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WEAK AND STRONG CONVERGENCE THEOREMS FOR WIDELY MORE GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, using strongly asymptotically invariant sequences, we first prove a weak convergence theorem of Mann's type [18] for widely more generalized hybrid mappings in a Hilbert space. Furthermore, using the idea of mean convergence by Shimizu and Takahashi [19, 20], we prove a strong convergence theorem of Halpern's type [6] for widely more generalized hybrid mappings in a Hilbert space. This theorem generalizes Hojo and Takahashi's strong convergence theorem [7] for generalized hybrid mappings.

1. Introduction

Let H be a real Hilbert space and let C be a non-empty subset of H. We denote by $F(T)$ the set of fixed points of T. Kocourek, Takahashi and Yao [14] introduced a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings [5], nonspreading mappings [15, 16] and hybrid mappings [24]. A mapping $T : C \to H$ is said to be *generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. We call such a mapping an (α, β)-*generalized hybrid* mapping. Hojo and Takahashi [7] proved the following strong convergence theorem.

Theorem 1.1 ([7]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a generalized hybrid mapping of C into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, \ldots$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu \in F(T)$, where P is the metric projection of H onto $F(T)$.

Very recently, Kawasaki and Takahashi [13] introduced a broader class of nonlinear mappings than the class of generalized hybrid mappings in a Hilbert space. A

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mapping T from C into H is said to be *widely more generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned}(1.1)\qquad \alpha\|Tx-Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned}$$

for all $x, y \in C$. Such a mapping T is called an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-*widely more generalized hybrid* mapping; see also [12]. In particular, an $(\alpha, \beta, \gamma, \delta, 0, 0, 0)$-widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [14] if $\alpha+\beta = -\gamma-\delta = 1$. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However, a widely more generalized hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. In [13], Kawasaki and Takahashi proved fixed point theorems and nonlinear ergodic theorems of Baillon's type [2] for such new mappings in a Hilbert space. In particular, by using their fixed point theorems, they proved directly Browder and Petryshyn's fixed point theorem [3] for strict pseudo-contractive mappings and Kocourek, Takahashi and Yao's fixed point theorem [14] for super generalized hybrid mappings.

In this paper, using strongly asymptotically invariant sequences, we first prove a weak convergence theorem of Mann's type [18] for widely more generalized hybrid mappings in a Hilbert space. Furthermore, using the idea of mean convergence by Shimizu and Takahashi [19, 20], we prove a strong convergence theorem of Halpern's type [6] for widely more generalized hybrid mappings in a Hilbert space. This theorem generalizes Hojo and Takahashi's strong convergence theorem [7] for generalized hybrid mappings.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [23], we have that for any $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$(2.1)\qquad \|y\|^2 - \|x\|^2 \leq 2\langle y - x, y\rangle,$$

$$(2.2)\qquad \|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2.$$

Furthermore, we know that for $x, y, u, v \in H$

$$(2.3)\qquad 2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2.$$

Let C be a non-empty subset of H. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. Let C be a non-empty, closed and convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y\in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u\rangle \geq 0$$

for all $x \in H$ and $u \in C$. Furthermore, we know that

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle \tag{2.4}$$

for all $x, y \in H$; see [23] for more details. For proving main results in this paper, we also need the following lemmas proved in [25] and [1].

Lemma 2.1 (Takahashi and Toyoda [25])**.** *Let D be a nonempty closed convex subset of H. Let P be the metric projection from H onto D. Let $\{u_n\}$ be a sequence in H. If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for any $u \in D$ and $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in D$.*

Lemma 2.2 (Aoyama-Kimura-Takahashi-Toyoda [1])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Let ℓ^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(\ell^\infty)^*$ (the dual space of ℓ^∞). Then we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \ldots) \in \ell^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on ℓ^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean μ is called a Banach limit on ℓ^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on ℓ^∞. If μ is a Banach limit on ℓ^∞, then for $f = (x_1, x_2, x_3, \ldots) \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \ldots) \in \ell^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [22] for the proof of existence of a Banach limit and its other elementary properties. For $f \in \ell^\infty$, define $\ell_1 : \ell^\infty \to \ell^\infty$ as follows:

$$\ell_1 f(k) = f(1 + k), \quad \forall k \in \mathbb{N}.$$

A sequence $\{\mu_n\}$ of means on ℓ^∞ is said to be *strongly asymptotically invariant* if

$$\|\ell_1^* \mu_n - \mu_n\| \to 0,$$

where ℓ_1^* is the adjoint operator of ℓ_1. See [4] for more details. The following definition which was introduced by Takahashi [21] is crucial in the fixed point theory. Let h be a bounded function of $\mathbb{N}$ into H. Then, for any mean μ on ℓ^∞, there exists a unique element $h_\mu \in H$ such that

$$\langle h_\mu, z\rangle = (\mu)_k \langle h(k), z\rangle, \quad \forall z \in H.$$

Such a h_μ is contained in $\overline{\text{co}}\{h(k) : k \in \mathbb{N}\}$, where $\overline{\text{co}}A$ is the closure of convex hull of A. In particular, let T be a mapping of a subset C of a Hilbert space H into itself such that $\{T^k x : k \in \mathbb{N}\}$ is bounded for some $x \in C$. Putting $h(k) = T^k x$ for all $k \in \mathbb{N}$, we have that there exists $z_0 \in H$ such tat

$$\mu_k \langle T^k x, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H.$$

We denote such z_0 by $T_\mu x$.

From Kawasaki and Takahashi [13], we also know the following fixed point theorem for widely more generalized hybrid mappings in a Hilbert space.

Theorem 2.3 ([13]). *Let H be a Hilbert space, let C be a non-empty, closed and convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself, i.e., there exist $\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta \in \mathbb{R}$ such that*

$$\alpha\|Tx-Ty\|^2 + \beta\|x-Ty\|^2 + \gamma\|Tx-y\|^2 + \delta\|x-y\|^2 + \varepsilon\|x-Tx\|^2 + \zeta\|y-Ty\|^2 + \eta\|(x-Tx)-(y-Ty)\|^2 \leq 0$$

for all $x, y \in C$. Suppose that it satisfies the following condition (1) or (2):

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$ *and* $\zeta+\eta \geq 0$*;*
(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$ *and* $\varepsilon+\eta \geq 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0,1,\ldots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$ under the conditions (1) and (2).

3. Weak convergence theorems of Mann's type

In this section, we prove a weak convergence theorem of Mann's type [18] for widely more generalized hybrid mappings in a Hilbert space. Before proving the result, we need the following lemma proved by Kawasaki and Takahashi [13]; see also [8].

Lemma 3.1 ([13]). *Let C be a non-empty, closed and convex subset of a Hilbert space H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself with $F(T) \neq \emptyset$ which satisfies the condition (1) or (2):*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta > 0$ *and* $\zeta+\eta \geq 0$*;*
(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma > 0$ *and* $\varepsilon+\eta \geq 0$.

Then T is quasi-nonexpansive.

If $T : C \to H$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [11]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \to z$. Since C is weakly closed, we have $z \in C$. Furthermore, from

$$\|z-Tz\| \leq \|z-z_n\| + \|z_n - Tz\| \leq 2\|z-z_n\| \to 0,$$

z is a fixed point of T and so $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0,1]$, put $z = \alpha x + (1-\alpha)y$. Then we have from (2.2) that

$$\begin{aligned}
\|z-Tz\|^2 &= \|\alpha x + (1-\alpha)y - Tz\|^2 \\
&= \alpha\|x-Tz\|^2 + (1-\alpha)\|y-Tz\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&\leq \alpha\|x-z\|^2 + (1-\alpha)\|y-z\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&= \alpha(1-\alpha)^2\|x-y\|^2 + (1-\alpha)\alpha^2\|x-y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&= \alpha(1-\alpha)(1-\alpha+\alpha-1)\|x-y\|^2 \\
&= 0
\end{aligned}$$

and hence $Tz = z$. This implies that $F(T)$ is convex.

Using Lemma 3.1 and the technique developed by Ibaraki and Takahashi [9, 10], we can prove the following weak convergence theorem.

Theorem 3.2. *Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let $T : C \to C$ be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$- widely more generalized hybrid mapping with $F(T) \neq \emptyset$ which satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma > 0$, $\varepsilon+\eta \geq 0$ *and* $\zeta+\eta \geq 0$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta > 0$, $\zeta+\eta \geq 0$ *and* $\varepsilon+\eta \geq 0$.

Let P be the mertic projection of H onto $F(T)$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on ℓ^∞. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n, \quad n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to $v \in F(T)$, where $v = \lim_{n\to\infty} Px_n$.

Proof. Since $T : C \to C$ be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping, we have that

$$\begin{aligned}\alpha\|Tx-Ty\|^2 &+ \beta\|x-Ty\|^2+\gamma\|Tx-y\|^2+\delta\|x-y\|^2 \\ &+\varepsilon\|x-Tx\|^2+\zeta\|y-Ty\|^2+\eta\|(x-Tx)-(y-Ty)\|^2 \leq 0\end{aligned} \tag{3.1}$$

for any $x, y \in C$. Since $T : C \to H$ is quasi-nonexpansive, we have from Lemma 3.1 that $F(T)$ is closed and convex. Furthermore, we have that for any $x \in C$ and $z \in F(T)$

$$\begin{aligned}\|T_{\mu_n}x-z\|^2 &= \langle T_{\mu_n}x-z, T_{\mu_n}x-z\rangle \\ &= (\mu_n)_k\langle T^kx-z, T_{\mu_n}x-z\rangle \\ &\leq \|\mu_n\|\sup_k|\langle T^kx-z, T_{\mu_n}x-z\rangle| \\ &\leq \sup_k\|T^kx-z\|\cdot\|T_{\mu_n}x-z\| \\ &\leq \sup_k\|x-z\|\cdot\|T_{\mu_n}x-z\| \\ &= \|x-z\|\cdot\|T_{\mu_n}x-z\|.\end{aligned}$$

and hence

$$\|T_{\mu_n}x-z\| \leq \|x-z\|. \tag{3.2}$$

Using (3.2), we have that for any $z \in F(T)$,

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq \|\alpha_nx_n+(1-\alpha_n)T_{\mu_n}x_n-z\|^2 \\ &\leq \alpha_n\|x_n-z\|^2+(1-\alpha_n)\|T_{\mu_n}x_n-z\|^2 \\ &\leq \alpha_n\|x_n-z\|^2+(1-\alpha_n)\|x_n-z\|^2 \\ &= \|x_n-z\|^2\end{aligned}$$

for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty}\|x_n-z\|^2$ exists. Then $\{x_n\}$ is bounded. We also have from (2.2) that

$$\|x_{n+1}-z\|^2 \leq \|\alpha_nx_n+(1-\alpha_n)T_{\mu_n}x_n-z\|^2$$

$$
\begin{aligned}
&= \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|Tx_n - z\|^2 - \alpha_n(1-\alpha_n)\|T_{\mu_n}x_n - x_n\|^2 \\
&\le \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|T_{\mu_n}x_n - x_n\|^2 \\
&= \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|T_{\mu_n}x_n - x_n\|^2.
\end{aligned}
$$

Thus we have

$$\alpha_n(1-\alpha_n)\|T_{\mu_n}x_n - x_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Since $\lim_{n\to\infty}\|x_n - z\|^2$ exists and $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, we have that

$$\|T_{\mu_n}x_n - x_n\| \to 0. \tag{3.3}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. From (3.3), we also have that $T_{\mu_{n_i}}x_{n_i} \rightharpoonup v$. Let us show that v is a fixed point of T. We obtain from (3.1) that for any $x, z \in C$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
&\alpha\|Tx - T^{n+1}z\|^2 + \beta\|x - T^{n+1}z\|^2 + \gamma\|Tx - T^nz\|^2 + \delta\|x - T^nz\|^2 \\
&\quad + \varepsilon\|x - Tx\|^2 + \zeta\|T^nz - T^{n+1}z\|^2 + \eta\|(x - Tx) - (T^nz - T^{n+1}z)\|^2 \le 0
\end{aligned}
$$

for any $n \in \mathbb{N}\cup\{0\}$ and $x \in C$. By (2.3) we obtain that

$$
\begin{aligned}
&\|(x - Tx) - (T^nz - T^{n+1}z)\|^2 \\
&\quad = \|x - Tx\|^2 + \|T^nz - T^{n+1}z\|^2 - 2\langle x - Tx, T^nz - T^{n+1}z\rangle \\
&\quad = \|x - Tx\|^2 + \|T^nz - T^{n+1}z\|^2 + \|x - T^nz\|^2 + \|Tx - T^{n+1}z\|^2 \\
&\qquad - \|x - T^{n+1}z\|^2 - \|Tx - T^nz\|^2.
\end{aligned}
$$

Thus we have that

$$
\begin{aligned}
&(\alpha+\eta)\|Tx - T^{n+1}z\|^2 + (\beta-\eta)\|x - T^{n+1}z\|^2 + (\gamma-\eta)\|Tx - T^nz\|^2 \\
&\quad + (\delta+\eta)\|x - T^nz\|^2 + (\varepsilon+\eta)\|x - Tx\|^2 + (\zeta+\eta)\|T^nz - T^{n+1}z\|^2 \le 0.
\end{aligned}
$$

From

$$
\begin{aligned}
&(\gamma-\eta)\|Tx - T^nz\|^2 \\
&\quad = (\alpha+\gamma)(\|x - Tx\|^2 + \|x - T^nz\|^2 - 2\langle x - Tx, x - T^nz\rangle) \\
&\qquad - (\alpha+\eta)\|Tx - T^nz\|^2,
\end{aligned}
$$

we have that

$$
\begin{aligned}
&(\alpha+\eta)\|Tx - T^{n+1}z\|^2 + (\beta-\eta)\|x - T^{n+1}z\|^2 \\
&\quad + (\alpha+\gamma)(\|x - Tx\|^2 + \|x - T^nz\|^2 - 2\langle x - Tx, x - T^nz\rangle) \\
&\quad - (\alpha+\eta)\|Tx - T^nz\|^2 + (\delta+\eta)\|x - T^nz\|^2 \\
&\quad + (\varepsilon+\eta)\|x - Tx\|^2 + (\zeta+\eta)\|T^nz - T^{n+1}z\|^2 \le 0
\end{aligned}
$$

and hence

$$
\begin{aligned}
&(\alpha+\eta)(\|Tx - T^{n+1}z\|^2 - \|Tx - T^nz\|^2) + (\beta-\eta)\|x - T^{n+1}z\|^2 \\
&\quad - 2(\alpha+\gamma)\langle x - Tx, x - T^nz\rangle + (\alpha+\gamma+\delta+\eta)\|x - T^nz\|^2 \\
&\quad + (\alpha+\gamma+\varepsilon+\eta)\|x - Tx\|^2 + (\zeta+\eta)\|T^nz - T^{n+1}z\|^2 \le 0.
\end{aligned}
$$

By $\alpha+\beta+\gamma+\delta \ge 0$, we have that

$$-(\beta-\eta) = -(\beta+\delta) + \delta + \eta \le \alpha+\gamma+\delta+\eta.$$

From this inequality and $\zeta + \eta \geq 0$ we obtain that

$$\begin{aligned}(\alpha+\eta)&(\|Tx - T^{n+1}z\|^2 - \|Tx - T^n z\|^2)\\ &+(\beta-\eta)(\|x - T^{n+1}z\|^2 - \|x - T^n z\|^2)\\ &-2(\alpha+\gamma)\langle x - Tx, x - T^n z\rangle + (\alpha+\gamma+\varepsilon+\eta)\|x - Tx\|^2 \leq 0.\end{aligned} \tag{3.4}$$

From (3.4), we have that

$$\begin{aligned}(\alpha+\eta)&(\|Tz - T^{k+1}x_n\|^2 - \|Tz - T^k x_n\|^2)\\ &+(\beta-\eta)(\|z - T^{k+1}x_n\|^2 - \|z - T^k x_n\|^2)\\ &-2(\alpha+\gamma)\langle z - Tz, z - T^k x_n\rangle + (\alpha+\gamma+\varepsilon+\eta)\|z - Tz\|^2 \leq 0\end{aligned}$$

for any $k \in \mathbb{N} \cup \{0\}$ and $z \in C$. We apply μ_n to both sides of this inequality. We have that

$$\begin{aligned}(\alpha+\eta)&(\mu_n)_k(\|Tz - T^{k+1}x_n\|^2 - \|Tz - T^k x_n\|^2)\\ &+(\beta-\eta)(\mu_n)_k(\|z - T^{k+1}x_n\|^2 - \|z - T^k x_n\|^2)\\ &-2(\alpha+\gamma)(\mu_n)_k\langle z - Tz, z - T^k x_n\rangle + (\alpha+\gamma+\varepsilon+\eta)\|z - Tz\|^2 \leq 0\end{aligned}$$

and hence

$$\begin{aligned}-|\alpha+\eta|&\|\mu_n - \ell_1^*\mu_n\| \sup_{k\in\mathbb{N}} \|Tz - T^k x_n\|^2\\ &- |\beta-\eta|\|\mu_n - \ell_1^*\mu_n\| \sup_{k\in\mathbb{N}} \|z - T^k x_n\|^2\\ &- 2(\alpha+\gamma)\langle z - Tz, z - T_{\mu_n} x_n\rangle + (\alpha+\gamma+\varepsilon+\eta)\|z - Tz\|^2 \leq 0.\end{aligned} \tag{3.5}$$

Replacing n by n_i in (3.5), we have that

$$\begin{aligned}-|\alpha+\eta|&\|\mu_{n_i} - \ell_1^*\mu_{n_i}\| \sup_{k\in\mathbb{N}} \|Tz - T^k x_{n_i}\|^2\\ &-|\beta-\eta|\|\mu_{n_i} - \ell_1^*\mu_{n_i}\| \sup_{k\in\mathbb{N}} \|z - T^k x_{n_i}\|^2\\ &-2(\alpha+\gamma)\langle z - Tz, z - T_{\mu_{n_i}} x_{n_i}\rangle + (\alpha+\gamma+\varepsilon+\eta)\|z - Tz\|^2 \leq 0.\end{aligned}$$

Letting $i \to \infty$, we have from $T_{\mu_{n_i}} x_{n_i} \rightharpoonup v$ that

$$-2(\alpha+\gamma)\langle z - Tz, z - v\rangle + (\alpha+\gamma+\varepsilon+\eta)\|z - Tz\|^2 \leq 0.$$

Putting $z = v$, we have that

$$(\alpha+\gamma+\varepsilon+\eta)\|v - Tv\|^2 \leq 0.$$

Since $\alpha+\gamma+\varepsilon+\eta > 0$, we have that $v \in F(T)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$. To complete the proof, we show $v_1 = v_2$. We know that $v_1, v_2 \in F(T)$ and hence $\lim_{n\to\infty} \|x_n - v_1\|^2$ and $\lim_{n\to\infty} \|x_n - v_2\|^2$ exist. Put

$$a = \lim_{n\to\infty} (\|x_n - v_1\|^2 - \|x_n - v_2\|^2).$$

Note that for $n = 1, 2, \dots,$

$$\|x_n - v_1\|^2 - \|x_n - v_2\|^2 = 2\langle x_n, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2.$$

From $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$, we have

$$a = 2\langle v_1, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2 \tag{3.6}$$

and

$$a = 2\langle v_2, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain $0 = 2\langle v_2 - v_1, v_2 - v_1\rangle$. Thus we obtain $v_2 = v_1$. This implies that $\{x_n\}$ converges weakly to an element $v \in F(T)$. Since $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $z \in F(T)$ and $n \in \mathbb{N}$, we obtain from Lemma 2.1 that $\{Px_n\}$ converges strongly to an element $p \in F(T)$. On the other hand, we have from the property of P that

$$\langle x_n - Px_n, Px_n - u\rangle \geq 0$$

for all $u \in F(T)$ and $n \in \mathbb{N}$. Since $x_n \rightharpoonup v$ and $Px_n \to p$, we obtain

$$\langle v - p, p - u\rangle \geq 0$$

for all $u \in F(T)$. Putting $u = v$, we obtain $p = v$. This means $v = \lim_{n\to\infty} Px_n$.

Similarly, we can obtain the desired result for the case of $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$, $\zeta + \eta \geq 0$ and $\varepsilon + \eta \geq 0$. This completes the proof. □

Using Theorem 3.2, we can show the following weak convergence theorem of Mann's type for generalized hybrid mappings in a Hilbert space.

Theorem 3.3. *Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on ℓ^∞. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_{\mu_n} x_n, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges weakly to an element $v \in F(T)$.

Proof. Since $T : C \to C$ is a generalized hybrid mapping, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - Ty\|^2 + (1 - \beta)\|x - Ty\|^2$$

for all $x, y \in C$. We have that an (α, β)-generalized hybrid mapping is an $(\alpha, 1 - \alpha, -\beta, -(1 - \beta), 0, 0, 0)$-widely more generalized hybrid mapping which satisfies the condition (2) in Theorem 3.2. Therefore, we have the desired result from Theorem 3.2. □

4. Strong convergence theorem

In this section, using the idea of mean convergence by Shimizu and Takahashi [19] and [20], we prove the following strong convergence theorem for widely more generalized hybrid mappings in a Hilbert space by using strongly asymptotically invariant sequences.

Theorem 4.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping of C into itself which satisfies the following condition (1) or (2):*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma>0$, $\varepsilon+\eta \geq 0$ *and* $\zeta+\eta \geq 0$;
(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta>0$, $\zeta+\eta \geq 0$ *and* $\varepsilon+\eta \geq 0$.

Let $\{\mu_n\}$ *be a strongly asymptotically invariant sequence of means on* ℓ^∞. *Let* $u \in C$ *and define sequences* $\{x_n\}$ *and* $\{z_n\}$ *in* C *as follows:* $x_1 = x \in C$ *and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = T_{\mu_n} x_n \end{cases}$$

for all $n = 1, 2, \ldots$, *where* $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *If* $F(T) \neq \emptyset$, *then* $\{x_n\}$ *and* $\{z_n\}$ *converge strongly to* Pu, *where* P *is the metric projection of* H *onto* $F(T)$.

Proof. Since $T: C \to C$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping, we have that

$$\begin{aligned} \alpha\|Tx-Ty\|^2 + \beta\|x-Ty\|^2 + \gamma\|Tx-y\|^2 + \delta\|x-y\|^2 \\ +\varepsilon\|x-Tx\|^2 + \zeta\|y-Ty\|^2 + \eta\|(x-Tx)-(y-Ty)\|^2 \leq 0 \end{aligned} \tag{4.1}$$

for any $x, y \in C$. Since $F(T) \neq \emptyset$, we have that for all $q \in F(T)$ and $n = 1, 2, 3, \ldots$,

$$\|z_n - q\| = \|T_{\mu_n} x_n - q\| \leq \|x_n - q\|. \tag{4.2}$$

Thus we have that

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n u + (1-\alpha_n) z_n - q\| \\ &\leq \alpha_n \|u - q\| + (1-\alpha_n)\|z_n - q\| \\ &\leq \alpha_n \|u - q\| + (1-\alpha_n)\|x_n - q\|. \end{aligned}$$

Hence, by induction, we obtain

$$\|x_n - q\| \leq \max\{\|u-q\|, \|x-q\|\}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ and $\{z_n\}$ are bounded. Since $\|T_{\mu_n} x_n - q\| \leq \|x_n - q\|$, we have also that $\{T_{\mu_n} x_n\}$ is bounded.

Since $\{T_n x_n\}$ is bounded, there exists a subsequence $\{T_{n_i} x_{n_i}\}$ of $\{T_n x_n\}$ such that $T_{n_i} x_{n_i} \rightharpoonup w \in H$. As in the proof of Theorem 3.2, we have that $w \in F(T)$. On the other hand, since $x_{n+1} - z_n = \alpha_n(u - z_n)$, $\{T_n x_n\}$ is bounded and $\alpha_n \to 0$, we have $\lim_{n\to\infty} \|x_{n+1} - T_n x_n\| = 0$. Let us show

$$\limsup_{n\to\infty} \langle u - Pu, x_{n+1} - Pu \rangle \leq 0.$$

We may assume without loss of generality that there exists a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n\to\infty} \langle u - Pu, x_{n+1} - Pu \rangle = \lim_{i\to\infty} \langle u - Pu, x_{n_i+1} - Pu \rangle$$

and $x_{n_i+1} \rightharpoonup v$. From $\|x_{n+1} - T_n x_n\| \to 0$, we have $T_{n_i} x_{n_i} \rightharpoonup v$. From the above argument, we have $v \in F(T)$. Since P is the metric projection of H onto $F(T)$, we have

$$\lim_{i\to\infty} \langle u - Pu, x_{n_i+1} - Pu \rangle = \langle u - Pu, v - Pu \rangle \leq 0.$$

This implies

$$\limsup_{n\to\infty} \langle u - Pu, x_{n+1} - Pu \rangle \leq 0. \tag{4.3}$$

Since $x_{n+1} - Pu = (1-\alpha_n)(z_n - Pu) + \alpha_n(u - Pu)$, from (2.1) we have

$$\begin{aligned}\|x_{n+1} - Pu\|^2 &= \|(1-\alpha_n)(z_n - Pu) + \alpha_n(u - Pu)\|^2 \\ &\le (1-\alpha_n)^2\|z_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle \\ &\le (1-\alpha_n)\|x_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle.\end{aligned}$$

Putting $s_n = \|x_n - Pu\|^2$, $\beta_n = 0$ and $\gamma_n = 2\langle u - Pu, x_{n+1} - Pu\rangle$ in Lemma 2.2, we have from $\sum_{n=1}^{\infty} \alpha_n = \infty$ and (4.3) that

$$\lim_{n\to\infty} \|x_n - Pu\| = 0.$$

By $\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0$, we also obtain $z_n \to Pu$ as $n \to \infty$.

Similarly, we can obtain the desired result for the case of $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \beta > 0$, $\zeta + \eta \ge 0$ and $\varepsilon + \eta \ge 0$. □

Using Theorem 4.1, we can show the following result obtained by Hojo and Takahashi [7]; see also [17].

Theorem 4.2 ([7]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a generalized hybrid mapping of C into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, ...,$ where $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu \in F(T)$, where P is the metric projection of H onto $F(T)$.

Proof. As in the proof of Theorem 3.3, a generalized hybrid mapping is a widely more generalized hybrid mapping. Define

$$\mu_n(f) = \frac{1}{n}\sum_{i=0}^{n-1} f(i)$$

for all $n \in \mathbb{N}$ and $f \in \ell^\infty$. We have that $\{\mu_n : n \in \mathbb{N}\}$ is a strongly asymptotically invariant sequence of means on ℓ^∞. Furthermore, we have that for any $x \in C$ and $n \in \mathbb{N}$,

$$T_{\mu_n} x = \frac{1}{n}\sum_{i=0}^{n-1} T^i x.$$

Therefore, we have the desired result from Theorem 4.1. □

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