(b) This is immediate from Lemma 3.1 and (a). $\square$

Corollary 3.10. *Let E be a nonzero reflexive Banach space and $f\colon\ E\times E^* \mapsto\ ]-\infty,\infty]$ be an LC–function. Then $M(f) = M(f^*)$.*

Proof. Let $u \in M(f)$ and $v \in M(f^*)$. Then, from the Fenchel–Young inequality,

$$q(v-u) = q(v) + q(u) - \lfloor u,v\rfloor = f^*(v) + f(u) - \lfloor u,v\rfloor \ge 0.$$

Thus (3.9.1) is satisfied, and we obtain from Theorem 3.9(a) that $M(f) \subset M(f^*)$. The result now follows since it is obvious from (3.2.1) that $M(f^*) \subset M(f)$. $\square$

The following simple lemma will be useful in our work on the surjectivity of various multifunctions. The idea goes back to [12, Theorem 10.7, p. 38].

Lemma 3.11. *Let E be a nonzero reflexive Banach space and $B,\ C \subset E\times E^*$. Then:*

(a) *If $B + \rho_2\, C = E\times E^*$ and $x \in E$ then there exist $(y,y^*) \in B$ and $(z,y^*) \in C$ such that $y + z = x$.*

(b) *If $B + \rho_1\, C = E\times E^*$ and $x^* \in E^*$ then there exist $(y,y^*) \in B$ and $(y,z^*) \in C$ such that $y^* + z^* = x^*$.*

Proof. In (a), there exist $(y,y^*) \in B$ and $(z,z^*) \in C$ such that $(y,y^*) + (z,-z^*) = (x,0)$, and (a) follows since this implies that $z^* = y^*$. The proof of (b), is similar. $\square$

Lemma 3.12 will be used in the three results following it.

Lemma 3.12. *Let $g\colon\ E \times E^* \mapsto\ ]-\infty, \infty]$ be an LC–function and $h\colon\ E \times E^* \mapsto\ ]-\infty, \infty]$ be defined by $h(u) := g(-u)$. Then h is an LC–function,* $\operatorname{dom} h = -\operatorname{dom} g$ *and $M(h) = -M(g)$.*

Proof. Immediate. □

Corollary 3.13 will be used in Theorem 6.1, which contains Rockafellar's surjectivity theorem.

Corollary 3.13. *Let E be a nonzero reflexive Banach space, $f\colon\ E\times E^* \mapsto\ ]-\infty, \infty]$ be an LC–function and $g\colon\ E \times E^* \mapsto \mathbb{R}$ be a continuous LC–function. Then:*

(a) *$M(f) + \rho_2\, M(g) = E \times E^*$ and $M(f) + \rho_1\, M(g) = E \times E^*$.*

(b) *If $x \in E$ then there exist $(y, y^*) \in M(f)$ and $(z, y^*) \in M(g)$ such that $y + z = x$.*

(c) *If $x^* \in E^*$ then there exist $(y, y^*) \in M(f)$ and $(y, z^*) \in M(g)$ such that $y^* + z^* = x^*$.*

(d) *$M(f) + G(-J) = E \times E^*$.*

(e) *If $x^* \in E^*$ then there exist $(y, y^*) \in M(f)$ and $(y, z^*) \in G(J)$ such that $y^* + z^* = x^*$.*

Proof. (a) is immediate from Theorem 3.8, Corollary 3.10 and Lemma 3.12, (b,c) follow from (a) and Lemma 3.11, and (d,e) are immediate from (a,c) with $g := j$. □

The remaining results in this section depend on Theorem 2.2 rather than Theorem 2.1. (Theorem 3.14 and Corollary 3.15 should be compared with Corollary 3.13.) Theorem 3.14 will be used in the surjectivity result, Theorem 6.2. Corollary 3.15 will be used in the surjectivity result, Theorem 6.4. Finally, Corollary 3.16 will be used in the sum theorem, Theorem 4.2. Theorem 3.14 is very close to [16, Corollary 7].

Theorem 3.14. *Let E be a nonzero reflexive Banach space and $f, g\colon\ E \times E^* \mapsto\ ]-\infty, \infty]$ be lower semicontiuous LC–functions. Then:*

(a) $\operatorname{int}\big[M(f) + \rho_2\, M(g)\big] = \operatorname{int}\big[\operatorname{dom} f + \rho_2 \operatorname{dom} g\big]$.

(b) $\operatorname{int}\big[M(f) + \rho_1\, M(g)\big] = \operatorname{int}\big[\operatorname{dom} f + \rho_1 \operatorname{dom} g\big]$.

Proof. Let u be an arbitrary element of $\operatorname{int}\big[\operatorname{dom} f + \rho_2 \operatorname{dom} g\big]$. From Lemma 3.6,

$$0 \in \operatorname{int}\big[\operatorname{dom} f_u + \rho_2 \operatorname{dom} g\big] = \operatorname{int}\big[\operatorname{dom} f_u - \operatorname{dom}(g \circ \rho_1)\big].$$

Arguing as in Theorem 3.8, but using Theorem 2.2 instead of Theorem 2.1, and then appealing to Corollary 3.10, we see that $u \in M(f^*) + \rho_2\, M(g^*) = M(f) + \rho_2\, M(g)$. Thus we have proved that $\operatorname{int}\big[\operatorname{dom} f + \rho_2 \operatorname{dom} g\big] \subset M(f) + \rho_2\, M(g)$, which clearly implies that $\operatorname{int}\big[\operatorname{dom} f + \rho_2 \operatorname{dom} g\big] \subset \operatorname{int}\big[M(f) + \rho_2\, M(g)\big]$. (a) now follows since the opposite inclusion is immediate, and Lemma 3.12 gives (b). □

Corollary 3.15. *Let E be a nonzero reflexive Banach space and $f, g\colon\ E \times E^* \mapsto\ ]-\infty, \infty]$ be lower semicontiuous LC–functions such that $\pi_2 \operatorname{dom} g = E^*$ and, for some $w^* \in E^*$, $E \times \{w^*\} \subset \operatorname{dom} f$. Then:*

(a) *$M(f) + \rho_2\, M(g) = E \times E^*$ and $M(f) + \rho_1\, M(g) = E \times E^*$.*

(b) *If $x \in E$ then there exist $(y, y^*) \in M(f)$ and $(z, y^*) \in M(g)$ such that $y + z = x$.*

(c) *If* $x^* \in E^*$ *then there exist* $(y, y^*) \in M(f)$ *and* $(y, z^*) \in M(g)$ *such that* $y^* + z^* = x^*$.

Proof. (a) Let (x, x^*) be an arbitrary element of $E \times E^*$. Since $\pi_2 \operatorname{dom} g = E^*$, there exists $y \in E$ such that $(y, w^* - x^*) \in \operatorname{dom} g$. The choice of w^* now implies that $(x - y, w^*) \in \operatorname{dom} f$. But then

$$(x, x^*) = (x-y, w^*) + (y, x^* - w^*) = (x-y, w^*) + \rho_2(y, w^* - x^*) \in \operatorname{dom} f + \rho_2 \operatorname{dom} g.$$

Thus we have proved that $\operatorname{dom} f + \rho_2 \operatorname{dom} g = E \times E^*$. Theorem 3.14 now implies that $M(f) + \rho_2 M(g) = E \times E^*$, and Lemma 3.12 that $M(f) + \rho_1 M(g) = E \times E^*$.

(b, c) These follow from (a) and Lemma 3.11. □

Corollary 3.16 below can also be deduced from [8, Corollary 3.7].

Corollary 3.16. *Let* E *be a nonzero reflexive Banach space,* $f,\ g\colon\ E \times E^* \mapsto\ ]-\infty, \infty]$ *be lower semicontinuous LC–functions and* $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big]$ *be a closed subspace of* E*. Then* $\big\{(x, s^* + t^*)\colon\ (x, s^*) \in M(f),\ (x, t^*) \in M(g)\big\}$ *is a maximal monotone subset of* $E \times E^*$

Proof. This is immediate from Lemma 3.7, Theorem 3.9 and Corollary 3.10. □

4. Fitzpatrick functions

Let E be a nonzero reflexive Banach space. In this section, we define the Fitzpatrick function of a nonempty subset of $E \times E^*$, characterize which convex functions on $E \times E^*$ are the Fitzpatrick functions of a maximal monotone set, and use Fitzpatrick functions and Corollary 3.16 to give a sufficient condition for the sum of maximal monotone multifunctions to be maximal monotone.

Let A be a nonempty subset of $E \times E^*$. We define the *Fitzpatrick function* $\varphi_A\colon\ E \times E^* \mapsto\ ]-\infty, \infty]$ associated with A by

$$\varphi_A(u) := \sup_{a \in A} \big[\lfloor u, a \rfloor - q(a)\big]. \tag{4.0.1}$$

(The function φ_A was introduced by Fitzpatrick in [5, Definition 3.1, p. 61].) From (3.0.2), A is monotone $\iff$ for all $u, a \in A$, $q(u - a) \geq 0$ $\iff$ for all $u, a \in A$, $\lfloor u, a \rfloor - q(a) \leq q(u)$ $\iff$ $\varphi_A \leq q$ on A. (This and many subtler criteria for monotonicity in terms of Fitzpatrick functions can be found in [7, Proposition 4, pp. 860–861] and [6, Proposition 2, p. 25].) Consequently, if A is monotone then $A \subset \operatorname{dom} \varphi_A$ and so, since $\operatorname{dom} \varphi_A$ is convex,

$$A \subset \operatorname{co} A \subset \operatorname{dom} \varphi_A, \tag{4.0.2.}$$

where "co" stands for "convex hull". In fact, $\operatorname{dom} \varphi_A$ can be much larger than A. For instance, let $A = G(J)$. It is easily verified that $\varphi_A \leq \frac{1}{2}\|\cdot\|^2 =: j$ on $E \times E^*$, and consequently $\operatorname{dom} \varphi_A = E \times E^*$.

Suppose now that A is maximal monotone. Then (4.0.1) implies that

$$\varphi_A \geq q \text{ on } E \times E^* \textit{and} \quad M(\varphi_A) = A \tag{4.0.3}$$

(see [5, Corollary 3.9, p. 62]). Combining this with (4.0.1), we derive that $\varphi_A(u) := \sup_{a \in M(\varphi_A)} \big[\lfloor u, a \rfloor - q(a)\big]$, in other words,

$$\varphi_A \text{ is an SLC–function} \tag{4.0.4}$$

(see Definition 3.4). Example 3.5 provides an example of an LC–function that is not the Fitzpatrick function of some maximal monotone set A.

Theorem 4.1 below can also be deduced from [5, Theorem 3.8, p. 62].

Theorem 4.1. *Let E be a nonzero reflexive Banach space. Then the mapping $A \mapsto \varphi_A$ is a bijection from the maximal monotone subsets of $E \times E^*$ onto the SLC-functions on $E \times E^*$. The inverse mapping is $M(\cdot)$.*

Proof. It is clear from (4.0.3) and (4.0.4) that the mapping $A \mapsto \varphi_A$ is an injection from the maximal monotone subsets of $E \times E^*$ into the SLC-functions on $E \times E^*$. Suppose, conversely, that $f\colon\ E \times E^* \mapsto \,]-\infty, \infty]$ is an SLC–function. Then, from Theorem 3.9 and Corollary 3.10, $M(f)$ is a maximal monotone subset of $E \times E^*$, and Definition 3.4 implies that $\varphi_{M(f)} = f$, so $M(\cdot)$ is an injection from the SLC–functions on $E \times E^*$ into the maximal monotone subsets of $E \times E^*$. The result is now immediate. □

Theorem 4.2. *Let E be a nonzero reflexive Banach space, $S, T\colon\ E \rightrightarrows E^*$ be maximal monotone and $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} \varphi_{G(S)} - \pi_1 \operatorname{dom} \varphi_{G(T)}\big]$ be a closed subspace of E. Then $S + T$ is maximal monotone.*

Proof. This is immediate from (4.0.4) and Corollary 3.16, with $f := \varphi_{G(S)}$ and $g := \varphi_{G(T)}$. □

Remark 4.3. Theorem 4.2 can be bootstrapped into the following result. See [14, Theorem 5.5, p. 13]. *Let E be a reflexive Banach space and $S, T\colon\ E \rightrightarrows E^*$ be maximal monotone. Suppose there exists a closed subspace F of E such that $D(S) - D(T) \subset F \subset \bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} \varphi_{G(S)} - \pi_1 \operatorname{dom} \varphi_{G(T)}\big]$. Then $S + T$ is maximal monotone. Furthermore, for all $\varepsilon > 0$, $D(S) - D(T) \subset \pi_1 \operatorname{dom} \varphi_{G(S)} - \pi_1 \operatorname{dom} \varphi_{G(T)} \subset (1 + \varepsilon)\big[D(S) - D(T)\big]$, and $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} \varphi_S - \pi_1 \operatorname{dom} \varphi_{G(T)}\big] = \bigcup_{\lambda>0} \lambda\big[D(S) - D(T)\big]$.*

We end this section with a simple result that will be used in Lemma 7.1.

Lemma 4.4. *If A is a maximal monotone subset of $E \times E^*$ and $u \in E \times E^*$ then $\varphi_{A-u} = \big(\varphi_A\big)_u$, and consequently $\operatorname{dom} \varphi_{A-u} = \operatorname{dom} \varphi_A - u$.*

Proof. Immediate, from Lemma 3.6. □

5. The fitpatrification of a monotone multifunction

If $S\colon\ E \rightrightarrows E^*$ is a multifunction then we use the standard notation

$$D(S) := \{x \in E\colon\ Sx \neq \emptyset\} = \pi_1 G(S) \quad \text{and} \quad R(S) := \textstyle\bigcup_{x \in E} Sx = \pi_2 G(S).$$

(4.0.2) implies that if S is monotone then $G(S) \subset \operatorname{co} G(S) \subset \operatorname{dom} \varphi_{G(S)}$, from which $D(S) \subset \operatorname{co} D(S) \subset \pi_1 \operatorname{dom} \varphi_{G(S)}$ and $R(S) \subset \operatorname{co} R(S) \subset \pi_2 \operatorname{dom} \varphi_{G(S)}$.

Definition 5.1. Let $S\colon\ E \rightrightarrows E^*$ be a nontrivial monotone multifunction. The *fitpatrification* of S is the multifunction $S_F\colon\ E \rightrightarrows E^*$ defined by $G\big(S_F\big) = \operatorname{dom} \varphi_{G(S)}$. It is clear from (4.0.2) that $G(S) \subset G\big(S_F\big)$. It is also clear that $\pi_1 \operatorname{dom} \varphi_{G(S)} = D\big(S_F\big)$ and $\pi_2 \operatorname{dom} \varphi_{G(S)} = R\big(S_F\big)$. The discussion following (4.0.2) tells us that $G\big(S_F\big)$ can be a much larger set than $G(S)$, since $G\big(J_F\big) = E \times E^*$. It is this observation that makes Lemmas 5.3 and 7.1 below rather surprising.

Lemma 5.2. *Let E be a nonzero reflexive Banach space and $S, T\colon\ E \rightrightarrows E^*$ be nontrivial and monotone. Then $R\big(S_F + T_F\big) \subset R\big((S+T)_F\big)$.*

Proof. Let x^* be an arbitrary element of $R\big(S_F + T_F\big)$. Then there exist $x \in E$, $p^* \in S_F(x)$ and $q^* \in T_F(x)$ such that $p^* + q^* = x^*$. Consequently, $\varphi_{G(S)}(x, p^*) < \infty$ and $\varphi_{G(T)}(x, q^*) < \infty$. If (y, y^*) is an arbitrary element of $G(S+T)$ then there exist $s^* \in Sy$ and $t^* \in Ty$ such that $s^* + t^* = y^*$. But then

$$\begin{aligned}\langle x, y^*\rangle + \langle y, x^*\rangle - \langle y, y^*\rangle &= \langle x, s^* + t^*\rangle + \langle y, p^* + q^*\rangle - \langle y, s^* + t^*\rangle \\ &= \big[\langle x, s^*\rangle + \langle y, p^*\rangle - \langle y, s^*\rangle\big] + \big[\langle x, t^*\rangle + \langle y, q^*\rangle - \langle y, t^*\rangle\big] \\ &\le \varphi_{G(S)}(x, p^*) + \varphi_{G(T)}(x, q^*).\end{aligned}$$

It follows by taking the supremum over $(y, y^*) \in G(S+T)$ that

$$\varphi_{G(S+T)}(x, x^*) \le \varphi_{G(S)}(x, p^*) + \varphi_{G(T)}(x, q^*) < \infty,$$

and so $x^* \in (S+T)_F x \subset R\big((S+T)_F\big)$. □

Lemma 5.3 below can also be deduced from [8, Corollary 3.7]. As pointed out in [13], it actually goes back to [12, Theorem 18.3, p. 67] and [14, Remark 5.6, pp. 13–14].

Lemma 5.3. *Let E be a nonzero reflexive Banach space and $U\colon\ E \rightrightarrows E^*$ be maximal monotone. Then* $\operatorname{int} D(U) = \operatorname{int} D\big(U_F\big)$ *and* $\operatorname{int} R(U) = \operatorname{int} R\big(U_F\big)$.

Proof. This follows from [13, Theorem 2.2], applied to $S := U$ and $S := U^{-1}$. □

Corollary 5.4 below is actually a partial result. The full result appears in Theorem 7.2.

Corollary 5.4. *Let E be a nonzero reflexive Banach space, $S, T\colon\ E \rightrightarrows E^*$ be monotone, and $S+T$ be maximal monotone. Then* $\operatorname{int} R\big(S_F + T_F\big) = \operatorname{int} R(S+T)$.

Proof. The inclusion "$\supset$" is immediate since $G(S_F) \supset G(S)$ and $G(T_F) \supset G(T)$, and the inclusion "$\subset$" follows from Lemmas 5.2 and 5.3. □

6. Surjectivity results

In this section, we prove various surjectivity results, including Rockafellar's surjectivity theorem (Theorem 6.1) and an abstract Hammerstein theorem (Theorem 6.5).

Compare the proof of Theorem 6.1 below with that of [12, Theorem 10.6, p. 37]. There is an extensive discussion the issues raised following that reference and in [12, Remark 10.8, pp. 38–39]. Theorem 6.1(b) is "Rockafellar's surjectivity theorem" — see [11, Proposition 1, p. 77] for the original proof depending ultimately on Brouwer's fixed–point theorem and an Asplund renorming. The proof given here is a simplification of that given in [14, Theorem 3.1(b), p. 8]. We note that this reference also provides the exact value of $\min\big\{\|x\|\colon\ x \in E,\ (S+J)x \ni 0\big\}$ in terms of $\varphi_{G(S)}$.

Theorem 6.1. *Let E be a nonzero reflexive Banach space and $S\colon\ E \rightrightarrows E^*$ be monotone. Then:*

(a) *S is maximal monotone* $\iff G(S)+G(-J)=E\times E^*$.
(b) *If S is maximal monotone then $S+J$ is surjective.*

Proof. (a)($\Longleftarrow$) is immediate (and is valid even when E is not reflexive), and (a)($\Longrightarrow$) and (b) follow from (4.0.4), (4.0.3) and Corollary 3.13(d,e) with $f:=\varphi_{G(S)}$. □

Since $G\big(J_F\big)=E\times E^*$, the following result generalizes Theorem 6.1(b). We note that it ultimately uses Theorem 2.2, which is a much harder result than Theorem 2.1. It can also be deduced from [16, Theorem 3 and Corollary 4].

Theorem 6.2. *Let E be a nonzero reflexive Banach space, $S,T\colon\ E\rightrightarrows E^*$ be maximal monotone and $G\big(S_F\big)+\rho_1\,G\big(T_F\big)=E\times E^*$. Then $S+T$ is surjective.*

Proof. This follows from (4.0.4), Theorem 3.14(b), (4.0.3) and Lemma 3.11(b), with $f:=\varphi_{G(S)}$ and $g:=\varphi_{G(T)}$. □

In Theorem 6.3, we relax the condition of maximal monotonicity for S and T, and assume it instead for $S+T$.

Theorem 6.3. *Let E be a nonzero reflexive Banach space, $S,T\colon\ E\rightrightarrows E^*$ be monotone, $S+T$ be maximal monotone and S_F+T_F be surjective. Then $S+T$ is surjective.*

Proof. This is immediate from Corollary 5.4. □

We now reverse the direction of T.

Theorem 6.4. *Let E be a nonzero reflexive Banach space and $S\colon\ E\rightrightarrows E^*$ and $T\colon\ E^*\rightrightarrows E$ be maximal monotone. Suppose that $D\big(T_F\big)=E^*$ and $\bigcap_{x\in E}S_F(x)\neq\emptyset$. Then:*
(a) *If I_E is the identity map on E, $(I_E+TS)(E)=E$.*
(b) *If I_{E^*} is the identity map on E^*, $(I_{E^*}+ST)(E^*)=E^*$.*

Proof. Let $f:=\varphi_{G(S)}$ and $g(x,x^*):=\varphi_{G(T)}(x^*,x)$, so that $\pi_2\operatorname{dom}g=E^*$.
(a) Let x be an arbitrary element of E. From (4.0.4) and Corollary 3.15(b), there exist $(y,y^*)\in M(f)$ and $(z,y^*)\in M(g)$ such that $y+z=x$. From (4.0.3), $(y,y^*)\in G(S)$ and $(y^*,z)\in G(T)$, so $z\in Ty^*\subset TSx$ and $x=y+z\in(I_E+TS)(y)\subset(I_E+TS)(E)$.
(b) Let x^* be an arbitrary element of E^*. From (4.0.4) and Corollary 3.15(c), there exist $(y,y^*)\in M(f)$ and $(y,z^*)\in M(g)$ such that $y^*+z^*=x^*$. From (4.0.3), $(y,y^*)\in G(S)$ and $(z^*,y)\in G(T)$, so $y^*\in Sy\subset STz^*$ and $x^*=z^*+y^*\in(I_{E^*}+ST)(E^*)(z^*)\subset(I_{E^*}+ST)(E^*)$. □

The next result is a considerable generalization of [17, Theorem 32.O, p. 909] which, in turn, was applied to Hammerstein integral equations. See Remark 6.6 below.

Theorem 6.5. *Let E be a nonzero reflexive Banach space, $S\colon\ E\rightrightarrows E^*$ and $T\colon\ E^*\rightrightarrows E$ be maximal monotone. Suppose that* **either** *$D\big(T_F\big)=E^*$ and $\bigcap_{x\in E}S_F(x)\neq\emptyset$* **or** *$D\big(S_F\big)=E$ and $\bigcap_{x^*\in E^*}T_F(x^*)\neq\emptyset$. Then $(I_E+TS)(E)=E$.*

Proof. The first case has already been established in Theorem 6.4(a), while the second case follows from Theorem 6.4(b), with E replaced by E^* and the roles of S and T interchanged. $\square$

Remark 6.6. We now compare Theorem 6.5 and [17, Theorem 32.O]. [17] assumes that E is a Hilbert space, while Theorem 6.5 assumes that E is a reflexive Banach space. Lemma 5.3 tells us that the assumptions $D(T) = E^*$ ($D(S) = E$, respectively) of [17] are equivalent to the assumptions $D\big(T_F\big) = E^*$ ($D\big(S_F\big) = E$, respectively) of Theorem 6.5. Finally, [17] assumes that $\bigcap_{x\in E} S_F(x) \supset R(S)$ $\big(\bigcap_{x^*\in E^*} T_F(x^*) \supset R(T)$, respectively$\big)$ while Theorem 6.5 makes the weaker assumption that $\bigcap_{x\in E} S_F(x) \neq \emptyset$ $\big(\bigcap_{x^*\in E^*} T_F(x^*) \neq \emptyset$, respectively).

7. Brezis–Haraux approximation

Let E be a nonzero reflexive Banach space and $S\colon\ E \rightrightarrows E^*$ and $T\colon\ E \rightrightarrows E^*$ be monotone. In this section, we consider when we cas assert that $R\big(S+T\big)$ is close to $R(S)+R(T)$ in the sense of the *Brezis–Haraux condition:*

$$\operatorname{int} R(S+T) = \operatorname{int}\big[R(S)+R(T)\big] \quad \text{and} \quad \overline{R(S+T)} = \overline{R(S)+R(T)}. \tag{7.0.1}$$

Lemma 7.1. *Let E be a nonzero reflexive Banach space and $U\colon\ E \rightrightarrows E^*$ be maximal monotone. Then*

$$R\big(U_F\big) \subset \overline{R(U)}. \tag{7.1.1}$$

In fact, $\overline{R(U)} = \overline{R\big(U_F\big)}$ and (by considering U^{-1}), $\overline{D(U)} = \overline{D\big(U_F\big)}$.

Proof. By virtue of Lemma 4.4, it suffices to prove that $0 \in R\big(U_F\big) \Longrightarrow 0 \in \overline{R(U)}$. Let $0 \in R\big(U_F\big)$ and $\varepsilon \in (0,1)$. Choose $z \in E$ such that $\varphi_{G(U)}(z,0) < \infty$, and let $M \geq \varphi_{G(U)}(z,0)$ and $M \geq \|z\| \geq 0$. Choose $\lambda > 0$ so that

$$\lambda M < \varepsilon^2/5 < 1. \tag{7.1.2}$$

Since U/λ is also maximal monotone, from Theorem 6.1(a), $G(U/\lambda) + G(-J) \ni 0$, hence there exists $(u,u^*) \in G(U)$ such that $\langle u, u^*/\lambda\rangle = -\|u^*/\lambda\|^2$, and so $-\langle u,u^*\rangle = \|u^*\|^2/\lambda$. We also have $\langle z,u^*\rangle \geq -\|z\|\|u^*\| \geq -M\|u^*\|$. The definition of $\varphi_{G(U)}$ gives

$$\langle z,u^*\rangle - \langle u,u^*\rangle \leq \varphi_{G(U)}(z,0) \leq M. \tag{7.1.3}$$

Substituting in for the two expressions on the left hand side of (7.1.3), we obtain $\|u^*\|^2 - \lambda M\|u^*\| - \lambda M \leq 0$. (7.1.2) now implies that

$$\begin{aligned}\|u^*\| &\leq \tfrac{1}{2}\big(\lambda M + \sqrt{\lambda^2M^2 + 4\lambda M}\big) \leq \sqrt{\lambda^2M^2+4\lambda M}\\ &= \sqrt{\lambda M(\lambda M+4)} < \sqrt{(\varepsilon^2/5)(1+4)} = \varepsilon.\end{aligned}$$

This establishes that $0 \in \overline{R(U)}$, and completes the proof of (7.1.1). The rest is immediate from (7.1.1) and the fact that $R(U) \subset R\big(U_F\big)$. $\square$

Theorem 7.2. *Let E be a nonzero reflexive Banach space, $S,T\colon\ E \rightrightarrows E^*$ be monotone, and $S+T$ be maximal monotone. Then*

$$\operatorname{int} R(S+T) = \operatorname{int} R\big(S_F+T_F\big) \textit{and} \quad \overline{R(S+T)} = \overline{R\big(S_F+T_F\big)}.$$

Proof. The first equality has already been established in Corollary 5.4, and the second follows in like fashion from Lemmas 5.2 and 7.1. □

Theorem 7.3. *Let E be a nonzero reflexive Banach space, $S,T\colon\ E \rightrightarrows E^*$ be monotone, $S+T$ be maximal monotone and $R(S)+R(T) \subset R\big(S_F+T_F\big)$. Then the Brezis–Haraux condition,* (7.0.1)*, is satisfied.*

Proof. It follows from Lemma 5.2 that $R(S)+R(T) \subset R\big((S+T)_F\big)$, and then from Lemmas 5.3 and 7.1 that $\operatorname{int}\big[R(S)+R(T)\big] \subset \operatorname{int} R\big((S+T)_F\big) = \operatorname{int} R(S+T)$ and $\overline{R(S)+R(T)} \subset \overline{R\big((S+T)_F\big)} = \overline{R(S+T)}$. The reverse inclusions in (7.0.1) are clear since $R(S+T) \subset R(S)+R(T)$. □

Definition 7.4. We say that a monotone multifunction $S\colon\ E \rightrightarrows E^*$ is *rectangular* if

$$D(S) \times R(S) \subset G\big(S_F\big).$$

It is easily seen that S is rectangular $\iff$ S is "3*–monotone" in the sense of [17, Definition 32.40(c), p. 901] $\iff$ S satisfies property "$(**)$" of [3, p. 166] (when E is a Hilbert space). It follows from [17, Proposition 32.41, p. 902] that if S is monotone with bounded range, or monotone and strongly coercive, or there exists a proper convex function $f\colon\ E \mapsto\ ]-\infty,\infty]$ such that $S = \partial f$, then S is rectangular. This last observation can also be seen directly from the result that appears in [2, Proposition 2.1] that $\operatorname{dom} f \times \operatorname{dom} f^* \subset \operatorname{dom} \varphi_{\partial f}$. Now if E is reflexive and S is maximal monotone then, from Lemmas 5.3 and 7.1,

$$\operatorname{int} G\big(S_F\big) \subset \operatorname{int} D\big(S_F\big) \times \operatorname{int} R\big(S_F\big) = \operatorname{int} D(S) \times \operatorname{int} R(S) \subset \operatorname{int}\big[D(S) \times (R(S)\big]$$

and

$$\overline{G\big(S_F\big)} \subset \overline{D\big(S_F\big)} \times \overline{R\big(S_F\big)} = \overline{D(S)} \times \overline{R(S)} = \overline{D(S) \times R(S)}$$

So, in this case, if S is rectangular then $G\big(S_F\big)$ is almost a rectangle.

The fact that either (7.5.1) or (7.5.2) implies (7.0.1) in Corollary 7.5 below was proved by Brezis and Haraux in Hilbert spaces in [3], with applications to Hammerstein integral equations, partial differential equations with nonlinear boundary conditions, and nonlinear periodic equations of evolution. These results were extended by Reich in [9, Theorem 2.2, p. 315] to the case where E is a nonzero reflexive Banach space. In Corollary 7.5, we do not need to use an Asplund renorming, in contrast to the proofs of the results quoted above.

Corollary 7.5. *Let E be a nonzero reflexive Banach space, $S,T\colon\ E \rightrightarrows E^*$ be monotone, and $S+T$ be maximal monotone. If either*

(7.5.1) *S and T are both rectangular,*

or

(7.5.2) *$D(S) \subset D(T)$ and T is rectangular,*

then $R(S)+R(T) \subset R\big(S_F+T_F\big)$. Consequently, from Theorem 7.3, the Brezis–Haraux condition, (7.0.1)*, is satisfied.*

Proof. Let x^* be an arbitrary element of $R(S) + R(T)$. Then there exist $s^* \in R(S)$ and $t^* \in R(T)$ such that $s^* + t^* = x^*$. If (7.5.1) is satisfied then, since $S + T$ is maximal monotone, there exists $x \in D(S) \cap D(T)$, from which $(x, s^*) \in D(S) \times R(S) \subset G(S_F)$ and $(x, t^*) \in D(T) \times R(T) \subset G(T_F)$, so $x^* = s^* + t^* \in (S_F + T_F)(x) \subset R(S_F + T_F)$. If (7.5.2) is satisfied then we choose $x \in E$ so that $(x, s^*) \in G(S) \subset G(S_F)$, from which $x \in D(S) \subset D(T)$, and so $(x, t^*) \in D(T) \times R(T) \subset G(T_F)$, giving $x^* \in R(S_F + T_F)$ again. □

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References

[1] H. Attouch and H. Brezis, *Duality for the sum of convex funtions in general Banach spaces.*, Aspects of Mathematics and its Applications, J. A. Barroso, ed., Elsevier Science Publishers (1986), 125–133.

[2] H. H. Bauschke, D. A. McLaren and H. S. Sendov, *Fitzpatrick functions of subdifferentials: inequalities and examples*, preprint, March 8, 2005.

[3] H. Brezis and A. Haraux, *Image d'une somme d'opérateurs monotone at applications*, Israel J. Math. **23**(1976), 165–186.

[4] R. S. Burachik and B. F. Svaiter, *Maximal monotonicity, conjugation and the duality product*, Proc. Amer. Math. Soc. **131** (2003), 2379–2383.

[5] S. Fitzpatrick, *Representing monotone operators by convex functions*, Workshop/ Miniconference on Functional Analysis and Optimization (Canberra, 1988), 59–65, Proc. Centre Math. Anal. Austral. Nat. Univ., **20**, Austral. Nat. Univ., Canberra, 1988.

[6] J-E. Martínez-Legaz and B. F. Svaiter, *Monotone Operators Representable by l.s.c. Convex Functions*, Set–Valued Anal., **13** (2005) 21–46.

[7] J.–P. Penot, *The relevance of convex analysis for the study of monotonicity*, Nonlinear Anal. **58** (2004), 855–871.

[8] J.–P. Penot and C. Zălinescu, *Some problems about the representation of monotone operators by convex functions*, ANZIAM J. **47** (2005), 1–20.

[9] S. Reich, *The range of sums of accretive and monotone operators* , J. Math. Anal. Appl. **68** (1979), 310–317.

[10] R. T. Rockafellar, *Extension of Fenchel's duality theorem for convex functions*, Duke Math. J. **33** (1966), 81–89.

[11] —–, *On the Maximality of Sums of Nonlinear Monotone Operators*, Trans. Amer. Math. Soc. **149**(1970),75-88.

[12] S. Simons, *Minimax and monotonicity*, Lecture Notes in Mathematics **1693** (1998), Springer–Verlag.

[13] —–, *Dualized and scaled Fitzpatrick functions*, Proc. Amer. Math. Soc., in press.

[14] S. Simons and C. Zălinescu, *Fenchel duality, Fitzpatrick functions and maximal monotonicity*, J. of Nonlinear and Convex Anal., **6** (2005), 1–22.

[15] C. Zălinescu, *Convex analysis in general vector spaces*, (2002), World Scientific.

[16] —–, *A new convexity property for monotone operators*, J. Convex Anal., in press.

[17] E. Zeidler, *Nonlinear functional analysis and its applications, II/B, Nonlinear monotone operators*, Springer-Verlag, New York, 1990.

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