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# FIXED POINTS OF APPROXIMABLE OR KAKUTANI MAPS IN GENERALIZED CONVEX SPACES

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Abstract. We study fixed point properties for various types of multimaps defined on generalized convex spaces and mutual relations of those properties. We obtain several fixed point theorems for approximable or Kakutani multimaps. Our approach is based on the study of approachable maps initiated mainly by Ben-El-Mechaiekh and on other results on generalized convex spaces.

## 1. Introduction

The celebrated Brouwer fixed point theorem in 1912 was generalized by Schauder in 1930 to Banach spaces. In 1935, Schauder conjectured that his theorem might hold for any topological vector spaces. Partial solutions to the conjecture were given by Tychonoff in 1935, Hukuhara in 1950, Klee in 1960, Fan in 1962, Idzik in 1987, Nhu in 1996, Aranđelović in 1996, and others. For references, see [41,43,48]. However, the following form of the Schauder conjecture is still open:

**Conjecture 1.** *Let $E$ be a Hausdorff topological vector space, $C$ a convex subset of $E$, and $f$ a continuous function from $C$ into $C$. If $f$ is compact (that is, $f(C)$ is contained in a compact subset of $C$), then $f$ has a fixed point $x_0 \in C$, that is, $x_0 = f(x_0)$.*

In 2001, Cauty [13] claimed the affirmativity of Conjecture 1. Later, it is known that his proof has a gap.

On the other hand, in 1941, Kakutani extended the Brouwer theorem to an upper semicontinuous (u.s.c.) multimap having nonempty compact convex values (which will be called a *Kakutani map*). This was also generalized by Bohnenblust and Karlin in 1950, Fan in 1952, Glicksberg in 1952, Himmelberg in 1972, Granas and Liu in 1986, Idzik in 1988, Park in 1988, Okon in 2002, and others for particular types of topological vector spaces; see [41,43,48]. However, the following is still open:

**Conjecture 2.** *Let $X$ be a compact convex subset of a Hausdorff topological vector space $E$. Then every Kakutani map $T : X \multimap X$ has a fixed point $x_0 \in X$, that is, $x_0 \in T(x_0)$.*

Later there have appeared other types of useful multimaps on topological vector spaces, for example, Fan-Browder maps, locally selectionable maps, approachable maps, approximable maps, and many others; see [38-43,45-49].

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In this paper, we are mainly concerned with fixed point properties for various types of multimaps defined on generalized convex spaces and mutual relations of those properties. From some of the most general cases where Conjectures 1 and 2 are known to be affirmative, we deduce several general fixed point theorems for approximable or Kakutani maps in topological vector spaces or in generalized convex spaces. Our approach is based on the study of approachable maps initiated mainly by Ben-El-Mechaiekh [3-11] and on other results on generalized convex spaces due to the author.

In Section 2, as preliminaries, we review the concepts of admissible (in the sense of Klee), weakly admissible, and convexly totally bounded (c.t.b.) subsets of topological vector spaces. Moreover, we recall the concepts of generalized convex (simply, $G$-convex) spaces, locally $G$-convex spaces, and $LG$-spaces. Note that these are essential in order to establish general forms of fixed point theorems on topological vector spaces.

Section 3 deals with basic fixed point theorems on $\Phi$-maps (or Fan-Browder maps) and locally selectionable maps having convex values. In Section 4, approachable multimaps defined on $G$-convex uniform spaces are considered. We obtain several fixed point theorems for compact closed approachable multimaps on subsets of topological vector spaces. Section 5 deals with Kakutani type multimaps defined on certain type of $G$-convex spaces. We obtain some abstract forms of fixed point theorems which include a large number of known results. Finally, in Section 6, some results in previous sections are applied to condensing maps on topological vector spaces.

## 2. Preliminaries

Let us say that a topological space $X$ has the (*compact*) *fixed point property* (simply, f.p.p.) if any (compact) continuous selfmap $f : X \to X$ has a fixed point $x_0 \in X$.

Throughout this paper, all topological spaces are Hausdorff otherwise explicitly stated, a t.v.s. means a topological vector space $E$, and $\mathcal{V}$ denotes a basis of neighborhoods of the origin 0 of $E$.

A nonempty subset $K$ of a t.v.s. $E$ is said to be *locally convex* if for each $x \in K$ there exists in $K$ a basis of neighborhoods $U_x$ of $x$ such that $U_x = W_x \cap K$ and $W_x$ is a convex subset of $E$.

Recall that a nonempty subset $X$ of a t.v.s. $E$ is said to be *admissible* (in the sense of Klee) provided that, for every nonempty compact subset $K$ of $X$ and every $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace $L$ of $E$.

A nonempty subset $K$ of $E$ is said to be *Klee approximable* if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to E$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace $L$ of $E$. Especially, for a subset $X$ of $E$, $K$ is said to be Klee approximable *into* $X$ whenever the range $h(K)$ is contained in a polytope in $X$.

**Examples.** Any compact locally convex subset of a t.v.s. is Klee approximable. Every nonempty convex subset of a locally convex t.v.s. is admissible. Examples of admissible t.v.s. are $l^p$, $L^p$, the Hardy spaces $H^p$ for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable functions on $[0,1]$, and others. Moreover, any locally convex subset of an $F$-normable t.v.s. and any compact convex locally convex subset of a t.v.s. is admissible. Note that an example of a nonadmissible nonconvex compact subset of the Hilbert space $l^2$ is known. For details, see Hadžić [19], Weber [61,62], and references therein.

In 1996, Nguyen To Nhu [30] defined the notion of weakly admissible compact convex subsets of a metrizable t.v.s. and showed that such subsets have the f.p.p. Arandelović [2] extended the notion of weak admissibility to arbitrary t.v.s. and gave a non-metrizable version of Nhu's result.

Let $X$ be a nonempty closed convex subset of a t.v.s. $E$. We say that $X$ is *weakly admissible* [2] if for every $V \in \mathcal{V}$ there exist closed convex subsets $X_1, X_2, \ldots, X_n$ of $X$ with $X = \mathrm{co}\,(\bigcup_{i=1}^n X_i)$ and continuous functions $f_i : X_i \to X \cap L,\ i = 1, 2, \ldots, n$, where $L$ is a finite dimensional subspace of $E$, such that $\sum_{i=1}^n (f_i(x_i) - x_i) \in V$ for every $x_i \in X_i$ and $i = 1, 2, \ldots, n$.

In 1988, Idzik [23] introduced the notion of convexly totally bounded subsets of a t.v.s. and established some (almost) fixed point theorems for Kakutani maps having relatively compact and convexly totally bounded ranges.

A subset $B$ of a t.v.s. $E$ is said to be *convexly totally bounded* (simply, c.t.b.) if for every $V \in \mathcal{V}$, there exist a finite subset $\{x_i\}_{i=1}^n \subset B$ and a finite family of convex subsets $\{C_i\}_{i=1}^n$ of $V$ such that $B \subset \bigcup_{i=1}^n (x_i + C_i)$.

**Examples.** Idzik [23] and others gave examples of c.t.b. sets as follows:

1. Every compact set in a locally convex t.v.s.
2. Any compact set which is locally convex.
3. Every compact convex subset of $E = l^p,\ 0 < p < 1$.
4. More generally, every compact convex subset of a t.v.s. $E$ on which $E^*$ separates points.

For more examples, see [16,61,62].

A *generalized convex space* or a $G$-*convex space* $(X, D; \Gamma)$ consists of a topological space $X$, a nonempty subset $D$ of $X$, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of $D$, $\Delta_n$ the standard $n$-simplex with vertices $\{e_i\}_{i=0}^n$, and $\Delta_J$ the face of $\Delta_n$ corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. It is possible to assume $\Gamma(A) = \phi_A(\Delta_n)$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$ and $(X; \Gamma) = (X, X; \Gamma)$. A subset $C$ of $X$ is said to be $\Gamma$-*convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$. For details on $G$-convex spaces, see [42-47,50-53], where basic theory was extensively developed and lots of examples of $G$-convex spaces were given.

A well-known subclass of $G$-convex spaces due to Horvath [21,22] can be generalized as follows:

A $G$-convex space $(X, D; \Gamma)$ is called a $C$-*space* (or an $H$-*space*) if each $\Gamma_A$ is $\omega$-connected (that is, $n$-connected for all $n \geq 0$) and $\Gamma_A \subset \Gamma_B$ for $A \subset B$ in $\langle D \rangle$.

The following is introduced in [47]:

A *locally G-convex space* is a $G$-convex space $(X, D; \Gamma)$ such that $X$ is a uniform space with a basis $\mathcal{U}$ for the symmetric entourages, $D$ is dense in $X$, and for each $U \in \mathcal{U}$ and each $x \in X$,

$$U[x] = \{x' \in X : (x, x') \in U\}$$

is $\Gamma$-convex.

We can also define a locally $C$-convex space as above.

A $G$-convex space $(X, D; \Gamma)$ is called an *LG-space* if $X$ is a uniform space with a basis $\mathcal{U}$ such that $D$ is dense in $X$, and for each $U \in \mathcal{U}$, $\{x \in X : C \cap U[x] \neq \emptyset\}$ is $\Gamma$-convex whenever $C \subset X$ is $\Gamma$-convex.

Similarly, we can define a *LC-space* $(X, D; \Gamma)$ if it is a $C$-space; for the case $X = D$, see [22], where a large number of examples were given.

## 3. Locally selectionable maps

A *multimap* (simply, a *map*) $T : X \multimap Y$ is a function from $X$ into the power set $2^Y$ of $Y$. $T(x)$ is called the *value* of $T$ at $x \in X$ and $T^-(y) := \{x \in X : y \in T(x)\}$ the *fiber* of $T$ at $y \in Y$. Let $T(A) := \bigcup\{T(x) : x \in A\}$ for $A \subset X$.

For topological spaces $X$ and $Y$, a map $T : X \multimap Y$ is said to be *closed* if its *graph* $\mathrm{Gr}(T) := \{(x, y) : x \in X,\ y \in T(x)\}$ is closed in $X \times Y$, and *compact* if its range $T(X)$ is contained in a compact subset of $Y$.

For any subset $X$ of a t.v.s., a map $T : X \multimap X$ is called a *Browder map* if it has nonempty convex values and open fibers. The well-known *Fan-Browder fixed point theorem* states that a Browder map from a compact convex subset of a t.v.s. (not necessarily Hausdorff) into itself has a fixed point [12].

For any topological space $Y$ and a $G$-convex space $(X, D; \Gamma)$, a map $T : Y \multimap X$ is called a $\Phi$-*map* (or a *Fan-Browder map*) if there exists a map $S : Y \multimap D$ such that

(i) for each $y \in Y$, $M \in \langle S(y) \rangle$ implies $\Gamma_M \subset T(y)$; and
(ii) $Y = \bigcup\{\mathrm{Int}\, S^-(x) : x \in D\}$.

If $X$ is a subset of a t.v.s. $E$, then Condition (i) can be replaced by the following:

(i)′ for each $y \in Y$, $\mathrm{co}\, S(y) \subset T(y)$.

The concept of $\Phi$-maps is originated from Horvath [21] and motivated by the works of Fan and Browder; see [12,40,42,43,46].

A $G$-convex space $(X, D; \Gamma)$ is called a $\Phi$-*space* if $X$ is a uniform space and for each entourage $U$ there is a $\Phi$-map $T : X \multimap X$ such that $\mathrm{Gr}(T) \subset U$. This concept is originated from Horvath [21], where a number of examples are given.

For a subset $X$ of a t.v.s. $E$, $X$ is a $\Phi$-*space* if for each neighborhood $V$ of 0 in $E$, there is a $\Phi$-map $T : X \multimap X$ such that $T(x) \subset x + V$ for each $x \in X$.

The following gives some of the most general partial solutions of Conjectures 1 and 2:

**Lemma 1.** *Let $X$ be a convex subset of a t.v.s. $E$ and $F : X \multimap X$ a Kakutani map. Then $F$ has a fixed point whenever one of the following holds:*

(1) (Idzik [23]) *$\overline{F(X)}$ is a compact c.t.b. subset of $X$.*
(2) (Okon [33-35]) *$X$ is compact and weakly admissible.*

(3) (Park [49]) $\overline{F(X)}$ *is a compact Klee approximable subset of* $X$.

Recall that Lemma 1 contains a large number of previously known partial solutions of Conjectures 1 and 2, see [2,19,21,24,25,30,41,43,48].

For $\Phi$-maps, we have the following strengthened form of the selection theorem in our previous work [42]:

**Lemma 2.** (1) *Let* $Y$ *be a normal space,* $(X, D; \Gamma)$ *a* $G$*-convex space, and* $S : Y \multimap D$ *a map such that* $Y = \bigcup\{\operatorname{Int} S^-(z) : z \in A\}$ *for some* $A \in \langle D \rangle$*. Then there exists a continuous function* $s : Y \to \Gamma_A$ *such that* $s(y) \in \Gamma(A \cap S(y))$ *for all* $y \in Y$*. In fact, if* $|A| = n + 1$*, then* $s = \phi_A \circ p$*, where* $\phi_A : \Delta_n \to \Gamma_A$ *and* $p : Y \to \Delta_n$ *are continuous functions.*

(2) *Let* $Y$ *be a paracompact space,* $(X, D; \Gamma)$ *a* $C$*-space, and* $T : Y \multimap X$ *a* $\Phi$*-map. Then* $T$ *has a continuous selection* $f : Y \to X$ *(that is,* $f(y) \in T(y)$ *for all* $y \in Y$*).*

**Lemma 3.** (1) *An* $LG$*-space* $(X, D; \Gamma)$ *is locally* $G$*-convex if every singleton is* $\Gamma$*-convex (that is,* $\Gamma_{\{x\}} = \{x\}$ *for each* $x \in D$*).*

(2) *Every locally* $G$*-convex space is a* $\Phi$*-space.*

(3) *For a subset* $X$ *of a t.v.s.* $E$*, if* $X$ *is a* $\Phi$*-space, then it is admissible.*

*Proof.* (1) For each symmetric entourage $U$ and any $x \in X$,

$$\begin{aligned} U[x] &= \{x' \in X : (x, x') \in U\} \\ &= \{x' \in X : x \in U[x']\} \\ &= \{x' \in X : \{x\} \cap U[x'] \neq \emptyset\}. \end{aligned}$$

Since $\{x\}$ is $\Gamma$-convex and $(X, D; \Gamma)$ is an $LG$-space, $U[x]$ is $\Gamma$-convex. Therefore, $(X, D; \Gamma)$ is locally $G$-convex.

(2) See [45, Lemma 4].

(3) For each symmetric neighborhood $V$ of the origin 0, there exist multimaps $S : X \multimap X$ and $T : X \multimap X$ such that

(i) for each $x \in X$, $\operatorname{co} S(x) \subset T(x)$;
(ii) $X = \bigcup\{\operatorname{Int} S^-(y) : y \in X\}$; and
(iii) $T(x) \subset x + V$ for each $x \in X$.

Let $K$ be a nonempty compact subset of $X$. Then, it follows from Lemma 2(1) that $T|_K$ has a continuous selection $h : K \to X$ such that

(iv) $h(K) \subset \operatorname{co} N$ for some $N \in \langle X \rangle$ with $|N| = n + 1$; and
(v) there exist continuous functions $p : K \to \Delta_n$ and $\phi_N : \Delta_n \to \operatorname{co} N$ such that $h = \phi_N \circ p$.

Moreover, $h(x) \in T(x)$ for all $x \in K$ implies

(vi) $x - h(x) \in x - T(x) \subset V$ for all $x \in K$.

Therefore, $X$ is admissible. □

For topological spaces $X$ and $Y$, a multimap $T : X \multimap Y$ is said to be *selectionable* if it has a continuous selection $f : X \to Y$, and *locally selectionable* if for each $x_0 \in X$, there exist an open neighborhood $V_0$ of $x_0$ and a continuous function $f_0 : V_0 \to Y$ such that $f_0(x) \in T(x)$ for all $x \in V_0$; see [42].

Any continuous function is selectionable and there are lots of examples of selectionable maps due to Michael and others. Any selectionable map is locally selectionable.

The following is given [42, Theorems 4 and 5]:

**Lemma 4.** *Let $X$ be a paracompact topological space and $Y$ a convex subset of a t.v.s. $E$. Then*

(1) *any $\Phi$-map $T : X \multimap Y$ is locally selectionable; and*
(2) *any locally selectionable map $T : X \multimap Y$ having convex values is selectionable.*

We say that a $G$-convex space $(X, D; \Gamma)$ has the (*compact*) $\Phi$-*fixed point property* (simply, $\Phi$-f.p.p.) if any (compact) $\Phi$-map $T : X \multimap X$ has a fixed point.

The following is given in [40]:

**Theorem 5.** (1) *A compact $G$-convex space $(X, D; \Gamma)$ has the $\Phi$-f.p.p.*

(2) *If a paracompact $C$-space $(X, D; \Gamma)$ has the f.p.p., then it has the $\Phi$-f.p.p.*

It is known that the converse of (2) is not true. Note that (2) follows from Lemma 2(2) and generalizes a result of Komiya [26].

We say that a nonempty subset $X$ of a t.v.s. has the (*compact*) $\mathbb{L}$-*fixed point property* (simply, $\mathbb{L}$-f.p.p.) if any (compact) locally selectionable map $T : X \multimap X$ having convex values has a fixed point.

**Theorem 6.** *Let $E$ be a t.v.s. whose nonempty paracompact convex subsets have the compact f.p.p. Then any nonempty convex subset $X$ of $E$ has the compact $\mathbb{L}$-f.p.p.*

*Proof.* Let $T : X \multimap X$ be a compact and locally selectionable map having convex values. Since $\overline{T(X)}$ is a compact subset of a convex set $X$, $L :=\mathrm{co}\,\overline{T(X)}$ is a paracompact convex subset of $X$ by the well-known Fournier-Granas argument [17]. In fact, since $\overline{T(X)}$ is compact, as in [17], $L :=\mathrm{co}\,\overline{T(X)}$ is a $\sigma$-compact subset of $X$ and hence $L$ is Lindelöf. Since $L$ is regular as a subset of a t.v.s., we know that $L$ is paracompact. By Lemma 4(2), $T|_L : L \multimap L$ is selectionable and has a continuous selection $s : L \to L$. Since $s(L) \subset T(L) \subset \overline{T(X)} \subset L$, $s$ is compact and has a fixed point $x_0 \in L \subset X$ such that $x_0 = s(x_0) \in T(x_0)$. This completes our proof. □

*Remarks.* 1. If Conjecture 1 is true, the hypothesis is satisfied. Moreover, Lemma 1 gives several particular cases of Theorem 6.

2. For the $\Phi$-f.p.p instead of $\mathbb{L}$-f.p.p., Theorem 6 is obtained in [40].

3. For the $\Phi$-f.p.p., a particular form of Theorem 6 was appeared in [36,37] whenever $X$ is closed and $E$ is locally convex.

## 4. Approachable maps on $G$-convex spaces

In 1992, Ben-El-Mechaiekh et al. [6] introduced the class $\mathbb{A}$ of approachable multimaps as follows:

Let $X$ and $Y$ be uniform spaces (with respective bases $\mathcal{U}$ and $\mathcal{V}$ of symmetric entourages). A multimap $T : X \multimap Y$ is said to be *approachable* whenever $T$ admits a $W$-approximative continuous selection $s : X \to Y$ for each $W$ in the basis $\mathcal{W}$ of the product uniformity on $X \times Y$; that is, $\mathrm{Gr}(s) \subset W[\mathrm{Gr}(F)]$, where

$$W[A] := \bigcup_{z \in A} W[z] = \{z' \in X \times Y : W[z'] \cap A \neq \emptyset\}$$

for any $A \subset X \times Y$, and

$$W[z] := \{z' \in X \times Y : (z, z') \in W]$$

for $z \in X \times Y$.

A multimap $T : X \multimap Y$ is said to be *approximable* if its restriction $T|_K$ to any compact subset $K$ of $X$ is approachable.

The following is due to Ben-El-Mechaiekh et al. [6, Proposition 3.9]:

**Lemma 7.** *Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are uniform spaces. If either*
- (i) *$X$ is paracompact and $(Y, D; \Gamma)$ is an LC-space; or*
- (ii) *$X$ is compact and $(Y, D; \Gamma)$ is an LG-space,*

*then every u.s.c. map $F : X \multimap Y$ with nonempty $\Gamma$-convex values is approachable.*

We adopt the following definitions from [10,11]:

Let $X$ and $Y$ be subsets of t.v.s. $E$ and $F$, respectively, and $T : X \multimap Y$ a multimap. Given two open neighborhoods $U$ and $V$ of the origin 0 of $E$ and $F$, respectively, a $(U, V)$-*approximative continuous selection* of $T$ is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (T[(x + U) \cap X] + V) \cap Y \quad \text{for every } \ x \in X.$$

$T$ is said to be *approachable* if it admits a $(U, V)$-approximative continuous selection for every $U$ and $V$ as above; and $T$ *approximable* if its restriction $T|_K$ to any compact subset $K$ of $X$ is approachable.

Note that an approachable map is always approximable. Recall that Ben-El-Mechaiekh et al. [3-11] established a large number of properties and examples of approachable or approximable maps.

**Examples.** We give some examples of approachable maps $T : X \to Y$ as follows:
1. Any selectionable multimap is approximable.
2. A locally selectionable map $T$ with convex values is approximable whenever $Y$ is a convex subset of a t.v.s.
3. An u.s.c. map $T$ with nonempty convex values is approachable whenever $X$ is paracompact and $Y$ is a convex subset of a locally convex t.v.s.; see Lemma 7(1).
4. An u.s.c. map $T$ with nonempty compact contractible values is approachable whenever $X$ is a finite polyhedron.
5. An u.s.c. map $T$ with nonempty compact values having trivial shape (that is, contractible in each neighborhood in $Y$) is approachable whenever $X$ is a finite polyhedron.

For 1 and 2, see Section 2; and for 3-5, see [5].

Let $\mathbb{A}(X, Y)$ denote the class of all u.s.c. approachable maps $T : X \multimap Y$ with compact values, $\mathbb{A}^\kappa(X, Y)$ the class of all u.s.c. approximable maps $T : X \multimap Y$ with compact values, and $\mathbb{A}^\kappa_c(X, Y)$ the class of all finite compositions $T : X \multimap Y$

of u.s.c. approximable maps with compact values, where the intermediate spaces are uniform ones. As usual, $\mathbb{C}(X,Y)$ denotes the class of all continuous functions $f : X \to Y$. Recall that the class $\mathbb{A}_c^\kappa$ is an example of the admissible class $\mathfrak{A}_c^\kappa$ and the better admissible class $\mathfrak{B}^\kappa$ due to the author [41,43].

We say that a uniform space $X$ has the (*compact*) *approachable fixed point property* (simply, $\mathbb{A}$-f.p.p.) if any (compact) map $T \in \mathbb{A}(X,X)$ has a fixed point; and the *approximable fixed point property* (simply, $\mathbb{A}^\kappa$-f.p.p.) if any map $T \in \mathbb{A}^\kappa(X,X)$ has a fixed point. Similarly, the $\mathbb{A}_c^\kappa$-f.p.p. can be defined.

**Theorem 8** (Park [45]). *A* $\Phi$*-space* $(X,D;\Gamma)$ *has the compact* $\mathbb{A}$*-f.p.p.*

Since every single-valued continuous function is approachable, we have the following:

**Corollary 8.1.** *A* $\Phi$*-space* $(X,D;\Gamma)$ *has the compact f.p.p.*

This was first obtained by Horvath [21, Section 4, Theorem 4] for a $C$-space $(X;\Gamma)$ and applied to fixed point and coincidence problems, the Ky Fan type minimax inequality, and the von Neumann-Sion type minimax equality. In the end of [21], its author generalized Ky Fan's extension of the Tychnoff fixed point theorem. Note that this can be done also for $G$-convex spaces.

**Theorem 9.** *For a subset* $X$ *of a t.v.s.* $E$*, the following are equivalent:*

(1) $X$ *has the compact f.p.p.*
(2) $X$ *has the compact* $\mathbb{A}$*-f.p.p.*

*Further, if* $X$ *is paracompact and convex, then* (1) *and* (2) *are equivalent to the following:*

(3) $X$ *has the compact* $\mathbb{L}$*-f.p.p.*

*Proof.* (1) $\Rightarrow$ (2): For any $U \in \mathcal{V}$, there exists a $V \in \mathcal{V}$ such that $V + V \subset U$. Let $T \in \mathbb{A}(X,X)$ be a compact map. Then we have a continuous function $s : X \to X$ satisfying

$$s(x) \in T[(x+V) \cap X] + V \quad \text{for every } x \in X.$$

We may assume that $V$ is symmetric and that $s(x) \in \overline{T(X)}$ for all $x \in X$. [Otherwise, by the regularity of $E$, we may have $[s(x) - U] \cap T[(x+U) \cap X] = \emptyset$ for some $U \in \mathcal{V}$.] Therefore, $s$ has a fixed point $x_0 \in X$ by (1). Then

$$x_0 = s(x_0) \in T[(x_0+V) \cap X] + V,$$

and hence there exist $y_U \in T[(x_0+V) \cap X]$ and $x_U \in (x_0+V) \cap X$ such that $y_U \in T(x_U)$ and $x_0 \in y_U + V$. Then we have $x_U - y_U \in (x_0+V) - (x_0-V) = V + V \subset U$. Since $\{y_U : U \in \mathcal{V}\}$ is a net in the compact set $\overline{T(X)}$, it has a subnet converging to some $\hat{x} \in \overline{T(X)}$. Then $\{x_U : U \in \mathcal{V}\}$ has a corresponding subnet converging to $\hat{x}$. Since the graph of $T$ is closed and $(x_U, y_U) \in \mathrm{Gr}(T)$, we have $(\hat{x}, \hat{x}) \in \mathrm{Gr}(T)$. This completes our proof.

(2) $\Rightarrow$ (1): Every single-valued continuous function is approachable.

(1) $\Rightarrow$ (3): If $X$ is paracompact and convex, Lemma 4(2) with $X = Y$ works.

(3) $\Rightarrow$ (1): Every single-valued map is locally selectionable with convex values. □

From Theorem 9 and its proof, we have the following:

**Corollary 9.1.** *A convex subset $X$ of a t.v.s. $E$ has the compact $\mathbb{A}$-f.p.p. whenever one of the following holds:*

(1) *$X$ is admissible (in the sense of Klee).*
(2) *Every compact subset of $X$ is c.t.b.*
(3) *$X$ is compact and weakly admissible.*

*Proof.* For (1) and (3), $X$ has the compact f.p.p. by Lemma 1(2),(3).

(2) As in the proof of Theorem 9, $\overline{s(X)} \subset \overline{T(X)}$ and hence $\overline{s(X)}$ is c.t.b. Therefore, by Lemma 1(1), $s$ has a fixed point. By following the proof of Theorem 9, we have the conclusion. □

*Remark.* For a locally convex t.v.s. $E$, a particular form of Corollary 9.1(1)-(3) appeared in [4, Theorem 2.4], [9, Corollary 3.4], [10, Corollary 7.3]. The following is another generalization:

Recall that a subset $X$ of a t.v.s. $E$ is said to be *almost convex* if for each $U \in \mathcal{V}$ and for any finite subset $\{x_1, x_2, \dots, x_n\}$ of $X$, there exists a finite subset $\{z_1, z_2, \dots, z_n\}$ of $X$ such that $z_i - x_i \in U$ for all $i$ and $\mathrm{co}\,\{z_i\}_{i=1}^n \subset X$.

**Corollary 9.2.** *An almost convex subset $X$ of a locally convex t.v.s. $E$ has* (1) *the compact f.p.p. and* (2) *the compact $\mathbb{A}$-f.p.p.*

*Remarks.* 1. (1) is given by Park and Tan [55].

2. It is known that, if $X$ is convex, then it has the compact $\mathbb{A}^\kappa$-f.p.p.; see [3].

From the definition, every approximable map with compact domain is approachable. Hence, we have the following:

**Corollary 9.3.** *A compact convex subset $X$ of a t.v.s. $E$ has the $\mathbb{A}^\kappa$-f.p.p. whenever one of the following holds:*

(1) *$X$ has the f.p.p.*
(2) *$X$ is c.t.b.*
(3) *$X$ is weakly admissible.*

For approximable maps having compact domains, we have some new results. We need the following:

**Lemma 10.** [10] *Let $X$ be a compact subset of a t.v.s., $Y$ a subset of a t.v.s., and $\Gamma$ a closed subset of $X \times Y$. Then the following statements are equivalent:*

(i) $\mathrm{Gr}(f) \cap \Gamma \neq \emptyset$ *for each* $f \in \mathbb{C}(X, Y)$*;*
(ii) $\mathrm{Gr}(T) \cap \Gamma \neq \emptyset$ *for each* $T \in \mathbb{A}_c^\kappa(X, Y)$*.*

From Lemma 10, we have a generalization of Corollary 9.3(1) as follows:

**Theorem 11.** *Let $X$ be a compact subset of a t.v.s. $E$. Then $X$ has the f.p.p. if and only if it has the $\mathbb{A}_c^\kappa$-f.p.p.*

*Proof.* Let $X = Y$ and $\Gamma = \{(x, x) : x \in X\}$ be the diagonal of $X \times X$ in Lemma 10. Since any $f \in \mathbb{C}(X, X)$ has a fixed point, Condition (i) of Lemma 10 holds and hence so does (ii). The converse is clear. This completes our proof. □

*Remark.* For a locally convex t.v.s., Theorem 11 generalizes [9, Corollary 3.6] and [10, Corollary 7.6].

**Corollary 11.1.** *Let $X$ be a finite polyhedron. Then any map $T : X \multimap X$ satisfying one of the following conditions has a fixed point:*

(i) *locally selectionable with convex values;*
(ii) *an u.s.c. map with nonempty compact contractible values;*
(iii) *an u.s.c. map with nonempty compact values having trivial shape.*

*Proof.* Note that $X$ can be imbedded in a t.v.s. and has the f.p.p. Since any map of type (i)-(iii) is approximable, Theorem 11 works. □

*Remark.* It is well-known that contractibility in (ii) can be extended to acyclicity in view of the Lefschetz fixed point theorem for acyclic maps.

## 5. The Kakutani maps on $G$-convex spaces

In this section, we obtain fixed point theorems for general Kakutani type multimaps defined on particular types of $G$-convex spaces.

Let $Y$ be a topological space and $(X, D; \Gamma)$ a $G$-convex space. A map $T : Y \multimap X$ is called a *Kakutani map* (or a $\mathbb{K}$-map) if $T$ is u.s.c. with nonempty compact $\Gamma$-convex values. Let $\mathbb{K}(Y, X)$ denote the class of all Kakutani maps $T : Y \multimap X$.

We say that a $G$-convex space $(X, D; \Gamma)$ has the (*compact*) *Kakutani fixed point property* (simply, $\mathbb{K}$-f.p.p.) if any (compact) Kakutani map $T : X \multimap X$ has a fixed point.

**Theorem 12** (Park [47]). *An $LG$-space $(X, D; \Gamma)$ has the compact $\mathbb{K}$-f.p.p. Moreover, if singletons are $\Gamma$-convex (that is, $\{x\} = \Gamma_{\{x\}}$ for $x \in D$), then it has the compact f.p.p.*

**Lemma 13** (Komiya [26]). *If a paracompact $LG$-space $(X, D; \Gamma)$ has the $\Phi$-f.p.p., then it has the $\mathbb{K}$-f.p.p.*

From Theorem 12 and Lemma 2(2), we have

**Theorem 14.** *Let $(X, D; \Gamma)$ be a paracompact $LC$-space such that every singleton is $\Gamma$-convex. Then $X$ has* (1) *the compact f.p.p.,* (2) *the compact $\Phi$-f.p.p., and* (3) *the compact $\mathbb{K}$-f.p.p.*

*Proof.* Since $(X, D; \Gamma)$ is an $LG$-space, Condition (3) holds by Theorem 12 (or [6, Corollary 4.7] or [45, Theorem 5]). Condition (1) follows from (3). Finally, Condition (2) follows from (1) by Lemma 2(2). □

*Remark.* Tarafdar [60, Theorem 2.1 and Corollary 2.2] showed that (1) holds for an $LC$-metric space $(X, d; \Gamma)$. A particular form of Theorem 14 was given by Horvath [22].

Moreover, from Lemma 1 and Theorem 9, we have

**Theorem 15.** *Let $X$ be a convex subset of a t.v.s. $E$. If $X$ is admissible or c.t.b., then $X$ has* (1) *the compact f.p.p.,* (2) *the compact $\mathbb{A}$-f.p.p., and* (3) *the compact $\mathbb{K}$-f.p.p.*

**Corollary 15.1.** *Let $X$ be a convex subset of a locally convex t.v.s. $E$. Then $X$ has* (1) *the compact f.p.p.,* (2) *the compact $\mathbb{A}$-f.p.p., and* (3) *the compact $\mathbb{K}$-f.p.p.*

*Remarks.* 1. Recall that (1) is due to Hukuhara, (2) to Ben-El-Mechaiekh, and (3) to Himmelberg; see [41,43]. In fact, $X$ has the compact $\mathbb{A}^\kappa$-f.p.p.; see [3].

2. More generally, if $X$ is an almost convex subset of a locally convex t.v.s. $E$, then $X$ has (1)-(3); see Corollary 9.2 and [56].

A metric space $(H, d)$ is said to be *hyperconvex* if for any collection of points $\{x_\alpha\}$ of $H$ and for any collection $\{r_\alpha\}$ of nonnegative reals such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, we have

$$\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset,$$

where $B(x, r)$ denotes the closed ball with center $x \in H$ and radius $r > 0$.

For any nonempty bounded subset $A$ of $H$, its *convex hull* $\operatorname{Co} A$ is defined by

$$\operatorname{Co} A = \bigcap\{B : B \text{ is a closed ball containing } A\}.$$

Let $\mathcal{A}(H) = \{A \subset H : A = \operatorname{Co} A\}$; that is, $A \in \mathcal{A}(H)$ iff $A$ is an intersection of closed balls.

The following is known by Horvath [22]:

**Lemma 16.** *Any hyperconvex metric space $(H, d)$ is a complete metric LC-space $(H; \Gamma)$ with $\Gamma_A = \operatorname{Co} A$ for each $A \in \langle H \rangle$.*

From Lemma 2(2), Theorems 8 and 14, and Lemma 16, we have the following:

**Corollary 16.1.** *Let $(H, d)$ be a hyperconvex metric space. Then X has* (1) *the compact f.p.p.,* (2) *the compact $\mathbb{A}$-f.p.p.,* (3) *the compact $\Phi$-f.p.p., and* (4) *the compact $\mathbb{K}$-f.p.p.*

*Remark.* Recall that (1) and (4) are due to Park; see [45].

Since any subset $A \in \mathcal{A}(H)$ of a hyperconvex metric space $H$ is hyperconvex, we have the following:

**Corollary 16.2.** *Let $(H, d)$ be a hyperconvex metric space and $X \in \mathcal{A}(H)$. Then X has* (1) *the compact f.p.p.,* (2) *the compact $\mathbb{A}$-f.p.p.,* (3) *the compact $\Phi$-f.p.p., and* (4) *the compact $\mathbb{K}$-f.p.p.*

**Theorem 17.** *Let $(X, D; \Gamma)$ be a compact G-convex uniform space such that any singleton is $\Gamma$-convex. Consider the following conditions:*

(1) *$X$ has the f.p.p.*
(2) *$X$ has the $\mathbb{A}_c^\kappa$-f.p.p.*
(3) *$X$ has the $\mathbb{K}$-f.p.p.*

*Then* (3) $\Rightarrow$ (1) $\Leftrightarrow$ (2).

*Proof.* (3) $\Rightarrow$ (1): Any continuous function $f : X \to X$ is a Kakutani map since $f(x)$ is $\Gamma$-convex for each $x \in X$.

(1) $\Rightarrow$ (2): Let $F \in \mathbb{A}_c^\kappa(X, X)$. Then we may assume $F \in \mathbb{A}(X, X)$. Then for each $W$ in the basis $\mathcal{W}$ of the product uniformity on $X \times X$, there exists a $W$-approximative continuous selection $s : X \to X$ of $F$; that is, $\operatorname{Gr}(s) \subset W[\operatorname{Gr}(F)]$. By (1), $s$ has a fixed point $x_W \in X$ and hence $(x_W, x_W) \in \operatorname{Gr}(s)$. Then $(x_W, x_W) \in W[\operatorname{Gr}(F)]$ and $z_W := (x_W, x_W) \in W[z]$ for some $z := (x, y) \in \operatorname{Gr}(F)$. Hence,

$(z, z_W) \in W$. Since $X \times X$ is compact, we may assume that the net $\{z_W\}$ converges to a point $z_0 := (x_0, x_0)$ in the diagonal $\Delta$ of $X$, and hence the corresponding net $\{z\}$ also converges to $z_0$. Since $z \in \mathrm{Gr}(F)$ and $\mathrm{Gr}(F)$ is closed, we have $z_0 \in \mathrm{Gr}(F)$ and hence $x_0 \in F(x_0)$.

(2) $\Rightarrow$ (1): Clear. □

*Remarks.* 1. Note that the part (1) $\Leftrightarrow$ (2) of Theorem 17 is comparable to Lemma 10.

2. There is an example of a compact $G$-convex space without f.p.p.; see [54].

3. We know that $X$ has the $\Phi$-f.p.p.; see Theorem 5(1).

**Corollary 17.1.** *Let $(X, D; \Gamma)$ be a compact LG-space such that any singleton is $\Gamma$-convex. Then $X$ has* (1) *the f.p.p.,* (2) *the $\mathbb{A}_c^\kappa$-f.p.p., and* (3) *the $\mathbb{K}$-f.p.p.*

*Proof.* Note that (3) holds by Theorem 12 or by Theorem 5(1) and Lemma 13. □

**Examples.** Let $X$ be a compact convex subset of a t.v.s. $E$. We give known cases when (1)-(3) of Theorem 17 hold as follows; for the references, see [43,45,47,48].

1. For a Euclidean space $E$, (1) is due to Brouwer and (1) $\Rightarrow$ (3) to Kakutani by a different method.

2. For a normed vector space $E$, (1) is due to Schauder and (3) to Bohnenblust and Karlin.

3. For a locally convex t.v.s. $E$, (1) is due to Tychonoff, (2) to Ben-El-Mechaiekh, and (3) to Fan and Glicksberg, independently.

4. For a t.v.s. $E$ having sufficiently many linear functionals, (1) is due to Fan and (3) to Granas and Liu.

5. For an admissible set $X$, (1) is due to Hahn and Pötter and (2) and (3) to Park.

6. If $X$ is locally convex, (1) is due to Rzepecki and (3) to Idzik.

7. For a c.t.b. set $X$, (1) and (3) are particular cases of a result of Idzik [23].

8. Further if $X$ is a $\Phi$-space, (1) is due to Horvath and (2) to Park; see Theorem 8.

9. For a weakly admissible set $X$, (1) is due to Nhu [30] and Arandelović [2], and (3) to Okon [33,35].

Recall that all *Roberts spaces* – that is, compact convex sets with no extreme points constructed by Roberts' method of needle point spaces – have the fixed point property; see Nhu et al. [29,31].

10. For a Roberts space $X$, (1) is due to Nhu et al. [29] and (3) to Okon [33-35].

*Remark.* All examples 1-10 are for (3) $\Rightarrow$ (1) $\Leftrightarrow$ (2) in Theorem 17. Examples 1-4 are for Corollary 17.1.

## 6. Condensing approximable maps

Various types of condensing maps are variants of compact maps. For such type of maps, we need the following:

**Lemma 18.** *Let $X$ be a closed convex subset of a t.v.s. $E$ with $x_0 \in X$ and $T : X \multimap X$ a map such that*

(C) $A \subset X,\ A = \overline{\mathrm{co}}\,(\{x_0\} \cup T(A))$ *implies $A$ is compact.*

*Then there exists a compact convex subset* $C_0$ *of* $X$ *such that* $T(C_0) \subset C_0$.

*Proof.* The proof is standard, but for completeness we give here. Let $\mathcal{H}$ be the family of all closed convex subsets $C$ of $X$ with $x_0 \in C$ such that $T(C) \subset C$. Note that $\mathcal{H}$ is not empty since $X \in \mathcal{H}$. Let $C_0 := \bigcap\{C : C \in \mathcal{H}\}$. Then $C_0$ is a nonempty closed convex subset of $X$ such that $T(C_0) \subset C_0$. It remains to show that $C_0$ is compact. Let

$$C_1 := \overline{\mathrm{co}}\,(\{x_0\} \cup T(C_0)).$$

Note that $x_0 \in C_0, T(C_0) \subset C_0$, and $C_0$ is closed and convex. So we have $C_1 \subset C_0$. Moreover, $T(C_1) \subset T(C_0) \subset C_1$. Therefore $C_1 \in \mathcal{H}$ and so $C_0 \subset C_1$. Hence,

$$C_0 = C_1 = \overline{\mathrm{co}}\,(\{x_0\} \cup T(C_0)),$$

whence $C_0$ is compact by condition (C). □

From Lemmas 4(2) and 18 and Theorem 17, we have the following:

**Theorem 19.** *Let $E$ be a t.v.s. whose nonempty compact convex subsets have the f.p.p., and $X$ a closed convex subsets of $E$ with $x_0 \in X$ such that Condition* (C) *holds. Then $X$ has* (1) *the f.p.p.,* (2) *the $\mathbb{L}$-f.p.p., and* (3) *the $\mathbb{A}_c^{\kappa}$-f.p.p.*

*Proof.* For any multimap $T : X \multimap X$, by Lemma 18, there exists a compact convex subset $C_0$ of $X$ such that $T(C_0) \subset C_0$. Then

(1) $C_0$ has the f.p.p. by the hypothesis.

(2) For any locally selectionable map $T : X \multimap X$ having convex values, $T|_{C_0}$ has a continuous selection $s : C_0 \to C_0$ by Lemma 4(2). Then $s$ has a fixed point.

(3) For the compact convex subset $C_0$, (3) is equivalent to (1) by Theorem 17.

This completes our proof. □

*Remarks.* 1. Any locally convex t.v.s. $E$ and any t.v.s. $E$ on which $E^*$ separates points on $E$ are examples of t.v.s. satisfying the hypothesis.

2. A result similar to Theorem 19(2) for the case $E$ on which $E^*$ separates points on $E$ appeared in [1, Theorem 4.2]. For a $\Phi$-map $T : X \multimap X$, Theorem 19(2) was appeared in [28,36,37] under the restriction that $E$ is locally convex.

Let $X$ be a closed convex subset of a t.v.s. $E$ and $C$ a lattice with a least element, which is denoted by 0. A function $\Psi : 2^X \to C$ is called a *measure* of *noncompactness* on $X$ provided that the following conditions hold for any $A, B \in 2^X$:

(1) $\Psi(A) = 0$ if and only if $A$ is relatively compact;
(2) $\Psi(\overline{\mathrm{co}}\,A) = \Psi(A)$; and
(3) $\Psi(A \cup B) = \max\{\Psi(A), \Psi(B)\}$.

It follows that $A \subset B$ implies $\Psi(A) \le \Psi(B)$.

The above notion is a generalization of the set-measure $\gamma$ and the ball-measure $\chi$ of noncompactness defined in terms of a family of seminorms or a norm.

For a measure $\Psi$ of noncompactness on $E$, a map $T : X \multimap E$ is said to be $\Psi$-*condensing* provided that if $A \subset X$ and $\Psi(A) \le \Psi(T(A))$, then $A$ is relatively compact; that is, $\Psi(A) = 0$.

From now on, we assume that $\Psi$ is a measure of noncompactness on the given set $X$ in a t.v.s. $E$ or on $E$ if necessary. Note that any map defined on a compact set or any compact map is $\Psi$-condensing. Especially, if $E$ is locally convex, then a compact map $T : X \multimap E$ is $\gamma$- or $\chi$-condensing whenever $X$ is complete or $E$ is quasi-complete.

The following in [28] is a simple consequence of Lemma 18:

**Lemma 20.** *Let $X$ be a nonempty closed convex subset of a t.v.s. $E$ and $T : X \multimap X$ a $\Psi$-condensing map. Then there exists a nonempty compact convex subset $K$ of $X$ such that $T(K) \subset K$.*

From Theorem 11 and Lemma 20, we have the following particular form of Theorem 19:

**Theorem 21.** *Let $E$ be a t.v.s. whose nonempty compact convex subsets have the f.p.p., and $X$ a closed convex subset of $E$. Then any $\Psi$-condensing map $T \in \mathbb{A}_c^\kappa(X, X)$ has a fixed point.*

*Proof.* By Lemma 20, there is a nonempty compact convex subset $K$ of $X$ such that $T(K) \subset K$. Then, by Theorem 11, $T|_K \in \mathbb{A}_c^\kappa(K, K)$ has a fixed point $x_0 \in K$. This completes our proof. □

*Remark.* It is known that the Leray-Schauder alternative holds for a $\Psi$-condensing closed approximable maps; see Park [39] and Ben-El-Mechaiekh et al. [7].

**Corollary 21.1.** *Let $E$ be a t.v.s. whose nonempty compact convex subsets have the f.p.p., and $X$ a closed convex subset of $E$. Then any $\Psi$-condensing locally selectionable map $T : X \multimap X$ with convex values has a fixed point.*

**Corollary 21.2.** *Let $E$ be a locally convex t.v.s. and $X$ a closed convex subset of $E$. Then any $\Psi$-condensing Kakutani map $T : X \multimap X$ has a fixed point.*

In the remainder of this section, we list more than ten papers in chronological order, from which we can deduce particular forms of Theorem 21:

Darbo [15]: Recall that Kuratowski defined the measure of noncompactness, $\alpha(A)$, of a bounded subset $A$ of a metric space $(X, d)$:

$$\alpha(A) = \inf \{\varepsilon > 0 : A \text{ can be covered by a finite number of sets of diameter less than or equal to } \varepsilon\}.$$

Let $T : X \to X$ be a continuous function. Darbo calls $T$ an *$\alpha$-contraction* if given any bounded set $A$ in $X$, $T(A)$ is bounded in $X$ and

$$\alpha[T(A)] \le k\alpha(A),$$

where the constant $k$ fulfils the inequality $0 \le k < 1$. Darbo [15] showed that if $G$ is a closed, bounded, convex subset of a Banach space $X$ and $T : G \to G$ is an $\alpha$-contraction, then $T$ has a fixed point.

Sadovskii [59]: Introduced the notion of condensing maps in Banach spaces and obtained a form of Theorem 21 extending the above result of Darbo.

Lifšic and Sadovskii [27]: The above result was extended to a locally convex t.v.s. $E$.

Himmelberg, Porter, and Van Vleck [20]: A form of Theorem 21 for a locally convex t.v.s.

Daneš [14], Furi and Vignoli [18], and Nussbaum [32] obtained particular forms of Theorem 21 for a Banach space $E$.

Reich [57] extended Sadovskii's theorem to a locally convex t.v.s.

Reinermann [58]: A form of Theorem 21 for a Banach space and $T = f \in \mathbb{C}(X, X)$.

Mehta, Tan, and Yuan [28]: Particular forms of Theorem 21 for the Fan-Browder type maps and the Kakutani maps, respectively, were obtained for a locally convex t.v.s.

*Remark.* The concept of compact multimaps has variants (not necessarily generalizations) in that of various types of condensing maps (pseudo-condensing or countably condensing maps or of Mönch type). It is well-known that the theory of such types of condensing maps reduces to that of compact maps. Therefore, our theorems might be applied to those types of condensing maps.

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