Journal of Nonlinear and Convex Analysis
Volume 6, Number 1, 2005, 55–64

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

A GENERIC RESULT IN MINIMAX OPTIMIZATION

ALEXANDER J. ZASLAVSKI

Abstract. In this paper we study a class of minimax problems on a complete metric space for which a cost function f is a maximum of functions $f_1, \ldots, f_n$. Using porosity notion we show that for most of problems there exists a unique solution. Moreover, if x is this unique solution, then the set of all $j \in \{1, \ldots, n\}$ satisfying $f_j(x) = f(x)$ is a singleton.

1. Introduction

The study of minimax problems is one of central topics in optimization theory. See, for example, [5-7] and the references mentioned therein. In this paper we consider a class of minimax problems

$$\max\{f_1(x), \ldots, f_n(x)\} \to \min,\ x \in X,$$

where n is a given natural number, X is a complete metric space and $f_1, \ldots, f_n$ are continuous functions. Using the porosity notion we show that for most problems there exists a unique point of minimum $z \in X$ and a unique integer $j \in \{1, \ldots, n\}$ such that

$$f_j(z) > f_i(z) \text{ for all } i \in \{1, \ldots, n\} \setminus \{j\}.$$

Here, instead of considering a certain property for a single minimax problem, we investigate it for a class of minimax problems and show that this property holds for most of the problems in the class. This approach has already been successfully applied in many areas of Analysis. See, for example, [1-4, 8-10].

Before we continue we recall the concept of porosity [4].

Let (Y, d) be a complete metric space. We denote by $B_d(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called porous with respect to d (or just porous if the metric is understood) if there exist $\alpha \in (0, 1]$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$ there exists $z \in Y$ for which

$$B_d(z, \alpha r) \subset B_d(y, r) \setminus E.$$

A subset of the space Y is called σ-porous with respect to d (or just σ-porous if the metric is understood) if it is a countable union of porous (with respect to d) subsets of Y.

The following definition was introduced in [10].

Assume that Y is a nonempty set and $d_1, d_2 : Y \times Y \to [0, \infty)$ are metrics which satisfy $d_1(x, y) \le d_2(x, y)$ for all $x, y \in Y$.

A subset $E \subset Y$ is called porous with respect to the pair (d_1, d_2) (or just porous if the pair of metrics is understood) if there exist $\alpha \in (0, 1]$ and $r_0 > 0$ such that

2000 *Mathematics Subject Classification.* 49J35, 54E52.
Key words and phrases. Complete metric space, minimax problem, porous set.

for each $r \in (0, r_0]$ and each $x \in Y$ there exists $y \in Y$ for which

$$d_2(y, x) \le r, \ B_{d_1}(y, \alpha r) \cap E = \emptyset.$$

A subset of the space Y is called σ-porous with respect to (d_1, d_2) (or just σ-porous if the pair of metrics is understood) if it is a countable union of porous (with respect to (d_1, d_2)) subsets of Y.

In [10] we use this generalization of the porosity notion because in applications a space is usually endowed with a pair of metrics and one of them is weaker than the other. Note that porosity of a set with respect to one of these two metrics does not imply its porosity with respect to the other metric. However, the following proposition proved in [10] shows that if a subset $E \subset Y$ is porous with respect to (d_1, d_2), then E is porous with respect to any metric which is weaker than d_2 and stronger than d_1.

Proposition 1.1. *Let k_1 and k_2 be positive numbers and $d : Y \times Y \to [0, \infty)$ be a metric such that $k_1 d(x, y) \ge d_1(x, y)$ and $k_2 d(x, y) \le d_2(x, y)$ for all $x, y \in Y$. Assume that a set $E \subset Y$ is porous with respect to (d_1, d_2). Then E is porous with respect to d.*

2. Well-posedness of optimization problems

We use the convention that $\infty/\infty = 1$. For each function $f : X \to [-\infty, \infty]$, where X is nonempty, we set

$$\inf(f) = \inf\{f(x) : \ x \in X\}.$$

We consider a metric space (X, ρ) which is called the domain space and a topological space $\mathcal{A}$ with the topology τ which is called the data space [8]. We always consider the set X with the topology generated by the metric ρ.

We assume that with every $a \in \mathcal{A}$ a lower semicontinuous function f_a on X is associated with values in $\bar{R} = [-\infty, \infty]$.

Let $a \in \mathcal{A}$. We say that the minimization problem for f_a on (X, ρ) is well-posed if $\inf(f_a)$ is finite and attained at a unique point $x_a \in X$ and the following assertion holds:

For each $\epsilon > 0$ there exists $\delta > 0$ such that if $z \in X$ satisfies $f_a(z) \le \inf(f_a) + \delta$, then $\rho(x_a, z) \le \epsilon$.

The following notion was introduced in [8].

Let $a \in \mathcal{A}$. We say that the minimization problem for f_a on (X, ρ) is strongly well-posed with respect to $(\mathcal{A}, \tau)$ if $\inf(f_a)$ is finite and attained at a unique point $x_a \in X$ and the following assertion holds:

For each $\epsilon > 0$ there exists a neighborhood $\mathcal{V}$ of a in $\mathcal{A}$ with the topology τ and $\delta > 0$ such that for each $b \in \mathcal{V}$, $\inf(f_b)$ is finite and if $z \in X$ satisfies $f_b(z) \le \inf(f_b) + \delta$, then $\rho(x_a, z) \le \epsilon$ and $|f_b(z) - f_a(x_a)| \le \epsilon$.

If $(\mathcal{A}, d)$ is a metric space and τ is a topology generated by the metric d, then "strongly well-posedness with respect to $(\mathcal{A}, \tau)$" will be sometimes replaced by "strongly well-posedness with respect to $(\mathcal{A}, d)$".

Assume that $d_1, d_2 : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be metrics such that $d_1(a, b) \le d_2(a, b)$ for all $a, b \in \mathcal{A}$. In our study we use the following hypotheses about the functions introduced in [10].

(H1) If $a \in \mathcal{A}$, $\inf(f_a)$ is finite, $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence and the sequence $\{f_a(x_n)\}_{n=1}^{\infty}$ is bounded, then the sequences $\{x_n\}_{n=1}^{\infty}$ converges in X.

(H2) For each $\epsilon > 0$ and each integer $m \geq 1$ there exist numbers $\delta > 0$ and $r_0 > 0$ such that the following property holds:

For each $a \in \mathcal{A}$ satisfying $\inf(f_a) \leq m$ and each $r \in (0, r_0]$ there exist $\bar{a} \in \mathcal{A}$ and $\bar{x} \in X$ such that

$$d_2(a, \bar{a}) \leq r, \ \inf(f_{\bar{a}}) \leq m + 1$$

and that for each $z \in X$ satisfying

$$f_{\bar{a}}(z) \leq \inf(f_{\bar{a}}) + \delta r$$

the inequality $\rho(z, \bar{x}) \leq \epsilon$ holds.

(H3) For each integer $n \geq 1$ there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$, each $a, b \in \mathcal{A}$ satisfying $d_1(a, b) \leq \alpha r$ and each $x \in X$ satisfying $\min\{f_a(x), f_b(x)\} \leq n$ the relation $|f_a(x) - f_b(x)| \leq r$ is valid.

The following variational principle was established in [10, Theorem 3.2].

Theorem 2.1. *Assume that (H1), (H2) and (H3) hold and that* $\inf(f_a)$ *is finite for each* $a \in \mathcal{A}$*. Then there exists a set* $\mathcal{B} \subset \mathcal{A}$ *such that* $\mathcal{A} \setminus \mathcal{B}$ *is* σ*-porous with respect to* (d_1, d_2) *and that for each* $a \in \mathcal{B}$ *the minimization problem for* f_a *on* (X, ρ) *is strongly well-posed with respect to* $(\mathcal{A}, d_1)$.

3. A porosity result

Let (X, ρ) be a complete metric space and n be a natural number.

Denote by $\mathcal{M}$ the set of all functions $f = (f_1, \dots, f_n) : \ X \to R^n$ such that for each $i \in \{1, \dots, n\}$ the function $f_i : X \to R^1$ is continuous and the function

$$x \to \max\{f_i(x) : \ i = 1, \dots, n\}, \ x \in X$$

is bounded from below on X.

For $f = (f_1, \dots, f_n), \ g = (g_1, \dots, g_n) \in \mathcal{M}$ set

$$\tilde{d}(f, g) = \sup\{|f_i(x) - g_i(x)| : \ x \in X, \ i = 1, \dots, n\}, \tag{3.1}$$

$$d(f, g) = \tilde{d}(f, g)(1 + \tilde{d}(f, g))^{-1}.$$

Here we use the convention that $\infty/\infty = 1$.

It is not difficult to see that the metric space $(\mathcal{M}, d)$ is complete.

For each $f = (f_1, \dots, f_n) \in \mathcal{M}$ define a function $\max\{f_1, \dots, f_n\} : \ X \to R^1$ by

$$(\max\{f_1, \dots, f_n\})(x) = \max\{f_1(x), \dots, .f_n(x)\}, \ x \in X.$$

In this section we prove the following porosity result.

Theorem 3.1. *There exists a set* $\mathcal{F} \subset \mathcal{M}$ *such that its complement* $\mathcal{M} \setminus \mathcal{F}$ *is* σ*-porous with respect to* d *and that for each* $f = (f_1, \dots, f_n) \in \mathcal{F}$ *the minimization problem for* $\max\{f_1, \dots, f_n\}$ *on* (X, ρ) *is strongly well-posed with respect to* $(\mathcal{M}, d)$.

Proof. We prove the theorem as a realization of the variational principle established by Theorem 2.1.

Set $\mathcal{A} = \mathcal{M}$. For each $a = (a_1, \dots, a_n) \in \mathcal{A}$ define

$$f_a = \max\{a_1, \dots, a_n\}.$$

Set $d_1 = d_2 = d$. We show that (H1), (H2) and (H3) hold. It is not difficult to see that (H1) and (H3) are true. We show that (H2) holds.

Let $\epsilon \in (0,1)$. Choose a positive number δ such that

$$\delta < \epsilon/4. \tag{3.2}$$

Assume that

$$f = (f_1, \dots, f_n) \in \mathcal{M} \text{ and } r \in (0,1]. \tag{3.3}$$

Choose $\bar{x} \in X$ such that

$$\max\{f_1, \dots, f_n\}(\bar{x}) \le \inf(\max\{f_1, \dots, f_n\}) + \delta r \tag{3.4}$$

and define $\bar{f} = (\bar{f}_1, \dots, \bar{f}_n) \in \mathcal{M}$ by

$$\bar{f}_i(x) = f_i(x) + 2\delta r \epsilon^{-1} \min\{\rho(x, \bar{x}), 1\},\ x \in X,\ i = 1, \dots, n. \tag{3.5}$$

By (3.1), (3.5) and (3.2),

$$d(f, \bar{f}) \le \tilde{d}(f, \bar{f}) \le 2\delta r \epsilon^{-1} \le r. \tag{3.6}$$

It follows from (3.5), (3.2) and (3.3) that for each $x \in X$

$$\begin{aligned}\max\{\bar{f}_1, \dots, \bar{f}_n\}(x) &= 2\delta r \epsilon^{-1} \min\{\rho(x, \bar{x}), 1\} + \max\{f_1, \dots, f_n\}(x) \\ &\le \max\{f_1, \dots, f_n\}(x) + 1\end{aligned}$$

and

$$\inf(\max\{\bar{f}_1, \dots, \bar{f}_n\}) \le \inf\{\max\{f_1, \dots, f_n\}) + 1. \tag{3.7}$$

Assume that $x \in X$ satisfies

$$\max\{\bar{f}_1, \dots, \bar{f}_n\}(x) \le \inf(\max\{\bar{f}_1, \dots, \bar{f}_n\}) + \delta r. \tag{3.8}$$

By (3.5), (3.8) and (3.4)

$$\begin{aligned}\max\{f_1, \dots, f_n\}(x) &+ 2\delta r \epsilon^{-1} \min\{\rho(x, \bar{x}), 1\} \\ &= \max\{\bar{f}_1, \dots, \bar{f}_n\}(x) \\ &\le \max\{\bar{f}_1(\bar{x}), \dots, \bar{f}_n(\bar{x})\} + \delta r \\ &= \max\{f_1(\bar{x}), \dots, f_n(\bar{x})\} + \delta r \le \max\{f_1(x), \dots, f_n(x)\} + 2\delta r,\end{aligned}$$

$$2\delta r \epsilon^{-1} \min\{\rho(x, \bar{x}), 1\} \le 2\delta r,$$

$$\rho(x, \bar{x}) \le \epsilon.$$

Thus (H2) holds. This completes the proof of the theorem. □

4. Main results

Let (X, ρ) be a complete metric space and n be a natural number. Consider the complete metric space $(\mathcal{M}, d)$ introduced in Section 3.

Denote by $\mathcal{F}_0$ the set of all $f = (f_1, \dots, f_n) \in \mathcal{M}$ which have the following property:

(P1) There are $\delta, \epsilon > 0$ and $j \in \{1, \dots, n\}$ such that if $x \in X$ satisfies

$$\max\{f_1, \dots, f_n\}(x) < \inf(\max\{f_1, \dots, f_n\}) + \delta,$$

then

$$f_j(x) > f_i(x) + \epsilon \text{ for all } i \in \{1, \dots, n\} \setminus \{j\}.$$

Proposition 4.1. *Assume that $f = (f_1, \dots, f_n) \in \mathcal{M}$, the minimization problem for the function $\max\{f_1, \dots, f_n\}$ on (X, ρ) is well-posed, $\bar{x} \in X$ satisfies*

$$\max\{f_1(\bar{x}), \dots, f_n(\bar{x})\} = \inf(\max\{f_1, \dots, f_n\})$$

and $j \in \{1, \dots, n\}$ satisfies

$$f_j(\bar{x}) > f_i(\bar{x}) \text{ for all } i \in \{1, \dots, n\} \setminus \{j\}.$$

Then $f = (f_1, \dots, f_n) \in \mathcal{F}_0$.

Proof. Choose $\Delta > 0$ such that

$$f_j(\bar{x}) > f_i(\bar{x}) + 4\Delta \text{ for all } i \in \{1, \dots, n\} \setminus \{j\}. \tag{4.1}$$

Since $f_1, \dots, f_n$ are continuous functions there is $\epsilon > 0$ such that if $x \in X$ satisfies $\rho(x, \bar{x}) \le \epsilon$, then

$$|f_i(x) - f_i(\bar{x})| \le \Delta/8, \ i = 1, \dots, n. \tag{4.2}$$

Since the minimization problem for the function $\max\{f_1, \dots, f_n\}$ is well-posed on (X, ρ) there is $\delta > 0$ such that if $x \in X$ satisfies

$$\max\{f_1, \dots, f_n\}(x) \le \inf(\max\{f_1, \dots, f_n\}) + \delta, \tag{4.3}$$

then

$$\rho(x, \bar{x}) \le \epsilon. \tag{4.4}$$

Assume now that $x \in X$ satisfies (4.3). Then (4.4) holds. By (4.4) and the definition of ϵ the inequality (4.2) is valid for $i = 1, \dots, n$. It follows from (4.2) and (4.1) that for all $i \in \{1, \dots, n\} \setminus \{j\}$

$$\begin{aligned} f_j(x) &\ge f_j(\bar{x}) - \Delta/8 \ge f_i(\bar{x}) + 4\Delta - \Delta/8 \\ &\ge f_i(x) - \Delta/8 + 4\Delta - \Delta/8 \ge f_i(x) + 3\Delta. \end{aligned}$$

Thus $f \in \mathcal{F}_0$. Proposition 4.1 is proved. □

Proposition 4.2. *The set $\mathcal{F}_0$ is open.*

Proof. Let $f = (f_1, \dots, f_n) \in \mathcal{F}_0$ and let $\epsilon, \delta > 0$, $j \in \{1, \dots, n\}$ be as guaranteed by property (P1). We may assume that

$$\delta < \min\{1, \epsilon\}/4. \tag{4.5}$$

Assume that $g = (g_1, \dots, g_n) \in \mathcal{M}$ satisfies

$$d(g, f) \le \delta/16. \tag{4.6}$$

We show that $g \in \mathcal{F}_0$. (3.1) and (4.6) imply that

$$\tilde{d}(f,g) = d(f,g)(1-d(f,g))^{-1} \le 2\delta/16 = \delta/8 \tag{4.7}$$

and

$$|f_i(x) - g_i(x)| \le \delta/8,\ x \in X,\ i = 1,\dots,n. \tag{4.8}$$

Assume that $x \in X$ satisfies

$$\max\{g_1(x),\dots.g_n(x)\} \le \inf(\max\{g_1,\dots,g_n\}) + \delta/8. \tag{4.9}$$

It follows from (4.8) and (4.9) that

$$\begin{aligned}\max\{f_1(x),\dots,f_n(x)\} &\le \max\{g_1(x),\dots,g_n(x)\} + \delta/8\\ &\le \inf(\max\{g_1,\dots,g_n\}) + \delta/4\\ &\le \inf(\max\{f_1,\dots,f_n\}) + 3\delta/8.\end{aligned}$$

By this inequality and (P1) with ϵ, δ

$$f_j(x) > f_i(x) + \epsilon \text{ for all } i \in \{1,\dots,n\} \setminus \{j\}. \tag{4.10}$$

It follows from (4.8), (4.10) and (4.5) that for al $i \in \{1,\dots,n\} \setminus \{j\}$

$$\begin{aligned}g_j(x) &\ge f_j(x) - \delta/8 \ge f_i(x) + \epsilon - \delta/8\\ &\ge g_i(x) - \delta/8 + \epsilon - \delta/8 \ge g_i(x) + \epsilon/2.\end{aligned}$$

Thus $g \in \mathcal{F}_0$. Proposition 4.2 is proved. □

Proposition 4.3. *The set $\mathcal{F}_0$ is an everywhere dense subset of $(\mathcal{M}, d)$.*

Proof. Let $g = (g_1,\dots,g_n) \in \mathcal{M}$, $\epsilon > 0$. We show that there is $f = (f_1,\dots,f_n) \in \mathcal{F}_0$ such that $\tilde{d}(f,g) \le \epsilon$. By Theorem 3.1 there exists $h = (h_1,\dots,h_n) \in \mathcal{M}$ such that

$$\tilde{d}(g,h) \le \epsilon/4 \tag{4.11}$$

and the minimization problem for $\max\{g_1,\dots,g_n\}$ is strongly well-posed on (X,ρ) with respect to $(\mathcal{M}, d)$. There is $\bar{x} \in X$ such that

$$\max\{h_1(\bar{x}),\dots,h_n(\bar{x})\} = \inf(\max\{h_1,\dots,h_n\}) \tag{4.12}$$

and the following property holds:

(P2) For each $\gamma > 0$ there is $\delta > 0$ such that if $z \in X$ satisfies

$$\max\{h_1(z),\dots,h_n(z)\} \le \inf(\max\{h_1,\dots,h_n\}) + \delta,$$

then $\rho(z,\bar{x}) \le \gamma$.

There is $j \in \{1,\dots,n\}$ such that

$$h_j(\bar{x}) = \max\{h_1(\bar{x}),\dots,h_n(\bar{x})\}. \tag{4.13}$$

Since the functions $h_1,\dots,h_n$ are continuous there is

$$\delta_0 \in (0, \min\{1,\epsilon\}/8) \tag{4.14}$$

such that the following property holds:

(P3) If $i \in \{1,\dots,n\}$ and $x \in X$ satisfies $\rho(x,\bar{x}) \le \delta_0$, then $|h_i(x) - h_i\bar{x})| \le \epsilon/8$.

Define a continuous function $\psi : [0,\infty) \to [0,1]$ by

$$\psi(t) = 1,\ t \in [0,\delta_0/2],\ \psi(t) = 2 - (2t)/\delta_0,\ t \in [\delta_0/2, \delta_0], \tag{4.15}$$

$$\psi(t) = 0,\ t \in [\delta_0, \infty)$$

and set

$$\phi(x) = \psi(\rho(x, \bar{x})),\ x \in X. \tag{4.16}$$

Define $f = (f_1, \dots, f_n) \in \mathcal{M}$ as follows. For $i \in \{1, \dots, n\} \setminus \{j\}$ set

$$f_i(x) = (1 - \phi(x))h_i(x) + \phi(x)[h_i(\bar{x}) + \rho(x, \bar{x}) - \epsilon/2],\ x \in X \tag{4.17}$$

and define

$$f_j(x) = (1 - \phi(x))h_j(x) + \phi(x)[h_j(\bar{x}) + \rho(x, \bar{x}) - \epsilon/4],\ x \in X. \tag{4.18}$$

By (4.17) and (4.18) for $i \in \{1, \dots, n\}$ and $x \in X$

$$|f_i(x) - h_i(x)| \le \phi(x)[|h_i(\bar{x}) - h_i(x)| + \rho(x, \tilde{x}) + \epsilon/2]. \tag{4.19}$$

By (4.19), (4.15), (4.16), property (P3) and (4.14)

$$\begin{aligned}
&\sup\{|f_i(x) - h_i(x)| :\ x \in X,\ i = 1, \dots, n\} \\
&\quad \le \epsilon/2 + \sup\{\phi(x)[|h_i(x) - h_i(\bar{x})| + \rho(x, \bar{x})] :\ x \in X,\ i = 1, \dots, n\} \\
&\quad = \epsilon/2 + \sup\{\phi(x)[|h_i(x) - h_i(\bar{x})| + \rho(x, \bar{x})] : \\
&\qquad\qquad i = 1, \dots, n \text{ and } x \in X \text{satisfies } \rho(x, \bar{x}) \le \delta_0\} \\
&\quad \le \epsilon/2 + \delta_0 + \sup\{|h_i(x) - h_i(\bar{x})| : \\
&\qquad\qquad i = 1, \dots, n \text{ and } x \in X \text{ satisfies } \rho(x, \bar{x}) \le \delta_0\} \\
&\quad \le \epsilon/2 + \delta_0 + \epsilon/8 \\
&\quad \le \epsilon/2 + \epsilon/4
\end{aligned}$$

and

$$\tilde{d}(f, h) \le \epsilon/2 + \epsilon/4.$$

Combined with (4.11) this implies that

$$\tilde{d}(g, f) \le \tilde{d}(g, h) + \tilde{d}(h, f) \le \epsilon. \tag{4.20}$$

We show that $f \in \mathcal{F}_0$. By (4.18), (4.15) and (4.16)

$$f_j(\bar{x}) = h_j(\bar{x}) - \epsilon/4, \tag{4.21}$$

$$f_i(\bar{x}) = h_i(\bar{x}) - \epsilon/2,\ i \in \{1, \dots, n\} \setminus \{j\}. \tag{4.22}$$

In view of (4.13), (4.21) and (4.22)

$$f_j(\bar{x}) \ge f_i(\bar{x}) + \epsilon/4,\ i \in \{1, \dots, n\} \setminus \{j\}. \tag{4.23}$$

Assume that $x \in X$ saisfies

$$\rho(x, \bar{x}) \ge \delta_0. \tag{4.24}$$

It follows from (4.24), (4.17), (4.18), (4.15), (4.16), (4.13) and (4.21) that for all $i \in \{1, \dots, n\}$

$$f_i(x) = h_i(x),$$

$$\begin{aligned}
\max\{f_i(x) :\ i = 1, \dots, n\} &= \max\{h_i(x) :\ i = 1, \dots, n\} \\
&\ge \max\{h_i(\bar{x}) :\ i = 1, \dots, n\} = h_j(\bar{x}) = f_j(\bar{x}) + \epsilon/4.
\end{aligned}$$

Thus (4.23) implies that

$$\max\{f_i(x):\ i=1,\dots,n\}\geq f_j(\bar{x})+\epsilon/4=\max\{f_i(\bar{x}):\ i=1,\dots,n\}+\epsilon/4 \tag{4.25}$$

for all $x\in X$ satisfying (4.24).

Assume that $x\in X$ satisfies

$$0<\rho(x,\bar{x})<\delta_0. \tag{4.26}$$

Then by property (P3)

$$|h_i(x)-h_i(\bar{x})|\leq\epsilon/8. \tag{4.27}$$

It follows from (4.18), (4.27), (4.13), (4.15) and (4.16) that

$$\begin{aligned}\max\{f_i(x):\ i=1,\dots,n\}&\\ &\geq f_j(x)\\ &=(1-\phi(x))h_j(x)+\phi(x)[h_j(\bar{x})+\rho(x,\bar{x})-\epsilon/4]\\ &\geq(1-\phi(x))[h_j(\bar{x})-\epsilon/8]+\phi(x)[h_j(\bar{x})+\rho(x,\bar{x})-\epsilon/4]\\ &=\phi(x)[\rho(x,\bar{x})-\epsilon/4]-(\epsilon/8)(1-\phi(x))+h_j(\bar{x})\\ &=\phi(x)\rho(x,\bar{x})+h_j(\bar{x})-\epsilon/8-(\epsilon/8)\phi(x)\\ &\geq\phi(x)\rho(x,\bar{x})+h_j(\bar{x})-\epsilon/4.\end{aligned} \tag{4.28}$$

By (4.26), (4.15), (4.16), the inequality

$$\phi(x)\rho(x,\bar{x})>0,$$

(4.28), (4.21) and (4.23)

$$\begin{aligned}\max\{f_i(x):\ i=1,\dots,n\}&>h_j(\bar{x})-\epsilon/4\\ &=f_j(\bar{x})=\max\{f_i(\bar{x}):\ i=1,\dots,n\}.\end{aligned} \tag{4.29}$$

(4.25) and (4.29) imply that $\bar{x}$ is a unique solution for the minimization problem for $\max\{f_1,\dots,f_n\}$ on X.

We show that the minimization problem for $\max\{f_1,\dots,f_n\}$ on (X,ρ) is well-posed.

Assume that

$$\begin{aligned}\{x_k\}_{k=1}^{\infty}\subset X,\ \lim_{k\to\infty}&(\max\{f_1,\dots,f_n\})(x_k)\\ &=\inf(\max\{f_1,\dots,f_n\})=\max\{f_1,\dots,f_n\}(\bar{x}).\end{aligned} \tag{4.30}$$

We show that

$$\lim_{i\to\infty}\rho(x_i,\bar{x})=0.$$

Let us assume the converse. Then there are a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ and $\Delta>0$ such that

$$\rho(x_{i_k},\bar{x})\geq\Delta \text{ for all natural numbers } k. \tag{4.31}$$

Since (4.25) holds for all $x\in X$ satisfying (4.24) it follows from (4.30) that $\rho(x_i,\bar{x})<\delta_0$ for all sufficiently large i. We may assume without loss of generality that

$$\rho(x_i,\bar{x})<\delta_0 \text{ for all natural numbres } i. \tag{4.32}$$

By (4.21), (4.23), (4.30), (4.32), (4.31), (4.28), (4.15) and (4.16)

$$\begin{aligned} h_j(\bar{x}) - \epsilon/4 &= \limsup_{k\to\infty} \max\{f_1(x_{i_k}), \dots, f_n(x_{i_k})\} \\ &\geq \limsup_{k\to\infty} \phi(x_{i_k})\rho(x_{i_k}, \bar{x}) + h_j(\bar{x}) - \epsilon/8 - (\epsilon/8)\phi(x_{i_k}) \\ &= h_j(\bar{x}) - \epsilon/4 + \limsup_{k\to\infty} [\phi(x_{i_k})(\rho(x_{i_k}, \bar{x}) + (\epsilon/8)(1 - \phi(x_{i_k}))], \end{aligned}$$

$$0 = \lim_{k\to\infty} [\phi(x_{i_k})\rho(x_{i_k}, \bar{x}) + (\epsilon/8)(1 - \phi(x_{i_k}))]. \tag{4.33}$$

By (4.15), (4.16), (4.33) and (4.31)

$$\lim_{k\to\infty} \phi(x_{i_k}) = 0$$

and

$$\lim_{k\to\infty} (1 - \phi(x_{i_k})) = 0.$$

The contradiction we have reached proves that $\lim_{i\to\infty} \rho(x_i, \bar{x}) = 0$ and that the minimization problem for $\max\{f_1, \dots f_n\}$ on (X, ρ) is well-posed. Since $\bar{x}$ is a unique solution for the minimization problem for $\max\{f_1, \dots, f_n\}$ on X and $\bar{x}$ satisfies (4.23) it follows from Proposition 4.1 that $f \in \mathcal{F}_0$.

This completes the proof of the proposition. □

Propositions 4.2 and 4.3 imply the following result.

Theorem 4.1. *The set $\mathcal{F}_0$ is an open everywhere dense subset of $(\mathcal{M}, d)$.*

Corollary 4.1. *Let the metric space (X, ρ) be a compact. Then for each $f = (f_1, \dots, f_n) \in \mathcal{F}_0$ there are $\epsilon > 0$ and $j \in \{1, \dots, n\}$ such that if $x \in X$ satisfies*

$$\max\{f_1, \dots, f_n\}(x) = \inf(\max\{f_1, \dots, f_n\}),$$

then

$$f_j(x) \geq f_i(x) + \epsilon \text{ for all } i \in \{1, \dots, n\} \setminus \{j\}.$$

Corollary 4.2. *Let the set $\mathcal{F}$ be as guaranteed by Theorem 3.1. Then for each $f = (f_1, \dots, f_n) \in \mathcal{F}_0 \cap \mathcal{F}$ there is a unique $x_f \in X$ such that*

$$\max\{f_1, \dots, f_n\}(x_f) = \inf(\max\{f_1, \dots, f_n\})$$

and $j \in \{1, \dots, n\}$ such that

$$f_j(x_f) > f_i(x_f) \text{ for all } i \in \{1, \dots, n\} \setminus \{j\}.$$

References

[1] D. Butnariu, S. Reich and A.J. Zaslavski, *Generic power convergence of operators in Banach spaces*, Numerical Functional Analysis and Optimization **20** (1999), 629–650.

[2] A. Cellina and C. Mariconda, *The existence question in the calculus of variations: A density result*, Proc. Amer. Math. Soc. **120** (1994), 1145–1150.

[3] S. Cobzas, *Generic existence of solutions for some perturbed optimization problems* J. Math. Anal. Appl. **243** (2000), 344–356.

[4] F.S. De Blasi and J. Myjak, *Sur la porosité des contractions sans point fixe*, C. R. Acad. Sci. Paris **308** (1989) 51–54.

[5] V.F. Demyanov and V.N. Malozemov, *Introduction to Minimax*, Dover Publications, Inc., New York, 1990.

[6] V.F. Demyanov and A.M. Rubinov, *Constructive Nonsmooth Analysis*, Peter Lang, Frankfurt am Main, 1995.
[7] V.F. Demyanov and V.L. Vasil'ev, *Nondifferentiable Optimization. Translations Series in Mathematics and Engineering*, Optimization Software, Inc., New York, 1985.
[8] A.D. Ioffe and A.J. Zaslavski, *Variational principles and well-posedness in optimization and calculus of variations*, SIAM J. Control Optim. **38** (2000), 566–581.
[9] S. Reich and A.J. Zaslavski, *Convergence of generic infinite products of nonexpansive and uniformly continuous operators*, Nonlinear Analysis: Theory, Methods and Applications **36** (1999), 1049–1065.
[10] A.J. Zaslavski, *Well-posedness and porosity in optimal control without convexity assumptions*, Calculus of Variations and Partial Differential Equations **13** (2001), 265–293.

Manuscript received April 1, 2004
revised June 17, 2004

Alexander J. Zaslavski
Department of Mathematics, The Technion-Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: ajzasl@tx.technion.ac.il