Journal of Nonlinear and Convex Analysis
Volume 7, Number 1, 2006, 115–122

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ON CONVEX FUNCTIONS WITH VALUES IN CONLINEAR SPACES

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ABSTRACT. The following result of Convex Analysis is well–known, see e.g. [2]: *If the function* $f: X \to [-\infty, +\infty]$ *is convex and some* $x_0 \in \operatorname{core}(\operatorname{dom} f)$ *satisfies* $f(x_0) > -\infty$*, then* f *never takes the value* $-\infty$. From a corresponding statement for convex functions with values in conlinear spaces a variety of results is deduced, among them the mentioned theorem, a theorem of Deutsch and Singer on the single–valuedness of convex set–valued maps as well as a result on the compact–valuedness of convex set–valued maps. We also discuss the possibility of embedding the image points of such a convex function into a linear space.

1. Introduction

Conlinear structures (that means the structure of a convex cone, see [5]) naturally occur in Optimization and Analysis. In many cases, conlinear structures can considered being convex cones in linear spaces. However, the concept of a convex cone is not appropriate in important cases because it is not possible to find a linear space wherein the conlinear structure is a convex cone. Therefore, we start introducing the concept of a conlinear space. We define convexity and convex functions with values in ordered conlinear spaces and we point out a basic principle for such functions. Then we show that this principle is the common basis for a variety of well–known assertions. Some other conclusions seem to be new.

We state a simple condition implying that the embedding of a conlinear space into a linear space is not possible. However, the principle tells us that in the special case of a convex set–valued map the image points are, essentially, part of a linear structure. This is the basis of a duality theory of convex set–valued maps which is based on the ordering relation "set inclusion", see [9].

2. Preliminaries

The concept of a conlinear space and similar concepts were already considered, for instance, by Godini [4] (almost linear spaces), Keimel and Roth [8] (cones) and Hamel [5]. In some of the cited references the axioms slightly differ from our ones, which are adopted to [5].

Let X be a set. On X let an addition $+: X \times X \to X$, a multiplication $\cdot: \mathbb{R}_+ \times X \to X$ by nonnegative real numbers and a neutral element $0_X \in X$ be defined such that for all $x, u, z \in X$ and $\alpha, \beta \in \mathbb{R}_+ := \{\gamma \in \mathbb{R} | \, \gamma \geq 0\}$ the following axioms are satisfied:

(C1) $(x+u)+z = x+(u+z)$;

2000 *Mathematics Subject Classification.* 90C25, 52A41, 52A99, 54C60, 26E25.

Key words and phrases. conlinear space, semilinear space, almost linear space, convex set–valued map.

(C2) $0_X + x = x$;
(C3) $x + u = u + x$;
(C4) $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$;
(C5) $1 \cdot x = x$;
(C6) $\alpha \cdot (x + u) = \alpha \cdot x + \alpha \cdot u$;
(C7) $0 \cdot x = 0_X$.

Then, X is called a *conlinear space*. Compare [1, page 141] for a special case and note that the concept of an almost linear spaces of [4] and that of a semilinear space in [5] additionally involve the multiplication by negative real numbers. The axioms imply that the neutral element is unique and $\alpha \cdot 0_X = 0_X$ for all $\alpha \geq 0$.

In the remainder of this section let X be a conlinear space. A subset $C \subseteq X$ is said to be *convex* if $x, u \in C$ implies $\lambda \cdot x + (1 - \lambda) \cdot u \in C$ for all $\lambda \in [0, 1]$ and a subset $K \subseteq X$ is said to be a *cone* if $x \in K$ implies $\alpha \cdot x \in K$ for all $\alpha > 0$.

Proposition 2.1. *A subset $\{x\} \subseteq X$, consisting of exactly one element $x \in X$, is convex if and only if the "second distributive law" holds, i.e.,*

(C8) $\forall \alpha, \beta \in \mathbb{R}_+ : \ \alpha \cdot x + \beta \cdot x = (\alpha + \beta) \cdot x$.

Proposition 2.2. *Let $X_c \subseteq X$ be the set of all points of X satisfying the second distributive law (C8). Then X_c is a convex cone in X with $0_X \in X_c$.*

Examples of conlinear spaces. (1) Every linear space V.

(2) Every convex cone $C \subseteq X$ of a conlinear space with $0_X \in C$.

(3) The collection $\hat{\mathcal{P}}(X)$ ($\mathcal{P}(X)$) of all (nonempty) subsets of X with the following operations: $A, B \in \hat{\mathcal{P}}(X)$, $\alpha \in \mathbb{R}_+$, $A + B := \{a + b |\ a \in A,\ b \in B\}$, $\alpha \cdot A := \{\alpha \cdot a |\ a \in A\}$, if $\alpha > 0$, $0 \cdot A := \{0_X\}$, see Remark 2.3 below.

(4) Let V be a topological linear space. The space $\hat{\mathcal{F}}(V)$ ($\mathcal{F}(V)$) of all (nonempty) closed subsets of V, where the addition is defined by $A+B := \operatorname{cl}\{a+b |\ a \in A,\ b \in B\}$ and the multiplication is as in the previous example.

(5) The spaces $\hat{\mathcal{P}}_c(X)$, $\mathcal{P}_c(X)$, $\hat{\mathcal{F}}_c(V)$ and $\mathcal{F}_c(V)$ (compare (3), (4), and Proposition 2.2).

(6) Let V be a separated topological linear space. The spaces $\hat{\mathcal{C}}(V) \subseteq \hat{\mathcal{F}}(V)$, ($\mathcal{C}(V) \subseteq \mathcal{F}(V)$) of all (nonempty) compact subsets of V, where the operations are defined as in (3).

(7) The spaces $\hat{\mathcal{C}}_c(V) \subseteq \hat{\mathcal{C}}(V)$, ($\mathcal{C}_c(V) \subseteq \mathcal{C}(V)$) of all (nonempty) convex compact subsets of V, where the operations are defined as in (3).

(8) The space $\mathcal{K}(X)$ of all cones $K \subseteq X$ with $0_X \in K$, and the space $\hat{\mathcal{K}}(X) := \mathcal{K}(X) \cup \{\emptyset\}$, where the operations are defined as in (3).

(9) The space of extended real numbers $\mathbb{R}^\star := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ with the extended operations: $x + (-\infty) = (-\infty) + x = -\infty$ for all $x \in \mathbb{R}^\star \setminus \{\infty\}$, $x + \infty = \infty + x = \infty$ for all $x \in \mathbb{R}^\star$, $\alpha \cdot \pm\infty = \pm\infty$ for all $\alpha > 0$ and $0 \cdot \pm\infty = 0$ (compare [14]).

(10) The space of extended real numbers $\mathbb{R}^\diamond := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ with the extended operations: $x + \infty = \infty + x = \infty$ for all $x \in \mathbb{R}^\diamond \setminus \{-\infty\}$, $x + (-\infty) = (-\infty) + x = -\infty$ for all $x \in \mathbb{R}^\diamond$ and the multiplication as above.

Remark 2.3. In $\hat{\mathcal{P}}(X)$ we use the convention $0 \cdot \emptyset = \{0\}$, although in the literature one usually sets $0 \cdot \emptyset = \emptyset$. Our convention is a natural extension of the rules

$0 \cdot (+\infty) = 0$ and $0 \cdot (-\infty) = 0$, which are usually used in the extended real numbers. A relationship between both conventions is given by the support function in (4) below. By our convention, (4) also holds for $A = \emptyset$.

Note that a subset $\{x\} \subseteq X$ consisting of exactly one element $x \in X$ can be a cone in X, even if $x \neq 0_X$. Such an element $x \in X$ with $x = \alpha \cdot x$ for all $\alpha > 0$ is called a *conical element*. Note that in [5] the term *cone* is used instead. This is because of the following relationship [5]: A subset $K \subseteq X$ is a cone in X if and only if K is a conical element in $\hat{\mathcal{P}}(X)$.

Of course, in every conlinear space X the neutral element 0_X is a conical element, therefore a conical element $x \neq 0_X$ is called *nontrivial*. Let A and B be two nonempty subsets of a conlinear space X. We say A *dominates* B, in short $A \succ B$, if $a \in A$,$b \in B$ implies $a + b \in A$. If there is some $\hat{x} \in X$ such that $\{\hat{x}\} \succ X$, then $\hat{x}$ is called *dominant* in X. A dominant element of a conlinear space X, if it exists, is a conical element and is uniquely defined. Indeed, let $\hat{x} \in X$ be dominant in X. If $\alpha \geq 1$ we have $\alpha\hat{x} = \hat{x} + (1 - \alpha)\hat{x} = \hat{x}$. If $\alpha \in (0, 1)$, we have $\frac{1}{\alpha}\hat{x} = \hat{x}$ and hence $\hat{x} = \alpha\hat{x}$. The uniqueness is obvious. Moreover, the union of all conical elements is a convex cone in X. The following proposition underlines the advantage of considering conlinear spaces instead of convex cones of linear spaces.

Proposition 2.4. *A conlinear space having a nontrivial conical element cannot be embedded into a linear space.*

Proof. Suppose the contrary, i.e., there exists a linear space L such that X is a convex cone in L with a conical element $\hat{x} \neq 0_X$. Then there must be an inverse element $\bar{x}$ and we have $\hat{x}+\bar{x} = 0_L$. It follows $0_L = \bar{x}+\hat{x} = \bar{x}+2\cdot\hat{x} = (\bar{x}+\hat{x})+\hat{x} = \hat{x}$, which contradicts the assumption. □

The preceding proposition shows that a lot of important examples of conlinear spaces, for instance, the spaces of Examples (3) to (10), cannot be treated as convex cones in a linear space. A sufficient condition for embedding a conlinear space into a linear space is discussed in Radström [12, Theorem 1].

In the following, let the conlinear space X be equipped with a partial ordering $\leq$ (i.e., a reflexive, transitive and antisymmetric relation on X). We say $(X, \leq)$ (shortly X) is an *ordered conlinear space* if it holds

$$(1) \qquad x_1 \leq x_2, x_3 \leq x_4 \quad \Rightarrow \quad \alpha \cdot (x_1 + x_3) \leq \alpha \cdot (x_2 + x_4)$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $\alpha \in \mathbb{R}_+$.

Proposition 2.5. *Let X be an ordered conlinear space. Then the largest (smallest) element of X, if it exists, is a conical element.*

Proof. Let $\hat{x}$ be the largest element of X, i.e., $x \leq \hat{x}$ for all $x \in X$. For given $\alpha > 0$, condition (1) yields $\alpha \cdot x \leq \alpha \cdot \hat{x}$ for all $x \in X$. Given some $u \in X$, we have $x := 1/\alpha \cdot u \in X$. Hence for all $\alpha > 0$ and all $u \in X$ it holds $u \leq \alpha \cdot \hat{x}$, i.e., $\alpha \cdot \hat{x}$ is the largest element of X. Since the largest element of a partially ordered set is uniquely defined we get $\alpha \cdot \hat{x} = \hat{x}$ for all $\alpha > 0$. The proof for the smallest element is analogous. □

Since a linear space has no nontrivial conical element, the preceding result means that a partially ordered linear space cannot be order complete (a partially ordered set is said to be order complete if every subset has supremum and infimum [15]). However, every Dedekind complete ordered conlinear space (a partially ordered set is said to be Dedekind complete if every subset which is bounded above (below) has a supremum (infimum) [15]) can be extended to an order complete ordered conlinear space. To see this extend the space by a new element defined to be the largest (smallest) one and being dominant in the extended space. After this, extend the space by a second new element defined to be the smallest (largest) one and being dominant in the extended space (compare Examples (9) and (10)).

Examples of ordered conlinear spaces. (11) Every partially ordered linear space.

(12) The spaces of Examples (3) to (8) equipped with the partial orderings $\subseteq$ and $\supseteq$.

(13) The extended real numbers of Examples (9) and (10) with the usual $\leq$ relation.

Now we are able to give the definition of a convex function. Let $(Y, \leq)$ be an ordered conlinear space and $C \subseteq X$. The set $\operatorname{epi} f := \{(x, y) \in C \times Y |\ f(x) \leq y\}$ is called *epigraph* of f. A function $f : C \to (Y, \leq)$ is said to be *convex* if its epigraph $\operatorname{epi} f$ is a convex subset of $X \times Y$. In this case, C must be convex. It is an easy task to show that a function $f : C \to (Y, \leq)$ is convex if and only if for all $\lambda \in [0, 1]$ and all $x, u \in C$ it holds

$$f(\lambda \cdot x + (1 - \lambda) \cdot u) \leq \lambda \cdot f(x) + (1 - \lambda) \cdot f(u).$$

A convex function $f : C \to Y$, defined on a subset $C \subseteq X$, can be extended to the whole space X, if $(Y, \leq)$ has a largest element $\hat{y}$ which is simultaneously dominant in Y. In this case, the extension $\hat{f} : X \to Y$, defined by $\hat{f}(x) := f(x)$ if $x \in C$ and $\hat{f}(x) = \hat{y}$ elsewhere, is convex. Moreover, the set $\operatorname{dom} f := \{x \in C |\ f(x) \neq \hat{y}\}$ is called the *effective domain* of f. In the spaces of Examples (3) to (8), equipped with the relation $\supseteq$, we have $\hat{y} = \emptyset$, if the empty set belongs to the space. If we take the relation $\subseteq$ instead, we have $\hat{y} = X$ (respectively $\hat{y} = V$) if the empty set does not belong to the space.

It is well–known that in the special case of a function $f : U \to (\mathcal{P}(V), \supseteq)$, where U and V are linear spaces, f is convex if and only if its "graph" $G(f) := \{(u, v) \in U \times V |\ v \in f(u)\}$ is a convex subset of $U \times V$.

3. A basic principle and its conclusions

The following theorem is the essential part of a lot of assertions concerning convex functions (and maps). It states that under certain assumptions to the conlinear structure and to the ordering structure, a convex function cannot attain values in a certain cone of its ordered conlinear image space.

In this section, let X be a linear space and $C \subseteq X$. The core or the algebraic interior of a subset $A \subseteq X$ is denoted by $\operatorname{core} A$ (compare [6]). As usual, for $f : C \to (Y, \leq)$ and $A \subseteq C$ we define $f(A) := \{y \in Y |\ \exists x \in A :\ \ y = f(x)\}$.

Theorem 3.1. *Let $(Y, \leq)$ be an ordered conlinear space, $S \subseteq Y$ a cone, $f : C \to (Y, \leq)$ a convex function and $A \subseteq C$ such that $S \succ f(A)$. If there exists $x_0 \in \operatorname{core} A$ such that $f(x_0) \not\leq s$ for all $s \in S$, then $f(x) \notin S$ for all $x \in C$.*

Proof. Assume $f(x) \in S$. Since $x_0 \in \operatorname{core} A$, we find some $x' \in A$ such that $x_0 = \lambda x' + (1 - \lambda)x$ for some $\lambda \in (0, 1)$. The convexity of f yields $f(x_0) \leq \lambda \cdot f(x') + (1 - \lambda) \cdot f(x) =: s$. Since S is a cone in Y and $\lambda > 0$, $S \succ f(A)$ implies $S \succ \lambda \cdot f(A)$. Consequently, we have $s \in S$. Hence $f(x_0) \leq s$ and $s \in S$ contradicting the assumption. This means $f(x) \notin S$ for all $x \in C$. □

The first corollary is a classical result for convex functions with values in the extended reals $\mathbb{R}^\star$ of Example (9). Note that, for instance, in [13] other calculus rules in the extended reals are used, but the same result is valid.

Corollary 3.2. *Let $f : C \to (\mathbb{R}^\star, \leq)$ be a convex function. If some point $x_0 \in \operatorname{core}(\operatorname{dom} f)$ satisfies $f(x_0) > -\infty$, then f never takes the value $-\infty$.*

Proof. $S = \{-\infty\}$, $A = \operatorname{dom} f$. □

In the following result, we set $\operatorname{lev}_f(\alpha) = \{x \in C |\ f(x) = \alpha\}$.

Corollary 3.3. *Let $f : C \to \mathbb{R}$ be a convex function. If $x_0 \in \operatorname{core} \operatorname{lev}_f(0)$, then $f(x) \geq 0$ for all $x \in C$.*

Proof. $S = \{y \in \mathbb{R} |\ y < 0\}$, $A = \operatorname{lev}_f(0)$. □

With the aid of the principle it is easy to obtain a vector–valued variant of the preceding assertion. Therein, $\operatorname{bd} K = \operatorname{cl} K \setminus \operatorname{int} K$ denotes the boundary of K.

Corollary 3.4. *Let $(Y, \leq_K)$ be a separated topological linear space partially ordered by a closed pointed (i.e., $K \cap -K = \{0_Y\}$) convex cone $K \subseteq Y$ having a nonempty interior, $f : C \to (Y, \leq_K)$ a convex function. If f takes values in $-\operatorname{bd} K$ on an algebraically open subset of C, then f never takes values in $-\operatorname{int} K$.*

Proof. $S = -\operatorname{int} K$, $A = f^{-1}(-\operatorname{bd} K)$ □

In vector optimization optimality conditions of the following type occur [7, Theorem 7.6]: If $\bar{x} \in D$ is a weakly minimal solution of the vector optimization problem $\min_{x \in D} f(x)$ of [7, page 153] and if $f : D \to (Y, \leq_K)$ has a directional variation $f'(\bar{x}) : D - \{\bar{x}\} \to (Y, \leq_K)$ at $\bar{x}$ with respect to $-\operatorname{core} K$ (i.e., whenever there is $x \in D$ with $x \neq \bar{x}$ and $f'(\bar{x})(x - \bar{x}) \in -\operatorname{core} K$, then there exists $\bar{\lambda} > 0$ with $\bar{x} + \lambda(x - \bar{x}) \in D$ for all $\lambda \in (0, \bar{\lambda}]$ and $\frac{1}{\lambda}\left(f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x})\right) \in -\operatorname{core} K$ for all $\lambda \in (0, \bar{\lambda}]$, [7, Definition 2.14]), then

$$\forall x \in D : \quad f'(\bar{x})(x - \bar{x}) \notin -\operatorname{core} K. \tag{2}$$

If the directional variation $f'(\bar{x}) : D - \{\bar{x}\} \to (Y, \leq_K)$ is convex and takes values in $-\operatorname{bd} K$ on an algebraically open set, and if $\operatorname{int} K \neq \emptyset$ (in particular this implies $\operatorname{core} K = \operatorname{int} K$), then, by Corollary 3.4, the optimality condition (2) is satisfied.

The following corollary is a result by Deutsch and Singer [3] on the single–valuedness of a convex set–valued map. In [3] a further conclusion, namely f must be affine on $\operatorname{dom} f$, is drawn and applications to metric projections and adjoints of set–valued maps are discussed.

Corollary 3.5. *Let V be a linear space and let $f : C \to (\hat{\mathcal{P}}(V), \supseteq)$ be convex. If f is single–valued in some point of $x_0 \in \operatorname{core}(\operatorname{dom} f)$, then f is single–valued everywhere in* $\operatorname{dom} f$.

Proof. $S = \{\text{"nonsingletons"}\}$, $A = \operatorname{dom} f$. □

Corollary 3.6. *Let V be a separated topological linear space and $f : C \to (\hat{\mathcal{F}}(V), \supseteq)$ be convex. If f is compact–valued at some point $x_0 \in \operatorname{core}(\operatorname{dom} f)$, then f is compact–valued everywhere on* $\operatorname{dom} f$.

Proof. $S = \{\text{"noncompacts"}\}$, $A = \operatorname{dom} f$. □

The following result by Zamfirescu [16] was published in the framework of a generalization of the mentioned result of Deutsch and Singer to so–called *star–shaped* functions. The same generalization could be done for all the assertions given here.

Corollary 3.7. *Let $V = \mathbb{R}^n$ and let $f : C \to (\hat{\mathcal{P}}(V), \supseteq)$ be convex. Then* $\dim f(x)$, *as a function of x, is constant on* $\operatorname{core}(\operatorname{dom} f)$ *and not larger elsewhere.*

Proof. Let $x_0 \in \operatorname{core}(\operatorname{dom} f)$ with $\dim f(x_0) = k$, $S = \{v \subseteq V |\ \dim v > k\}$, $A = \operatorname{dom} f$. Then the theorem yields $\dim f(x) \leq k$ for all $x \in C$. Now suppose there is some $x_1 \in \operatorname{core}(\operatorname{dom} f)$ such that $\dim f(x_1) = m < k$. Applying the theorem again we obtain $\dim f(x) \leq m < k$ for all $x \in C$ contradicting $\dim f(x_0) = k$. □

It follows a result on cone–valued maps.

Corollary 3.8. *Let Z be a conlinear space and let $f : C \to (\hat{\mathcal{K}}(Z), \supseteq)$ be convex. Then f is constant on* $\operatorname{core}(\operatorname{dom} f)$.

Proof. Let $x_0 \in \operatorname{core}(\operatorname{dom} f)$ with $f(x_0) = k_0$, $S = \{k \in \mathcal{K}(Z) |\ k \not\subseteq k_0\}$, $A = \operatorname{dom} f$. Then the theorem yields $f(x) \subseteq k_0$ for all $x \in C$. Now suppose there is some $x_1 \in \operatorname{core}(\operatorname{dom} f)$ with $f(x_1) = k_1 \subsetneq k_0$. Applying the theorem again we obtain $f(x) \subseteq k_1 \subsetneq k_0$ for all $x \in C$ contradicting $f(x_0) = k_0$. □

Let V be a locally convex space and V^* its topological dual. As usual, $\delta^*(\cdot\,|\,A) : V^* \to \mathbb{R}^\diamond$, $\delta^*(v^*|\,A) = \sup\{\langle v^*, a\rangle \mid a \in A\}$ is the support function of a convex set $A \subseteq V$. For $A, B \in \hat{\mathcal{F}}_c(V)$ (compare Example (5) and note that $\hat{\mathcal{F}}_c(V)$ is the space of closed convex subsets of V) it holds

$$\forall v^* \in V^* : \quad \delta^*(v^*|\,A + B) = \delta^*(v^*|\,A) + \delta^*(v^*|\,B), \tag{3}$$

and for $A \in \hat{\mathcal{F}}_c(V)$ and $\alpha \geq 0$ we have

$$\forall v^* \in V^* : \quad \delta^*(v^*|\,\alpha \cdot A) = \alpha \cdot \delta^*(v^*|\,A). \tag{4}$$

Hence, the map which assigns every $A \in \mathcal{F}_c(V)$ its support function is a homomorphism into the conlinear space Ψ of all functions $\psi : V^* \to \mathbb{R} \cup \{\infty\}$, where the conlinear operations are defined pointwise. Using a separation theorem, for instance [11, page 25], it can easily be seen that this homomorphism is injective, i.e., we have an embedding. Moreover, it is clear that a family of all functions $\psi : V^* \to \mathbb{R} \cup \{\infty\}$ having a certain fixed effective domain can considered to be a linear space L. Let $\mathcal{A} \subseteq \mathcal{F}_c(V)$ such that the support functions of all members A of $\mathcal{A}$ have the same effective domain. Then $\mathcal{A}$ can be embedded into a linear space.

The following corollary tells us that the support functions of the values of a convex function $f : C \to (\hat{\mathcal{F}}(V), \supseteq)$ have essentially the same effective domain.

Corollary 3.9. *Let $f : C \to (\hat{\mathcal{F}}(V), \supseteq)$ be convex. Then $x \longmapsto \operatorname{dom} \delta^*(\,\cdot \,|\, f(x)\,)$ is constant on* $\operatorname{core}(\operatorname{dom} f)$.

Proof. The convexity of f implies $f(x) \supseteq \lambda f(x) + (1-\lambda) f(x)$ for all $\lambda \in [0,1]$. Hence $f(x)$ is a convex subset of V for all $x \in \operatorname{dom} f$. Define $K_0 := \operatorname{dom} \delta^*(\,\cdot \,|\, f(x_0)\,)$ for some $x_0 \in \operatorname{core}(\operatorname{dom} f)$, $S := \{w \in \mathcal{F}_c(V) |\ \operatorname{dom} \delta^*(\,\cdot \,|\, w\,) \not\supseteq K_0\}$ and $A := \operatorname{dom} f$. Obviously, S is a cone in $\mathcal{F}(V)$.

Let $s \in S$ and $w \in f(A)$. Then we have $\operatorname{dom} \delta^*(\,\cdot \,|\, s\,) \not\supseteq K_0$ and since $w \neq \emptyset$ we have $\delta^*(\,\cdot \,|\, w\,) > -\infty$. Since s and w are nonempty, (3) is valid and it follows $s + w \in S$, i.e., the assumption $S \succ f(A)$ is satisfied.

For all $s \in S$ we have $f(x_0) \not\supseteq s$. Indeed, assuming that $\bar{s} \subseteq f(x_0)$ for some $\bar{s} \in S$ we obtain $\delta^*(\,\cdot \,|\, \bar{s}\,) \leq \delta^*(\,\cdot \,|\, f(x_0)\,)$ and hence $\operatorname{dom} \delta^*(\,\cdot \,|\, \bar{s}\,) \supseteq \operatorname{dom} \delta^*(\,\cdot \,|\, f(x_0)\,) = K_0$. This contradicts the definition of S. The theorem yields $f(x) \notin S$ for all $x \in C$. This means $\operatorname{dom} \delta^*(\,\cdot \,|\, f(x)\,) \supseteq K_0$ for all $x \in C$.

Assuming that $\operatorname{dom} \delta^*(\,\cdot \,|\, f(x_1)\,) = K_1 \supsetneq K_0$ for some $x_1 \in \operatorname{core}(\operatorname{dom} f)$ and applying the same procedure as above we get $\operatorname{dom} \delta^*(\,\cdot \,|\, f(x)\,) \supseteq K_1$ for all $x \in C$, in particular, $\operatorname{dom} \delta^*(\,\cdot \,|\, f(x_0)\,) \supseteq K_1 \supsetneq K_0$ contradicting the definition of K_0. Hence $x \longmapsto \operatorname{dom} \delta^*(\,\cdot \,|\, f(x)\,)$ is constant on $\operatorname{core}(\operatorname{dom} f)$. □

From the previous corollary and the considerations above, we may conclude that the set $f(\operatorname{core}(\operatorname{dom} f)) \subseteq \mathcal{F}(V)$ can be embedded into a linear space L, even though $\mathcal{F}(V)$ cannot be embedded (compare Proposition 2.4). Note that for different functions $f : C \to \hat{\mathcal{F}}(V)$ with the same X, $C \subseteq X$ and V the linear space L can be different, in particular, the neutral element of L does not coincides with the neutral element of $\hat{\mathcal{F}}(V)$, in general. In [9] we have shown that (at least in a finite dimensional context), if $G(f)$ is additionally closed, even the set $f(\operatorname{dom} f)$ can be embedded into a linear space. Note further that the assertions involving the ordering relation $\supseteq$ do not remain valid for $\subseteq$.

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Manuscript received July 22, 2003
revised January 25, 2005

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