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FARTHEST POINTS AND POROSITY

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Abstract. Given a nonempty, closed and bounded subset A of a complete geodesic space (X, ρ, M) and a point $x \in X$, we consider the problem of finding a farthest point from x in A. Denoting by $B(X)$ the family of all nonempty, closed and bounded subsets of X, we first endow $B(X)$ with a pair of natural metrics. We then define a corresponding metric space Q of pairs (A, x) and construct a subset Ω of Q with a σ-porous complement such that for each pair in Ω, this problem is well posed.

1. Introduction

Given a nonempty, closed and bounded subset A of a Banach space $(X, \|\cdot\|)$ and a point $x \in X$, we consider the maximization problem

$$\text{(P)} \qquad \max\{\| x - y \| : y \in A\}.$$

If X is a reflexive and locally uniformly convex Banach space, then according to a classical result of Asplund [1], the set of all points in X having a farthest point in A contains a dense G_δ subset of X. If A is a weakly compact subset of a general Banach space X, then according to a subsequent result of Lau [9], the set of all points having a farthest point in A again contains a dense G_δ subset of X. In this connection, see also [5] and [6]. A more recent result of De Blasi, Myjak and Papini [4] establishes well-posedness of problem (P) for uniformly convex X, closed and bounded A, and a generic $x \in X$. At this point we recall that problem (P) is said to be well posed if it has a unique solution, say a_0, and every maximizing sequence of (P) converges to a_0.

In the generic approach, instead of considering the existence of a solution to problem (P) for a single point $x \in X$, one investigates it for the whole space X and shows that solutions exist for most points in X. Such an approach is common in global analysis and the theory of dynamical systems (see, for example, [13], [16] and [17]). It has also been used in the study of the structure of extremals of variational and optimal control problems (see, for instance, [26] and [27]) and in approximation theory (see, for example, [7], [10], [18], [20] and [21]). A recent exposition of various genericity results in nonlinear analysis can be found in [23]. All those topological and functional analytic concepts which are used, but not defined, here are discussed in the books [8], [12] and [14].

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A more precise formulation of the aforementioned De Blasi-Myjak-Papini result involves the notion of porosity [25], which we now recall. Let (Y, d) be a metric space. We denote by $B(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called *porous* (with respect to the metric d) if there exist numbers $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$, there exists a point $z \in Y$ for which

$$B(z, \alpha r) \subset B(y, r) \setminus E.$$

A subset of the space Y is called *σ-porous* (with respect to the metric d) if it is a countable union of porous subsets of Y.

Since porous sets are nowhere dense, all σ-porous sets are of the first Baire category. If Y is a finite dimensional Euclidean space, then σ-porous sets are of Lebesgue measure zero. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have measure zero and are of the first category. Also, every Banach space contains a set of the first category which is not σ-porous.

To point out the difference between porous and nowhere dense sets, note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there are a point $z \in Y$ and a number $s > 0$ such that $B(z, s) \subset B(y, r) \setminus E$. If, however, E is also porous, then for small enough r, we can choose $s = \alpha r$, where $\alpha \in (0, 1)$ is a constant which only depends on E.

Using this terminology and denoting by $\mathcal{F}$ the set of all points such that the maximization problem (P) is well posed, we note that De Blasi, Myjak and Papini prove in [4] that the complement $X \setminus \mathcal{F}$ is, in fact, σ-porous in X.

However, a fundamental restriction in this result is that it only holds in special Banach spaces. On the other hand, many generic results in nonlinear functional analysis hold in any Banach space. Therefore the following natural question arises: can generic well-posedness results for farthest point problems in closed and bounded subsets be established in general Banach spaces? In the present paper we answer this question in the affirmative.

To this end, we change our point of view and consider another framework. The main feature of this framework is that the set A in problem (P) may also vary. Such a framework has already been used in several best approximation problems (see, for example, [3], [18], [19], [20], [21] and [22]). In our first result (Theorem 3.1 below) we fix x and consider the space $B(X)$ of all nonempty, closed and bounded subsets of X equipped with an appropriate complete metric, say $\widetilde{H}$. We then show that the collection of all sets $A \in B(X)$ for which problem (P) is well posed has a σ-porous complement. In the second result (Theorem 3.2) we consider the space of pairs $B(X) \times X$ with the metric $\widetilde{H}(A, B) + \| x - y \|$, where $A, B \in B(X)$ and $x, y \in X$. Once again, we show that the family of all pairs $(A, x) \in B(X) \times X$ for which problem (P) is well posed has a σ-porous complement.

The precise statements of these two theorems can be found in Section 3. Section 2 contains more information on porous sets and the class of geodesic spaces. Two auxiliary results are presented in Section 4. The proofs of Theorems 3.1 and 3.2 are given in Section 5. We conclude the paper with a short discussion concerning completeness (see Section 6).

2. Porous sets and geodesic spaces

In this section we provide more information on porous sets and the class of geodesic spaces.

Let (Y, ρ) be a metric space. We denote by $B_\rho(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. The following simple observation was made in [27].

Proposition 2.1. *Let E be a subset of the metric space (Y, ρ). Assume that there exist numbers $r_0 > 0$ and $\beta \in (0, 1)$ such that the following property holds:*
(P1) *For each $x \in Y$ and each $r \in (0, r_0]$, there exists a point $z \in Y \setminus E$ such that $\rho(x, z) \leq r$ and $B_\rho(z, \beta r) \cap E = \emptyset$.*
Then E is porous with respect to ρ.

As a matter of fact, property (P1) can be replaced by a weaker one [18].

Proposition 2.2. *Let E be a subset of the metric space (Y, ρ). Assume that there exist numbers $r_0 > 0$ and $\beta \in (0, 1)$ such that the following property holds:*
(P2) *For each $x \in E$ and each $r \in (0, r_0]$, there exists $z \in Y \setminus E$ such that $\rho(x, z) \leq r$ and $B_\rho(z, \beta r) \cap E = \emptyset$.*
Then E is porous with respect to ρ.

The following definition was introduced in [27].

Assume that a set Y is equipped with two metrics ρ_1 and ρ_2 such that $\rho_1(x, y) \leq \rho_2(x, y)$ for all $x, y \in Y$. We say that a set $E \subset Y$ is porous with respect to the pair (ρ_1, ρ_2) if there exist numbers $r_0 > 0$ and $\alpha \in (0, 1)$ such that for each $x \in E$ and each $r \in (0, r_0]$, there exists a point $z \in Y \setminus E$ such that $\rho_2(z, x) \leq r$ and $B_{\rho_1}(z, \alpha r) \cap E = \emptyset$.

Proposition 2.2 implies that if E is porous with respect to the pair (ρ_1, ρ_2), then it is porous with respect to both ρ_1 and ρ_2.

A set $E \subset Y$ is called σ-porous with respect to the pair of metrics (ρ_1, ρ_2) if it is a countable union of sets which are porous with respect to (ρ_1, ρ_2).

It turns out that our results are true not only for Banach spaces, but also for all complete geodesic spaces. We now recall the definition of this important class of spaces.

Let (X, ρ) be a metric space and let R^1 denote the real line. We say that a mapping $c : R^1 \longrightarrow X$ is a metric embedding of R^1 into X if $\rho(c(s), c(t)) = |s - t|$ for all real s and t. The image of R^1 under a metric embedding is called a *metric line.* The image of a real interval $[a, b] = \{t \in R^1 : a \leq t \leq b\}$ under such a mapping is called a *metric segment.*

Assume that (X, ρ) contains a family M of metric lines such that for each pair of distinct points x and y in X, there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment joining x and y. We denote this segment by $[x, y]$. For each $0 \leq t \leq 1$, there is a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \quad \text{and} \quad \rho(z, y) = (1 - t)\rho(x, y).$$

This point is denoted by $(1-t)x \oplus ty$. In this case we say that X, or more precisely (X, ρ, M), is a *geodesic space.*

It is clear that all normed linear spaces, as well as the hyperbolic spaces in the sense of Reich and Shafrir (see [15]) are geodesic spaces.

3. Main results

Let (X, ρ, M) be a complete geodesic space. For each $x \in X$ and each $A \subset X$, we set

$$\rho(x, A) := \inf\{\rho(x, y) : y \in A\},$$
$$e(x, A) := \sup\{\rho(x, y) : y \in A\}.$$

We also define

$$\operatorname{diam}(A) := \sup\{\rho(x, y) : x, y \in A\}. \tag{3.1}$$

We denote by $B(X)$ the family of all nonempty, closed and bounded subsets of X. For each A, $B \in B(X)$, we define

$$H(A, B) := \max\{\sup\{\rho(x, B) : x \in A\},\ \sup\{\rho(y, A) : y \in B\}\}$$

and

$$\widetilde{H}(A, B) := H(A, B)(1 + H(A, B))^{-1}.$$

It is not difficult to see that $\widetilde{H}$ is a metric on $B(X)$ and that the metric space $(B(X), \widetilde{H})$ is complete (*cf.* [24, page 253]).

Fix a point $\theta \in X$. For each natural number n and each A, $B \in B(X)$, we set

$$h_n(A, B) := \sup\{|\rho(x, A) - \rho(x, B)| : x \in X \text{ and } \rho(x, \theta) \leq n\} \tag{3.2}$$

and

$$h(A, B) := \sum_{n=1}^{\infty} [2^{-n} h_n(A, B)(1 + h_n(A, B))^{-1}].$$

Once again, it is not difficult to see that h is a metric on $B(X)$. Clearly,

$$\widetilde{H}(A, B) \geq h(A, B)$$

for all

$$A, B \in B(X).$$

We equip the set $B(X)$ with the pair of metrics $\widetilde{H}$ and h.

We now state our two main results. Their proofs will be given in Section 5.

Theorem 3.1. *Let (X, ρ, M) be a complete geodesic space and let $\tilde{x} \in X$. Then there exists a set $\Omega \subset B(X)$ such that its complement $B(X) \setminus \Omega$ is σ-porous with respect to the pair of metrics $(h, \widetilde{H})$, and such that for each $A \in \Omega$, the following property holds:*

(C1) *There exists a unique $\tilde{y} \in A$ such that $\rho(\tilde{x}, \tilde{y}) = \sup_{y \in A} \rho(\tilde{x}, y) = e(\tilde{x}, A)$. Moreover, for each $\varepsilon > 0$, there exists a number $\delta > 0$ such that if a point $x \in A$ satisfies $\rho(\tilde{x}, x) \geq e(\tilde{x}, A) - \delta$, then $\rho(x, \tilde{y}) \leq \varepsilon$.*

To state our second result we endow the Cartesian product $B(X) \times X$ with the pair of metrics d_1 and d_2 defined by

$$d_1((A,x),(B,y)) := h(A,B) + \rho(x,y),$$
$$d_2((A,x),(B,y)) := \widetilde{H}(A,B) + \rho(x,y),$$

where $x, y \in X$ and $A, B \in B(X)$.

Theorem 3.2. *Let (X, ρ, M) be a complete geodesic space. Then there exists a set $\Omega \subset B(X) \times X$ such that its complement $[B(X) \times X] \setminus \Omega$ is σ-porous with respect to the pair of metrics (d_1, d_2), and such that for each $(A, \tilde{x}) \in \Omega$, the following property holds:*

(C2) *There exists a unique $\tilde{y} \in A$ such that $\rho(\tilde{x}, \tilde{y}) = \sup_{y \in A} \rho(\tilde{x}, y) = e(\tilde{x}, A)$. Moreover, for each $\varepsilon > 0$, there exists a number $\delta > 0$ such that if a point $z \in X$ satisfies $\rho(\tilde{x}, z) \leq \delta$, a set $B \in B(X)$ satisfies $h(A, B) \leq \delta$, and a point $y \in B$ satisfies $\rho(z, y) \geq e(z, B) - \delta$, then $\rho(y, \tilde{y}) \leq \varepsilon$.*

In classical generic results the set A was fixed and x varied in a dense G_δ subset of X. In our two theorems the set A is also variable.

4. Auxiliary results

Let (X, ρ, M) be a complete geodesic space and let $B(X)$ be the family of all nonempty, closed and bounded subsets of X.

Lemma 4.1. *Let $A \in B(X)$, $\tilde{x} \in X$ and let r, $\varepsilon \in (0,1)$. Then there exists a point $\bar{x} \in X$ such that $\rho(\bar{x}, A) \leq r$ and the set $\widetilde{A} = A \cup \{\bar{x}\}$ has the following two properties :*

(i) *$\rho(\tilde{x}, \bar{x}) = e(\tilde{x}, \widetilde{A})$;*

(ii) *if $x \in \widetilde{A}$ and $\rho(\tilde{x}, x) \geq e(\tilde{x}, \widetilde{A}) - \frac{\varepsilon r}{4}$, then $\rho(\bar{x}, x) \leq \varepsilon$.*

Proof. We first take $x_1 \in A$ such that $\rho(\tilde{x}, x_1) > e(\tilde{x}, A) - \frac{\varepsilon r}{2}$.

Next, we define $\bar{x}$ by requiring that $x_1 \in \{\gamma\bar{x} \oplus (1-\gamma)\tilde{x} : 0 < \gamma < 1\}$, $\rho(\bar{x}, x_1) = r$, and $\rho(x_1, \tilde{x}) = \rho(\tilde{x}, \bar{x}) - r$. We now set $\widetilde{A} = A \cup \{\bar{x}\}$. Then $\rho(\tilde{x}, \bar{x}) = \rho(x_1, \tilde{x}) + r > e(\tilde{x}, A) - \frac{\varepsilon r}{2} + r > e(\tilde{x}, A)$. Therefore $\rho(\tilde{x}, \bar{x}) = e(\tilde{x}, \widetilde{A})$.

If $x \in \widetilde{A}$ and $\rho(\tilde{x}, x) \geq e(\tilde{x}, \widetilde{A}) - \frac{\varepsilon r}{4}$, then:

If $x = \bar{x}$ we get $\rho(x, \bar{x}) = \rho(\bar{x}, \bar{x}) = 0 < \varepsilon$. In the other case, where $x \neq \bar{x}$, then $x \in A$ and using our previous condition, we obtain

$$\begin{aligned}
\rho(\tilde{x}, x) &\geq e(\tilde{x}, \widetilde{A}) - \frac{\varepsilon r}{4} \\
&= \rho(\tilde{x}, \bar{x}) - \frac{\varepsilon r}{4} \\
&= \rho(\tilde{x}, x_1) + r - \frac{\varepsilon r}{4} \\
&> e(\tilde{x}, A) - \frac{\varepsilon r}{2} + r - \frac{\varepsilon r}{4} \\
&= e(\tilde{x}, A) + r - \frac{3}{4}\varepsilon r > e(\tilde{x}, A).
\end{aligned}$$

Thus we get a contradiction. So this case cannot, in fact, happen. This completes the proof of Lemma 4.1. $\square$

Before stating our next lemma we choose, for each number $\varepsilon \in (0,1)$ and each pair of natural numbers n and m, a number

$$\alpha(\varepsilon, n, m) \in (0, 2^{-4n-m-10}\varepsilon). \tag{4.1}$$

Lemma 4.2. *Let $A \in B(X)$, $\tilde{x} \in X$ and let r, $\varepsilon \in (0,1)$. Suppose that n and m are natural numbers, let*

$$\alpha := \alpha(\varepsilon, n, m), \tag{4.2}$$

and assume that

$$\rho(\tilde{x}, \theta) \leq n, \ \{z \in X : \rho(z, \theta) \leq n\} \cap A \neq \emptyset \ \text{ and } \ diam(A) < m. \tag{4.3}$$

Then there exists a point $\bar{x} \in X$ such that $\rho(\bar{x}, A) \leq r$ and the set $\widetilde{A} = A \cup \{\bar{x}\}$ has the following two properties:

$$\rho(\tilde{x}, \bar{x}) = e(\tilde{x}, \widetilde{A}); \tag{4.4}$$

if

$$\tilde{y} \in X, \ \rho(\tilde{y}, \tilde{x}) \leq \alpha r, \tag{4.5}$$

$$B \in B(X), \ h(\widetilde{A}, B) \leq \alpha r \text{ with } \operatorname{diam}(B) < m, \tag{4.6}$$

and

$$z \in B, \ \rho(\tilde{y}, z) \geq e(\tilde{y}, B) - \frac{\varepsilon r}{16}, \tag{4.7}$$

then

$$\rho(z, \bar{x}) \leq \varepsilon. \tag{4.8}$$

Proof. By Lemma 4.1, there exists a point $\bar{x} \in X$ such that

$$\rho(\bar{x}, A) \leq r \tag{4.9}$$

and such that for the set $\widetilde{A} = A \cup \{\bar{x}\}$, we have $\rho(\tilde{x}, \bar{x}) = e(\tilde{x}, \widetilde{A})$. We also have

$$\begin{aligned} \rho(\tilde{x}, \bar{x}) &\leq \rho(\tilde{x}, \theta) + \rho(\theta, a) + \rho(a, \bar{x}) \\ &\leq \rho(\tilde{x}, \theta) + \rho(\theta, a) + \rho(a, \bar{a}) + \rho(\bar{a}, \bar{x}) \\ &\leq n + n + \operatorname{diam}(A) + r + \tilde{\varepsilon} \leq 2n + 1 + m, \end{aligned}$$

where $a \in \{z \in X : \rho(z, \theta) \leq n\} \cap A$, while $\bar{a} \in A$ has the following property: $\rho(\bar{x}, \bar{a}) \leq \rho(\bar{x}, A) + \tilde{\varepsilon} \leq r + \tilde{\varepsilon} < 1$ for an appropriate $\tilde{\varepsilon} \in (0,1)$. Next, $\rho(\bar{x}, \theta) \leq \rho(\bar{x}, \tilde{x}) + \rho(\tilde{x}, \theta) \leq 2n + 1 + m + n = 3n + 1 + m \leq 4n + m$.

Using the definitions we presented at the beginning of Section 3, we see that

$$\frac{h_{4n+m}(\widetilde{A}, B)}{1 + h_{4n+m}(\widetilde{A}, B)} \leq 2^{4n+m} h(\widetilde{A}, B) \leq 2^{4n+m}\alpha r.$$

Therefore we get

$$h_{4n+m}(\widetilde{A}, B) \leq \frac{2^{4n+m}\alpha r}{1 - 2^{4n+m}\alpha r} \leq 2^{4n+m+1}\alpha r.$$

Since $\bar{x} \in \widetilde{A}$, it follows that $\rho(\bar{x}, B) \leq 2^{4n+m+1}\alpha r$, so that there exists $\bar{y} \in B$ such that $\rho(\bar{x}, \bar{y}) < 2^{4n+m+2}\alpha r$.

Hence we have

$$\begin{aligned}
e(\tilde{y}, B) &\geq \rho(\tilde{y}, \bar{y}) \geq |\rho(\tilde{y}, \bar{x}) - \rho(\bar{y}, \bar{x})| \geq \rho(\tilde{y}, \bar{x}) - \rho(\bar{y}, \bar{x}) \\
&\geq |\rho(\bar{x}, \tilde{x}) - \rho(\tilde{x}, \tilde{y})| - \rho(\bar{y}, \bar{x}) \geq \rho(\bar{x}, \tilde{x}) - \rho(\tilde{x}, \tilde{y}) - \rho(\bar{y}, \bar{x}) \\
&\geq e(\tilde{x}, \widetilde{A}) - \alpha r - 2^{4n+m+2}\alpha r.
\end{aligned}$$

We now assume that a point $z \in B$ satisfies the condition formulated in the statement of the lemma. Then
$\rho(\tilde{y}, z) \geq e(\tilde{y}, B) - \frac{\varepsilon r}{16} \geq e(\tilde{x}, \widetilde{A}) - \alpha r - 2^{4n+m+2}\alpha r - \frac{\varepsilon r}{16}$.

On the other hand,

$$\begin{aligned}
\rho(\tilde{y}, z) \leq \rho(\tilde{y}, B) + \operatorname{diam}(B) &\leq \rho(\tilde{y}, \tilde{x}) + \rho(\tilde{x}, \widetilde{A}) + h(\widetilde{A}, B) + \operatorname{diam}(B) \\
&\leq \alpha r + \rho(\tilde{x}, \widetilde{A}) + \alpha r + m \\
&\leq 2\alpha r + m + \rho(\tilde{x}, \widetilde{A}) \\
&\leq 2\alpha r + m + \rho(\tilde{x}, A) \\
&\leq 2\alpha r + m + \rho(\tilde{x}, \theta) + \rho(\theta, a) \\
&\leq 2\alpha r + m + n + n \leq 2n + m + 2\alpha r.
\end{aligned}$$

Continuing our computations, we obtain

$$\begin{aligned}
\rho(z, \theta) &\leq \rho(\tilde{y}, z) + \rho(\tilde{y}, \theta) \leq \rho(\tilde{y}, z) + \rho(\tilde{y}, \tilde{x}) + \rho(\tilde{x}, \theta) \\
&\leq 2n + m + 2\alpha r + \alpha r + n = 3n + m + 3\alpha r \leq 4n + m.
\end{aligned}$$

Therefore it follows from the definitions we presented at the beginning of Section 3 that

$$\rho(z, \widetilde{A}) = |\rho(z, \widetilde{A}) - \rho(z, B)| \leq h_{4n+m}(\widetilde{A}, B) \leq 2^{4n+m+1}\alpha r.$$

Hence there exists a point $\tilde{z} \in \widetilde{A}$ such that $\rho(z, \tilde{z}) \leq 2^{4n+m+2}\alpha r$.

We also have

$$\begin{aligned}
\rho(\tilde{x}, \tilde{z}) &\geq |\rho(\tilde{x}, z) - \rho(\tilde{z}, z)| \\
&\geq \rho(\tilde{x}, z) - \rho(\tilde{z}, z) \\
&\geq |\rho(z, \tilde{y}) - \rho(\tilde{y}, \tilde{x})| - \rho(\tilde{z}, z) \\
&\geq \rho(z, \tilde{y}) - \rho(\tilde{y}, \tilde{x}) - \rho(\tilde{z}, z) \\
&\geq e(\tilde{x}, \widetilde{A}) - \alpha r - 2^{4n+m+2}\alpha r - \frac{\varepsilon r}{16} - \alpha r - 2^{4n+m+2}\alpha r \\
&\geq e(\tilde{x}, \widetilde{A}) - 2\alpha r - 2 \cdot 2^{4n+m+2}\alpha r - \frac{\varepsilon r}{16} \\
&= e(\tilde{x}, \widetilde{A}) - 2\alpha r - 2^{4n+m+3}\alpha r - \frac{\varepsilon r}{16} \\
&\geq e(\tilde{x}, \widetilde{A}) - \frac{\varepsilon r}{8}.
\end{aligned}$$

Applying Lemma 4.1, we get $\rho(\bar{x}, \tilde{z}) \leq \frac{\varepsilon}{2}$.

So $\rho(z, \bar{x}) \leq \rho(z, \tilde{z}) + \rho(\tilde{z}, \bar{x}) \leq 2^{4n+m+2}\alpha r + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus Lemma 4.2 holds. $\square$

5. Proofs of Theorems 3.1 and 3.2

Proof of Theorem 3.1. For each integer $k \geq 1$, denote by Ω_k the set of all $A \in B(X)$ which have the following property:
(P3) There exist $x_A \in X$ and $\delta_A > 0$ such that if $x \in A$ satisfies $\rho(\tilde{x}, x) \geq e(\tilde{x}, A) - \delta_A$, then $\rho(x, x_A) \leq \frac{1}{k}$.

Clearly, $\Omega_{k+1} \subset \Omega_k$, $k = 1, 2, \ldots$. Set

$$\Omega := \bigcap_{k=1}^{\infty} \Omega_k.$$

We claim that $B(X) \setminus \Omega$ is σ-porous with respect to the pair of metrics $(h, \widetilde{H})$. To prove this, we show that $B(X) \setminus \Omega_k$ is σ-porous with respect to $(h, \widetilde{H})$ for all sufficiently large integers k.

Indeed, for each integer $n \geq k_0$, where $\rho(\tilde{x}, \theta) \leq k_0$, and each natural number m, we let

$$E_{nkm} := \{A \in B(X) \setminus \Omega_k : \{z \in X : \rho(z, \theta) \leq n\} \cap A \neq \emptyset, \ \operatorname{diam}(A) < m\}.$$

By Lemma 4.2, the set E_{nkm} is porous with respect to the pair $(h, \widetilde{H})$ for each integer $n \geq k_0$ and each natural number m. Therefore $B(X) \setminus \Omega_k$ is indeed σ-porous with respect to $(h, \widetilde{H})$ and so is $B(X) \setminus \Omega$.

Let $A \in \Omega$. We claim that A has property (C1). By the definition of Ω_k and property (P3), for each integer $k \geq 1$, there exist $x_k \in X$ and $\delta_k > 0$ such that the following property holds:

(P4) If a point $x \in A$ satisfies $\rho(\tilde{x}, x) \geq e(\tilde{x}, A) - \delta_k$, then $\rho(x, x_k) \leq \dfrac{1}{k}$.

Let a sequence $\{z_i\}_{i=1}^{\infty} \subset A$ be such that

$$\lim_{i \longrightarrow \infty} \rho(\tilde{x}, z_i) = e(\tilde{x}, A). \tag{5.1}$$

Fix an integer $k \geq 1$. It follows from property (P4) that for all large enough natural numbers i,

$$\rho(\tilde{x}, z_i) \geq e(\tilde{x}, A) - \delta_k \ \text{ and } \ \rho(z_i, x_k) \leq \frac{1}{k}.$$

Since k is an arbitrary natural number, we can conclude that $\{z_i\}$ ia a Cauchy sequence which converges to some $\tilde{y} \in A$. Clearly, $\rho(\tilde{x}, \tilde{y}) = e(\tilde{x}, A)$. If the maximizer $\tilde{y}$ were not unique, we would be able to construct a nonconvergent maximizing sequence $\{z_i\}_{i=1}^{\infty}$.

Thus this point $\tilde{y}$ is the unique solution to the maximization problem (P) with $x = \tilde{x}$ and any sequence $\{z_i\}_{i=1}^{\infty} \subset A$ satisfying $\lim_{i \longrightarrow \infty} \rho(\tilde{x}, z_i) = e(\tilde{x}, A)$ converges to $\tilde{y}$. This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. For each integer $k \geq 1$, denote by Ω_k the set of all $(A, x) \in B(X) \times X$ which have the following property:
(P5) There exist $\bar{x} \in X$ and $\bar{\delta} > 0$ such that if $x \in X$ satisfies $\rho(x, \tilde{x}) \leq \bar{\delta}$, $B \in B(X)$ satisfies $h(A, B) \leq \bar{\delta}$, and $y \in B$ satisfies $\rho(x, y) \geq e(x, B) - \bar{\delta}$, then $\rho(y, \bar{x}) \leq \frac{1}{k}$.

Clearly, $\Omega_{k+1} \subset \Omega_k$, $k = 1, 2, ...$ Set

$$\Omega = \bigcap_{k=1}^{\infty} \Omega_k.$$

We claim that $[B(X) \times X] \setminus \Omega$ is σ-porous with respect to the pair of metrics (d_1, d_2).

Indeed, for any natural numbers n, k and m, let

$$E_{nkm} := \{(A, x) \in [B(X) \times X] \setminus \Omega_k : \rho(x, \theta) \le n, \{z \in X : \rho(z, \theta) \le n\} \cap A \neq \emptyset,$$
$$\operatorname{diam}(A) < m\}.$$

By Lemma 4.2, the set E_{nkm} is porous with respect to (d_1, d_2) for all natural numbers n, k and m. Since

$$[B(X) \times X] \setminus \Omega = \bigcup_{k=1}^{\infty} [B(X) \times X] \setminus \Omega_k = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_{nkm},$$

the set $[B(X) \times X] \setminus \Omega$ is σ-porous with respect to (d_1, d_2) by definition.
Let $(A, \tilde{x}) \in \Omega$. We claim that $(A, \tilde{x})$ has property (C2).

By the definition of Ω_k and property (P5), for each integer $k \ge 1$, there exist a point $x_k \in X$ and a number $\delta_k > 0$ such that the following property holds:
(P6) If $x \in X$ satisfies $\rho(x, \tilde{x}) \le \delta_k$, $B \in B(X)$ satisfies $h(A, B) \le \delta_k$, and $y \in B$ satisfies $\rho(x, y) \ge e(x, B) - \delta_k$, then $\rho(y, x_k) \le 1/k$.

Let a sequence $\{z_i\}_{i=1}^{\infty} \subset A$ be such that

$$\lim_{i \to \infty} \rho(\tilde{x}, z_i) = e(\tilde{x}, A). \tag{5.2}$$

Fix an integer $k \ge 1$. It follows from property (P6) that for all large enough natural numbers i,

$$\rho(\tilde{x}, z_i) \ge e(\tilde{x}, A) - \delta_k \text{ and } \rho(z_i, x_k) \le 1/k.$$

Since k is an arbitrary natural number, we can conclude that $\{z_i\}_{i=1}^{\infty}$ is a Cauchy sequence which converges to some $\tilde{y} \in A$. Clearly, $\rho(\tilde{x}, \tilde{y}) = e(\tilde{x}, A)$. It is not difficult to see that $\tilde{y}$ is the unique solution to the maximization problem (P) with $x = \tilde{x}$.

Now let $\varepsilon > 0$ be given. Choose a natural number $k > 2/\min\{1, \varepsilon\}$. By property (P6),

$$\rho(\tilde{y}, x_k) \le 1/k. \tag{5.3}$$

Assume that a point $z \in X$ satisfies $\rho(z, \tilde{x}) \le \delta_k$, a set $B \in B(X)$ satisfies $h(A, B) \le \delta_k$ and a point $y \in B$ satisfies $\rho(z, y) \ge e(z, B) - \delta_k$. Then it follows from property (P6) that $\rho(y, x_k) \le 1/k$. When combined with (5.3), this inequality implies that $\rho(y, \tilde{y}) \le 2/k < \varepsilon$. This completes the proof of Theorem 3.2. □

6. Discussion

Denote by $CL(X)$ the family of nonempty and closed subsets of an unbounded complete metric space X (in our case X is a complete geodesic space). It follows from (the proof of) Theorem 8.4.2 in [11] (see also the discussion on page 79 in [2]) that our metric h is compatible with the Attouch-Wets topology τ_{AW} on $CL(X)$. Since the space $(B(X), \tau_{AW})$ is dense in $(CL(X), \tau_{AW})$ (see exercise 3 on page 84 of [2]), the space $(B(X), h)$ is incomplete.

However, the space $(B(X), H)$ is known to be complete (*cf.* [24, page 253]). Since $(B(X), h)$ is not necessarily complete, we cannot be sure in the setting of Theorem 3.1 that the set $B(X) \setminus \Omega$ is small in $(B(X), h)$, but we can conclude that it is small in $(B(X), \widetilde{H})$, which is also complete. The same logic applies in the setting of Theorem 3.2 to the set $[B(X) \times X] \setminus \Omega$, where we are not sure that this set is small in the space $B(X) \times X$ equipped with the metric $h(A, B) + \rho(x, y)$, but we do know that it is small in the space $B(X) \times X$ endowed with the metric $\widetilde{H}(A, B) + \rho(x, y)$.

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