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WEAK CONVERGENCE THEOREMS BY CESÁRO MEANS FOR NONEXPANSIVE MAPPINGS AND INVERSE-STRONGLY-MONOTONE MAPPINGS

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Abstract. In this paper, we introduce an iterative scheme by Cesáro means for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly-monotone mapping in a Hilbert space. Then we show that the sequence converges weakly to a common element of two sets. Using this result, we obtain the well-known nonlinear ergodic theorem which was proved by Baillon. Further we consider the problem of finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping and so on.

1. Introduction

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. A mapping S of C into itself is called *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. In 1975, Baillon [1] proved the first nonlinear ergodic theorem: Define

$$z_n = \frac{1}{n}\sum_{k=1}^{n} S^{k-1}x \tag{1.1}$$

for every $n = 1, 2, \ldots$ and $x \in C$ and suppose $F(S) \neq \emptyset$. Then the sequence $\{z_n\}$ generated by (1.1) converges weakly to some element of $F(S)$.

A mapping A of C into H is called *monotone* if for all $x, y \in C$, $\langle x-y, Ax-Ay\rangle \geq 0$. We denote by $VI(C,A)'$ the set of solutions $u \in C$ such that $\langle v-u, Av\rangle \geq 0$ for all $v \in C$. For finding an element of $VI(C,A)'$, Bruck [3] introduced the following iterative scheme: $x_{n+1} = P_C(x_n - \lambda_n Ax_n)$ and

$$z_n = \frac{\sum_{k=1}^{n}\lambda_k x_k}{\sum_{k=1}^{n}\lambda_k} \tag{1.2}$$

for every $n = 1, 2, \ldots$, where $x_1 = x \in C$ and $\{\lambda_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty}\lambda_n = \infty$ and $\sum_{n=1}^{\infty}\|\lambda_n Ax_n\|^2 < \infty$. He showed that the sequence $\{z_n\}$ generated by (1.2) converges weakly to some element of $VI(C,A)'$.

The *variational inequality problem* is to find a $u \in C$ such that

$$\langle v-u, Au\rangle \geq 0$$

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for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C,A)$. A mapping A of C into H is called *inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2$$

for all $x, y \in C$; see [4] and [8]. For such a case, A is called α-inverse-strongly-monotone. For finding an element of $F(S)\cap VI(C,A)$, Takahashi and Toyoda [16] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \lambda_n Ax_n) \tag{1.3}$$

for every $n = 1, 2, \ldots$, where $x_1 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that the sequence $\{x_n\}$ generated by (1.3) converges weakly to some element of $F(S)\cap VI(C,A)$.

In this paper, motivated by (1.1) and (1.3), we introduce an iterative scheme by Cesáro means for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly-monotone mapping in a Hilbert space. Then we show that the sequence converges weakly to a common element of two sets. Using this result, we obtain the well-known nonlinear ergodic theorem which was proved by Baillon [1]. Further we consider the problem of finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping and so on.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x-y\|$ for all $y \in C$. P_C is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.1}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the properties: $P_C x \in C$ and $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $y \in C$. In the context of the variational inequality problem, this implies that

$$u \in VI(C,A) \Longleftrightarrow u = P_C(u - \lambda Au) \tag{2.2}$$

for all $\lambda > 0$, where A is a monotone mapping of C into H. It is also known that H satisfies *Opial's condition* [10], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

If A is an α-inverse-strongly-monotone mapping of C into H, then it is obvious that A is $1/\alpha$-Lipschitz continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I-\lambda A)x - (I-\lambda A)y\|^2 &= \|(x-y) - \lambda(Ax-Ay)\|^2 \\ &= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2\end{aligned}$$

$$\leq \|x-y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2. \tag{2.3}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H.

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly-monotone mapping of C into H and let $N_C v$ be the *normal cone* to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$, and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see Theorem 3 of [12].

3. Weak Convergence theorem

In this section, we prove a weak convergence theorem for nonexpansive mappings and inverse-strongly-monotone mappings in a Hilbert space.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly-monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = SP_C(x_n - \lambda_n Ax_n), \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k \end{cases}$$

for every $n = 1, 2, \ldots,$ where $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$. Then $\{z_n\}$ converges weakly to $z \in F(S) \cap VI(C, A)$.

Proof. Put $y_n = P_C(x_n - \lambda_n Ax_n)$ for every $n = 1, 2, \ldots$. Let $u \in F(S) \cap VI(C, A)$. Since $I - \lambda_n A$ is nonexpansive and $u = P_C(u - \lambda_n Au)$ from (2.2), we have

$$\begin{aligned} \|y_n - u\| &= \|P_C(x_n - \lambda_n Ax_n) - P_C(u - \lambda_n Au)\| \\ &\leq \|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au)\| \\ &\leq \|x_n - u\| \end{aligned} \tag{3.1}$$

for every $n = 1, 2, \ldots$. From (2.2) and (2.3), we also have

$$\begin{aligned} \|y_n - u\|^2 &= \|P_C(x_n - \lambda_n Ax_n) - P_C(u - \lambda_n Au)\|^2 \\ &\leq \|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au)\|^2 \\ &\leq \|x_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Au\|^2 \\ &\leq \|x_n - u\|^2 + a(b - 2\alpha)\|Ax_n - Au\|^2 \end{aligned}$$

for every $n = 1, 2, \ldots$. So, we have

$$\|x_{n+1} - u\|^2 = \|Sy_n - u\|^2$$

$$\begin{aligned} &\le \|y_n - u\|^2 \le \|x_n - u\|^2 + a(b - 2\alpha)\|Ax_n - Au\|^2 \\ &\le \|x_n - u\|^2. \end{aligned} \tag{3.2}$$

Therefore, there exists $\lim_{n\to\infty} \|x_n - u\|$. Hence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded. Since

$$-a(b - 2\alpha)\|Ax_n - Au\|^2 \le \|x_n - u\|^2 - \|x_{n+1} - u\|^2,$$

we obtain $\|Ax_n - Au\| \to 0$. From (2.1), we have

$$\begin{aligned} \|y_n - u\|^2 &= \|P_C(x_n - \lambda_n Ax_n) - P_C(u - \lambda_n Au)\|^2 \\ &\le \langle (x_n - \lambda_n Ax_n) - (u - \lambda_n Au), y_n - u\rangle \\ &= \frac{1}{2}\{\|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au)\|^2 + \|y_n - u\|^2 \\ &\quad - \|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au) - (y_n - u)\|^2\} \\ &\le \frac{1}{2}\{\|x_n - u\|^2 + \|y_n - u\|^2 - \|(x_n - y_n) - \lambda_n(Ax_n - Au)\|^2\} \\ &= \frac{1}{2}\{\|x_n - u\|^2 + \|y_n - u\|^2 - \|x_n - y_n\|^2 \\ &\quad + 2\lambda_n\langle x_n - y_n, Ax_n - Au\rangle - \lambda_n^2\|Ax_n - Au\|^2\} \end{aligned}$$

and hence

$$\begin{aligned} \|y_n - u\|^2 \le \|x_n - u\|^2 &- \|x_n - y_n\|^2 \\ &+ 2\lambda_n\langle x_n - y_n, Ax_n - Au\rangle - \lambda_n^2\|Ax_n - Au\|^2. \end{aligned}$$

So, we have

$$\begin{aligned} \|x_{n+1} - u\|^2 &\le \|y_n - u\|^2 \\ &\le \|x_n - u\|^2 - \|x_n - y_n\|^2 \\ &\quad + 2\lambda_n\langle x_n - y_n, Ax_n - Au\rangle - \lambda_n^2\|Ax_n - Au\|^2 \end{aligned}$$

and hence

$$\|x_n - y_n\|^2 \le \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + 2\lambda_n\langle x_n - y_n, Ax_n - Au\rangle.$$

So, we obtain $\|x_n - y_n\| \to 0$.

As $\{z_n\}$ is bounded, we have that a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ converges weakly to z. We show $z \in F(S) \cap VI(C, A)$. Let us first show $z \in VI(C, A)$. Let

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v, w) \in G(T)$. Since $w \in Tv = Av + N_C v$, we have $w - Av \in N_C v$. From $x_k \in C$, we have

$$\langle v - x_k, w - Av\rangle \ge 0.$$

On the other hand, from $y_k = P_C(x_k - \lambda_k Ax_k)$, we have $\langle v - y_k, y_k - (x_k - \lambda_k Ax_k)\rangle \ge 0$ and hence

$$\left\langle v - y_k, \frac{y_k - x_k}{\lambda_k} + Ax_k \right\rangle \ge 0.$$

Therefore, we have

$$\begin{aligned}
\langle v - x_k, w \rangle &\geq \langle v - x_k, Av \rangle \\
&\geq \langle v - x_k, Av \rangle - \left\langle v - y_k, \frac{y_k - x_k}{\lambda_k} + Ax_k \right\rangle \\
&= \langle v - x_k, Av - Ax_k \rangle + \langle (v - x_k) - (v - y_k), Ax_k \rangle \\
&\quad - \left\langle v - y_k, \frac{y_k - x_k}{\lambda_k} \right\rangle \\
&\geq \langle y_k - x_k, Ax_k \rangle - \left\langle v - y_k, \frac{y_k - x_k}{\lambda_k} \right\rangle \\
&\geq \left(- \|Ax_k\| - \frac{\|y_k - v\|}{\lambda_k} \right) \|y_k - x_k\| \\
&\geq \left(- K - \frac{L}{a} \right) \|y_k - x_k\|
\end{aligned}$$

for every $k = 1, 2, \ldots$, where $K = \sup\{\|Ax_k\| : k \in \mathbf{N}\}$ and $L = \sup\{\|y_k - v\| : k \in \mathbf{N}\}$. Hence we have

$$\langle v - z_n, w \rangle \geq \left(- K - \frac{L}{a} \right) \frac{1}{n} \sum_{k=1}^{n} \|y_k - x_k\|.$$

Taking $n = n_i$, from $\|x_n - y_n\| \to 0$, we have $\langle v - z, w \rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we obtain $z \in T^{-1}0$ and hence $z \in VI(C, A)$. Let us show $z \in F(S)$. Let $u \in VI(C, A)$. From (3.1), we have

$$\|x_{k+1} - Su\| \leq \|y_k - u\| \leq \|x_k - u\|$$

for every $k = 1, 2, \ldots$. For $u \in VI(C, A)$, we have

$$\begin{aligned}
0 &\leq \|x_k - u\|^2 - \|x_{k+1} - Su\|^2 \\
&= \|x_k - Su\|^2 + 2\langle x_k - Su, Su - u \rangle \\
&\quad + \|Su - u\|^2 - \|x_{k+1} - Su\|^2
\end{aligned}$$

for every $k = 1, 2, \ldots$. Then

$$0 \leq \frac{1}{n}(\|x - Su\|^2 - \|x_{n+1} - Su\|^2) + 2\langle z_n - Su, Su - u \rangle + \|Su - u\|^2.$$

Taking $n = n_i$, we have, as $i \to \infty$,

$$0 \leq 2\langle z - Su, Su - u \rangle + \|Su - u\|^2.$$

Putting $u = z$, we obtain $0 \leq -\|Sz - z\|^2$ and hence $z \in F(S)$. This implies $z \in F(S) \cap VI(C, A)$.

Put $u_n = P_{F(S) \cap VI(C,A)} x_n$. Let $u \in F(S) \cap VI(C, A)$. From (3.2), we have

$$\begin{aligned}
\|x_{n+m} - u\|^2 &\leq \|x_{n+m-1} - u\|^2 \\
&\leq \|x_{n+m-2} - u\|^2 \leq \cdots \leq \|x_n - u\|^2
\end{aligned}$$

for every $n, m = 1, 2, \ldots$. Then we have

$$\|x_{n+m} - u_n\|^2 \leq \|x_n - u_n\|^2.$$

Since $u_{n+m} = P_{F(S)\cap VI(C,A)}x_{n+m}$, we have

$$\left\| x_{n+m} - \frac{u_n + u_{n+m}}{2} \right\| \geq \|x_{n+m} - u_{n+m}\|.$$

So, we have

$$\begin{aligned}
\|u_{n+m} - u_n\|^2 &= \|(u_{n+m} - x_{n+m}) + (x_{n+m} - u_n)\|^2 \\
&= 2\|u_{n+m} - x_{n+m}\|^2 + 2\|x_{n+m} - u_n\|^2 - 4\left\| x_{n+m} - \frac{u_n + u_{n+m}}{2} \right\|^2 \\
&\leq 2\|x_{n+m} - u_n\|^2 - 2\|x_{n+m} - u_{n+m}\|^2 \\
&\leq 2\|x_n - u_n\|^2 - 2\|x_{n+m} - u_{n+m}\|^2
\end{aligned}$$

for every $n, m = 1, 2, \ldots$. Therefore, $\{\|x_n - u_n\|\}$ is nonincreasing and hence there exists $\lim_{n\to\infty} \|x_n - u_n\|$. So, $\{u_n\}$ is a Cauchy sequence. Since $F(S) \cap VI(C, A)$ is closed, $\{u_n\}$ converges strongly to $w \in F(S) \cap VI(C, A)$.

Finally, we show $z = w$. Since $u_k = P_{F(S)\cap VI(C,A)}x_k$ and $z \in F(S) \cap VI(C, A)$, we have

$$\langle z - u_k, u_k - x_k \rangle \geq 0$$

for every $k = 1, 2, \ldots$. So, we have

$$\begin{aligned}
\langle z - w, x_k - u_k \rangle &= \langle z - u_k, x_k - u_k \rangle + \langle u_k - w, x_k - u_k \rangle \\
&\leq \|u_k - w\|\|x_k - u_k\| \leq M\|u_k - w\|
\end{aligned}$$

for every $k = 1, 2, \ldots$, where $M = \sup\{\|x_k - u_k\| : k \in \mathbf{N}\}$. Hence we have

$$\left\langle z - w, z_n - \frac{1}{n}\sum_{k=1}^{n} u_k \right\rangle \leq \frac{M}{n}\sum_{k=1}^{n} \|u_k - w\|.$$

Taking $n = n_i$, from $\|u_n - w\| \to 0$, we obtain $\langle z - w, z - w \rangle \leq 0$ as $i \to \infty$ and hence $z = w$. Therefore, we obtain $z_n \rightharpoonup z$. □

4. Applications

In this section, we prove some weak convergence theorems in a Hilbert space by using Theorem 3.1. We first prove a nonlinear ergodic theorem which was obtained by Baillon [1].

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H and let S be a nonexpansive mapping of C into itself such that $F(S) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} S^{k-1}x \end{cases}$$

for every $n = 1, 2, \ldots$. Then $\{z_n\}$ converges weakly to $z \in F(S)$.

Proof. In Theorem 3.1, put $Ax = 0$ for all $x \in C$. Then A is inverse-strongly-monotone. We have $C = VI(C, A)$ and

$$\begin{aligned} x_{n+1} &= SP_C(x_n - \lambda_n Ax_n) \\ &= Sx_n = S^n x. \end{aligned}$$

So, by Theorem 3.1, we obtain the desired result. □

Let C be a closed convex subset of a real Hilbert space H. Then a mapping $T : C \to C$ is called *strictly pseudocontractive* if there exists p with $0 \le p < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + p\|(I - T)x - (I - T)y\|^2$$

for all $x, y \in C$; see [11]. Put $A = I - T$. Then A is $(1 - p)/2$-inverse-strongly-monotone; for the proof, see [16]. Using Theorem 3.1, we consider the problem of finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping.

Theorem 4.2. *Let C be a closed convex subset of a real Hilbert space H. Let T be a p-strictly pseudocontractive mapping of C into itself and let S be a nonexpansive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = S((1 - \lambda_n)x_n + \lambda_n Tx_n), \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k \end{cases}$$

for every $n = 1, 2, \ldots,$ where $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 1 - p$. Then $\{z_n\}$ converges weakly to $z \in F(S) \cap F(T)$.

Proof. Put $A = I - T$. Then A is $(1 - p)/2$-inverse-strongly-monotone. We have $F(T) = VI(C, A)$ and

$$\begin{aligned} x_{n+1} &= SP_C(x_n - \lambda_n Ax_n) \\ &= SP_C(x_n - \lambda_n(I - T)x_n) \\ &= S((1 - \lambda_n)x_n + \lambda_n Tx_n). \end{aligned}$$

So, by Theorem 3.1, we obtain the desired result. □

Using Theorem 3.1, we also have the following:

Theorem 4.3. *Let H be a real Hilbert space. Let A be an α-inverse-strongly-monotone mapping of H into itself and let S be a nonexpansive mapping of H into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in H, \\ x_{n+1} = S(x_n - \lambda_n Ax_n), \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k \end{cases}$$

for every $n = 1, 2, \ldots,$ where $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$. Then $\{z_n\}$ converges weakly to $z \in F(S) \cap A^{-1}0$.

Proof. We have $A^{-1}0 = VI(H, A)$. So, putting $P_H = I$, by Theorem 3.1, we obtain the desired result. □

Remark. If A is strongly monotone and Lipschitz continuous, then A is inverse-strongly-monotone. See Yamada [17] for the case when S is a nonexpansive mapping of a Hilbert space H into itself and A is a strongly monotone and Lipschitz continuous mapping of H into itself.

Let f be a continuously Fréchet differentiable convex functional on H and let ∇f be the gradient of f. If ∇f is $1/\alpha$-Lipschitz continuous, then ∇f is α-inverse-strongly-monotone; see [2]. Using Theorem 3.1, we have the following:

Theorem 4.4. *Let C be a closed convex subset of a real Hilbert space H. Let f be a continuously Fréchet differentiable convex functional on H and let ∇f be the gradient of f such that $C \cap (\nabla f)^{-1}0 \neq \emptyset$. Suppose ∇f is $1/\alpha$-Lipschitz continuous. Let $\{z_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in H, \\ x_{n+1} = P_C(x_n - \lambda_n \nabla f(x_n)), \\ z_n = \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} x_k \end{cases}$$

for every $n = 1, 2, \ldots,$ where $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$. Then $\{z_n\}$ converges weakly to $z \in C \cap (\nabla f)^{-1}0$.

Proof. We know from [2] that ∇f is an α-inverse-strongly-monotone mapping and $(\nabla f)^{-1}0 = VI(H, \nabla f)$. We also have $C = F(P_C)$. So, putting $P_H = I$, by Theorem 3.1, we obtain the desired result. □

References

[1] J. B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris A-B **280** (1975), A1511-A1514.

[2] J. B. Baillon and G. Haddad, *Quelques propriétés des opérateurs angle-bornés et n-cycliquement monotones*, Israel J. Math. **26** (1977), 137-150.

[3] R. E. Bruck, *On the weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in Hilbert space*, J. Math. Anal. Appl. **61** (1977), 159-164.

[4] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (1967), 197-228.

[5] H. Iiduka and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly-monotone mappings*, Nonlinear Anal. **61** (2005), 341-350.

[6] H. Iiduka, W. Takahashi and M. Toyoda, *Approximation of solutions of variational inequalities for monotone mappings*, PanAmer. Math. J. **14** (2004), 49-61.

[7] J. L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math. **20** (1967), 493-517.

[8] F. Liu and M. Z. Nashed, *Regularization of nonlinear ill-posed variational inequalities and convergence rates*, Set-Valued Anal. **6** (1998), 313-344.

[9] K. Nakajo and W. Takahashi, *Strong convergence theorems by an improved splitting method*, Comm. Appl. Nonlinear Anal. **9** (2002), 99-107.

[10] Z. Opial, *Weak convergence of the sequence of successive approximation for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591-597.

[11] B. E. Rhoades, *Fixed-point iterations using infinite matrices*, Trans. Amer. Math. Soc. **196** (1974), 161-176.

[12] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75-88.
[13] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control and Optim. **14** (1976), 877-898.
[14] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000 (Japanese).
[15] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.
[16] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417-428.
[17] I. Yamada, *The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings*, in Inherently Parallel Algorithms for Feasibility and Optimization and Their Applications (D. Butnariu, Y. Censor and S. Reich, Eds.), Elsevier, 2001, pp. 473-504.

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