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FIXED POINTS OF PSEUDOCONTRACTIVE MAPPINGS BY A PROJECTION METHOD IN HILBERT SPACES

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Abstract. It is well-known that Mann's algorithm fails to converge for Lipschitzian pseudocontractions. The main purpose of this article is to construct iterative methods for finding the fixed points of pseudocontractive mappings in Hilbert spaces. Strong convergence results are given.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to C$ is called pseudocontractive (or a pseudocontraction) if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad x, y \in C. \tag{1.1}$$

It is known (and is easily seen) that T is pseudocontractive if and only if T satisfies the condition:

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad x, y \in C. \tag{1.2}$$

Construction of fixed points of nonlinear mappings is a classical and active area of nonlinear functional analysis, due to the fact that many nonlinear problems can be reformulated as fixed point equations of nonlinear mappings. The research of this area dates back to Picard's and Banach's time. As a matter of fact, the well-known Banach's contraction principle states that the Picard iterates $\{T^n x\}$ converge to the unique fixed point of T whenever T is a contraction of a complete metric space. However, if T is not a contraction (nonexpansive, say), then the Picard iterates $\{T^n x\}$ fail, in general, to converge; hence, other iterative methods are needed. In 1953, Mann [18] introduced the now called Mann's iterative method which generates a sequence $\{x_n\}$ via the averaged algorithm:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0$$

where $\{\alpha_n\}$ is a sequence in the unit interval $[0, 1]$, T is a self-mapping of a closed convex subset C of a Hilbert space H, and the initial guess x_0 is an arbitrary (but fixed) point of C.

Mann's algorithm has extensively been studied [1, 4, 5, 8, 9, 11–13, 16, 19, 20, 24, 26, 27, 29, 31], and in particular, it is known that if T is nonexpansive (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$) and if T has a fixed point, then the sequence $\{x_n\}$ generated

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by Mann's algorithm converges weakly to a fixed point of T provided the sequence $\{\alpha_n\}$ satisfies the condition: $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. This algorithm however does not converge in the strong topology, in general (see [1, Corollary 5.2]).

Browder and Petryshyn [2] studied weak convergence of Mann's algorithm for the class of strictly pseudocontractions. However, Mann's algorithm fails to converge for Lipschitzian pseudocontractions (see the counterexample of Chidume and Mutangadura [10]). It is therefore an interesting question of inventing iterative algorithms which generate a sequence converging in the norm topology to a fixed point of a Lipschitzian pseudocontraction (if any). The interest of pseudocontractions lies in their connection with monotone operators; namely, T is a pseudocontraction if and only if the complement $I - T$ is a monotone operator. Some related works, please refer to [35]- [22]. Especially, in order to find the minimum norm fixed point of pseudocontractions, Zegeye, Shahzad and Alghamdi [32] proved the following convergence result:

Theorem 1.1. *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T: K \to K$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $Fix(T) \neq \emptyset$. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - P_K[(1-t_n)x_n]), n \geq 1, \tag{1.3}$$

where $\{\lambda_n\}$, $\{\theta_n\}$ and $\{t_n\}$ are real sequences in $(0,1]$ satisfying the conditions: (C1) $\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} t_n = 0$; (C2) $\lambda_n(1+\theta_n) \leq 1, \sum_n \lambda_n\theta_n t_n = \infty$, $\lim_{n\to\infty} \lambda_n/\theta_n t_n = 0$; (C3) $\lim_{n\to\infty}[\theta_{n-1}-\theta_n]/\lambda_n\theta_n^2 t_n^2 = 0$ and $\lim_{n\to\infty}[t_{n-1} - t_n]/\lambda_n\theta_n t_n^2 = 0$. Then, the sequence $\{x_n\}$ converges strongly to the minimum-norm fixed point of T.

Remark 1.2. *We note that the restrictions (C2)-(C3) are complicated. That is to say, it is difficult to select the algorithm parameters.*

Inspired by the results in the literature, the main purpose of this article is to construct iterative method for finding the fixed points of pseudocontractive mappings. Under some mild conditions, strong convergence results are given. As a special case, the minimum norm fixed point of pseudocontractive mappings can be approached iteratively.

2. Preliminaries

A mapping $T: C \to C$ is called $L-$Lipschitzian if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|,$$

for all $x, y \in C$.

Recall that the (nearest point or metric) projection from H onto C, denoted P_C, assigns, to each $x \in H$, the unique point $P_C(x) \in C$ with the property

$$\|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}.$$

It is well known that the metric projection P_C of H onto C has the following basic properties:

(*a*) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$ for all $x, y \in H$;

(*b*) $\langle x - y, P_C(x) - P_C(y)\rangle \geq \|P_C(x) - P_C(y)\|^2$ for every $x, y \in H$;

(c) $\langle x - P_C(x), y - P_C(x)\rangle \le 0$ for all $x \in H$, $y \in C$.

It is well-known that in a real Hilbert space H, the following hold:

(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$, $\forall x, y \in H$;

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$, $\forall x, y \in H$ and $t \in [0,1]$.

In the sequel we shall use the following notations:

- $Fix(T)$ stands for the set of fixed points of T;
- $x_n \rightharpoonup x$ stands for the weak convergence of x_n to x;
- $x_n \to x$ stands for the strong convergence of x_n to x.

Lemma 2.1 ([34])**.** *Let H be a real Hilbert space, C a closed convex subset of H. Let $T : C \to C$ be a continuous pseudocontractive mapping. Then*

(i) $Fix(T)$ is a closed convex subset of C.

(ii) $(I - T)$ is demiclosed at zero.

Lemma 2.2 ([30])**.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([17])**.** *Let (s_n) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence (s_{n_i}) of (s_n) such that $s_{n_i} \le s_{n_i+1}$ for all $i \ge 0$. For every $n \ge n_0$, define an integer sequence $(\tau(n))$ as*

$$\tau(n) = \max\{k \le n : s_{n_i} < s_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \ge n_0$

$$\max\{s_{\tau(n)}, s_n\} \le s_{\tau(n)+1}.$$

3. Main results

In this section, we will introduce our algorithm and prove our main results.

Algorithm 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonlinear operator. For fixed $u \in H$ and $x_0 \in C$ arbitrarily, define a sequence $\{x_n\} \subset C$ by the following manner:*

$$\begin{cases} y_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ x_{n+1} = P_C[\alpha_n u + (1-\alpha_n-\beta_n)x_n + \beta_n T y_n], n \ge 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real number sequences and P_C is the metric projection.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be an L-Lipschitz pseudocontractve mapping with $Fix(T) \ne \emptyset$. Assume the parameters $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following restrictions:*

(i): $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii): $\alpha_n + \beta_n \le \gamma_n$;

(iii): $\liminf_{n\to\infty}\beta_n > 0$ *and* $\limsup_{n\to\infty}\gamma_n < \frac{1}{\sqrt{1+L^2}+1}$.
Then the sequence $\{x_n\}$ *defined by (3.1) converges strongly to* $P_{Fix(T)}u$.

Proof. Set $x^* = P_{Fix(T)}(u)$. Thus, from (3.1), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 = \|P_C[\alpha_n u + (1-\alpha_n-\beta_n)x_n + \beta_n T y_n] - x^*\|^2 \\
\leq\ & \|\alpha_n(u-x^*) + (1-\alpha_n-\beta_n)(x_n-x^*) + \beta_n(Ty_n - x^*)\|^2 \\
=\ & \left\|\alpha_n(u-x^*) + (1-\alpha_n)\left(\frac{1-\alpha_n-\beta_n}{1-\alpha_n}(x_n-x^*) + \frac{\beta_n}{1-\alpha_n}(Ty_n-x^*)\right)\right\|^2 \\
\leq\ & \alpha_n\|u-x^*\|^2 + (1-\alpha_n)\left\|\frac{1-\alpha_n-\beta_n}{1-\alpha_n}(x_n-x^*) + \frac{\beta_n}{1-\alpha_n}(Ty_n-x^*)\right\|^2 \\
=\ & \alpha_n\|u-x^*\|^2 + (1-\alpha_n)\left[\frac{1-\alpha_n-\beta_n}{1-\alpha_n}\|x_n-x^*\|^2 + \frac{\beta_n}{1-\alpha_n}\|Ty_n-x^*\|^2\right. \\
& \left.-\frac{\beta_n(1-\alpha_n-\beta_n)}{1-\alpha_n}\|x_n-Ty_n\|^2\right] \\
(3.2)=\ & \alpha_n\|u-x^*\|^2 + (1-\alpha_n-\beta_n)\|x_n-x^*\|^2 + \beta_n\|Ty_n-x^*\|^2 \\
& -\beta_n(1-\alpha_n-\beta_n)\|x_n-Ty_n\|^2.
\end{aligned}$$

By (1.2), we have

$$\|Tw-x^*\|^2 \leq \|w-x^*\|^2 + \|w-Tw\|^2,$$

for all $w \in C$.

Note that

$$\|x_n - y_n\| = \gamma_n\|x_n - Tx_n\|.$$

Hence,

$$\begin{aligned}
\|Ty_n - x^*\|^2 &= \|T[(1-\gamma_n)x_n + \gamma_n Tx_n] - x^*\|^2 \\
&\leq \|(1-\gamma_n)x_n + \gamma_n Tx_n - x^*\|^2 \\
&\quad + \|(1-\gamma_n)x_n + \gamma_n Tx_n - Ty_n\|^2 \\
&= \|(1-\gamma_n)(x_n-x^*) + \gamma_n(Tx_n-x^*)\|^2 \\
&\quad + \|(1-\gamma_n)(x_n-Ty_n) + \gamma_n(Tx_n-Ty_n)\|^2 \\
&= (1-\gamma_n)\|x_n-x^*\|^2 + \gamma_n\|Tx_n-x^*\|^2 \\
&\quad -\gamma_n(1-\gamma_n)\|x_n-Tx_n\|^2 \\
&\quad +(1-\gamma_n)\|x_n-Ty_n\|^2 + \gamma_n\|Tx_n-Ty_n\|^2 \\
&\quad -\gamma_n(1-\gamma_n)\|x_n-Tx_n\|^2 \\
&\leq (1-\gamma_n)\|x_n-x^*\|^2 + \gamma_n(\|x_n-x^*\|^2 + \|x_n-Tx_n\|^2) \\
&\quad -\gamma_n(1-\gamma_n)\|x_n-Tx_n\|^2 + (1-\gamma_n)\|x_n-Ty_n\|^2 \\
&\quad +\gamma_n L^2\|x_n-y_n\|^2 - \gamma_n(1-\gamma_n)\|x_n-Tx_n\|^2 \\
&\leq \|x_n-x^*\|^2 + (1-\gamma_n)\|x_n-Ty_n\|^2 \\
&\quad -\gamma_n(1-2\gamma_n-\gamma_n^2L^2)\|x_n-Tx_n\|^2. \qquad (3.3)
\end{aligned}$$

Since $\limsup_{n\to\infty} \gamma_n < \frac{1}{\sqrt{1+L^2}+1}$, without loss of generality, we may assume that $\gamma_n \le a < \frac{1}{\sqrt{1+L^2}+1}$ for all n. Then, we have $1 - 2\gamma_n - \gamma_n^2 L^2 > 0$. It follows from (3.2) and (3.3) that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \alpha_n\|u - x^*\|^2 + (1 - \alpha_n - \beta_n)\|x_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\ &\quad +\beta_n(1 - \gamma_n)\|x_n - Ty_n\|^2 - \gamma_n\beta_n(1 - 2\gamma_n - \gamma_n^2 L^2)\|x_n - Tx_n\|^2 \\ &\quad -\beta_n(1 - \alpha_n - \beta_n)\|x_n - Ty_n\|^2 \\ &\le \alpha_n\|u - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 \\ &\quad +\beta_n(\alpha_n + \beta_n - \gamma_n)\|x_n - Ty_n\|^2 \\ &\le \alpha_n\|u - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 \\ &\le \max\{\|u - x^*\|^2, \|x_n - x^*\|^2\}.\end{aligned}$$

By induction, we deduce

$$\|x_{n+1} - x^*\| \le \max\{\|u - x^*\|, \|x_0 - x^*\|\}.$$

Hence, $\{x_n\}$ is bounded.

Set $z_n = \alpha_n u + (1 - \alpha_n - \beta_n)x_n + \beta_n Ty_n$ for all n. Then, we can rewrite x_{n+1} in (3.1) as $x_{n+1} = P_C[\alpha_n u + (1 - \alpha_n)z_n]$ for all n. Thus,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|P_C[\alpha_n u + (1 - \alpha_n)z_n] - x^*\|^2 \\ &\le \|z_n - x^*\|^2 \\ &= \|(1 - \alpha_n)(x_n - x^*) - \beta_n(x_n - Ty_n) + \alpha_n(u - x^*)\|^2 \\ &\le \|(1 - \alpha_n)(x_n - x^*) - \beta_n(x_n - Ty_n)\|^2 + 2\alpha_n\langle u - x^*, z_n - x^*\rangle \\ &= \|(1 - \alpha_n)(x_n - x^*)\|^2 - 2\beta_n(1 - \alpha_n)\langle x_n - Ty_n, x_n - x^*\rangle \\ &\quad +\beta_n^2\|x_n - Ty_n\|^2 + 2\alpha_n\langle u - x^*, z_n - x^*\rangle.\end{aligned}$$

It is easy to verify that (3.3) is equivalent to

$$2\langle x_n - Ty_n, x_n - x^*\rangle \ge \gamma_n\|x_n - Ty_n\|^2 + \gamma_n(1 - 2\gamma_n - \gamma_n^2 L^2)\|x_n - Tx_n\|^2.$$

Therefore,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1 - \alpha_n)\|x_n - x^*\|^2 - \beta_n(1 - \alpha_n)\gamma_n\|x_n - Ty_n\|^2 \\ &\quad +\beta_n^2\|x_n - Ty_n\|^2 - \beta_n(1 - \alpha_n)\gamma_n(1 - 2\gamma_n - \gamma_n^2 L^2)\|x_n - Tx_n\|^2 \\ &\quad +2\alpha_n\langle u - x^*, z_n - x^*\rangle \\ &\le (1 - \alpha_n)\|x_n - x^*\|^2 - \beta_n(1 - \alpha_n)\gamma_n(1 - 2\gamma_n - \gamma_n^2 L^2)\|x_n - Tx_n\|^2 \\ &\quad +2\alpha_n\langle u - x^*, z_n - x^*\rangle. \qquad (3.4)\end{aligned}$$

It follows that

$$\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 + \beta_n(1 - \alpha_n)\gamma_n(1 - 2\gamma_n - \gamma_n^2 L^2)\|x_n - Tx_n\|^2 \le 2\alpha_n\langle u - x^*, z_n - x^*\rangle.$$

Since x_n is bounded, then y_n and Ty_n are all bounded. Consequently, z_n is bounded. So,

(3.5)

$$\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 + \beta_n(1 - \alpha_n)\gamma_n(1 - 2\gamma_n - \gamma_n^2 L^2)\|x_n - Tx_n\|^2$$

$$\leq \alpha_n M.$$

Next, we will prove that $x_n \to x^*$. We consider two possible cases.

Case 1. Assume $\{\|x_n - x^*\|\}$ is eventually decreasing, i.e., there exists $N > 0$ such that $\{\|x_n - x^*\|\}$ is decreasing for $n \geq N$. Then, $\lim_{n\to\infty} \|x_n - x^*\|\}$ exists and from (3.5) it follows that

$$\beta_n(1-\alpha_n)\gamma_n(1-2\gamma_n-\gamma_n^2L^2)\|x_n - Tx_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + M\alpha_n. \tag{3.6}$$

Since $\liminf_{n\to\infty} \beta_n > 0$ and $\limsup_{n\to\infty} \gamma_n < \frac{1}{\sqrt{1+L^2}+1}$, we have $\liminf_{n\to\infty} \beta_n(1-\alpha_n)\gamma_n(1-2\gamma_n-\gamma_n^2L^2) > 0$. Letting $n \to \infty$ in (3.6), we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Since

$$\begin{aligned}\|z_n - Tx_n\| &\leq \alpha_n\|u - Tx_n\| + \beta_n\|Ty_n - Tx_n\| \\ &\leq \alpha_n\|u - Tx_n\| + \beta_n L\|y_n - x_n\| \\ &\leq \alpha_n\|u - Tx_n\| + \beta_n\gamma_n L\|x_n - Tx_n\|,\end{aligned}$$

we get

$$\begin{aligned}\|z_n - x_n\| &\leq \|z_n - Tx_n\| + \|x_n - Tx_n\| \\ &\leq \|x_n - Tx_n\| + \alpha_n\|u - Tx_n\| + \beta_n\gamma_n L\|x_n - Tx_n\| \\ &= \alpha_n\|u - Tx_n\| + (\beta_n\gamma_n L + 1)\|x_n - Tx_n\|.\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$z_{n_k} \to \tilde{x} \in C \text{ and } \limsup_{n\to\infty}\langle u - x^*, z_n - x^*\rangle = \lim_{k\to\infty}\langle u - x^*, z_{n_k} - x^*\rangle.$$

Since, $x_n - z_n \to 0$, we also have $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to $\tilde{x} \in C$. From the demi-closed principle of T (Lemma 2.1), we have $\tilde{x} \in Fix(T)$. So,

$$\begin{aligned}\limsup_{n\to\infty}\langle u - x^*, z_n - x^*\rangle &= \lim_{k\to\infty}\langle u - x^*, z_{n_k} - x^*\rangle \\ &= \langle u - x^*, \tilde{x} - x^*\rangle \\ &\leq 0.\end{aligned}$$

From (3.6), we obtain

$$\|x_{n+1} - x^*\|^2 \leq (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle u - x^*, z_n - x^*\rangle. \tag{3.7}$$

This together with Lemma 2.2 imply that $\|x_n - x^*\| \to 0$.

Case 2. Assume $\omega_n = \{\|x_n - x^*\|\}$ is not eventually decreasing. That is, there exists an integer n_0 such that $\omega_{n_0} \leq \omega_{n_0+1}$. Thus, we can define an integer sequence $\{\tau_n\}$ for all $n \geq n_0$ as follows:

$$\tau(n) = \max\{k \in \mathbb{N} | n_0 \leq k \leq n, \omega_k \leq \omega_{k+1}\}.$$

Clearly, $\tau(n)$ is a non-decreasing sequence such that $\tau(n) \to +\infty$ as $n \to \infty$ and

$$\omega_{\tau(n)} \leq \omega_{\tau(n)+1},$$

for all $n \geq n_0$. In this case, we derive from (3.6) that

$$\lim_{n\to\infty} \|x_{\tau(n)} - Tx_{\tau(n)}\| = 0.$$

This implies that every weak cluster point of $\{x_{\tau(n)}\}$ is in the fixed points set $Fix(T)$; i.e., $\omega_w(x_{\tau(n)}) \subset Fix(T)$. On the other hand, we note that

$$\|z_{\tau(n)} - x_{\tau(n)}\| \to 0.$$

From which we can deduce that

$$\begin{aligned} \limsup_{n\to\infty}\langle u - x^*, z_{\tau(n)} - x^*\rangle &= \max_{\tilde{x}\in\omega_w(x_{\tau(n)})}\langle u - P_{Fix(T)}(u), \tilde{x} - P_{Fix(T)}(u)\rangle \\ &\leq 0. \end{aligned} \tag{3.8}$$

Since $\omega_{\tau(n)} \leq \omega_{\tau(n)+1}$, we have from (3.7) that

$$\omega_{\tau(n)} \leq 2\alpha_{\tau(n)})\langle u - x^*, z_{\tau(n)} - x^*\rangle. \tag{3.9}$$

Combining (3.8) and (3.9) yields

$$\limsup_{n\to\infty} \omega_{\tau(n)} \leq 0,$$

and hence

$$\lim_{n\to\infty} \omega_{\tau(n)} = 0.$$

From (3.7), we have

$$\limsup_{n\to\infty} \omega_{\tau(n)+1} \leq \limsup_{n\to\infty} \omega_{\tau(n)}.$$

Thus,

$$\lim_{n\to\infty} \omega_{\tau(n)+1} = 0.$$

From Lemma 2.3, we have

$$0 \leq \omega_n \leq \max\{\omega_{\tau(n)}, \omega_{\tau(n)+1}\}.$$

Therefore, $\omega_n \to 0$. That is, $x_n \to x^*$. This completes the proof. □

For $u = 0$ in Algorithm 3.1, we have the following iterative scheme.

Algorithm 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonlinear operator. For $x_0 \in C$ arbitrarily, define a sequence $\{x_n\} \subset C$ by the following manner:*

$$\begin{cases} y_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ x_{n+1} = P_C[(1-\alpha_n-\beta_n)x_n + \beta_n T y_n], n \geq 0, \end{cases} \tag{3.10}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real number sequences and P_C is the metric projection.

The following result is then a direct consequence of Theorem 3.2.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be an L-Lipschitz pseudocontractve mapping with $Fix(T) \neq \emptyset$. Assume the parameters $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following restrictions:*

(i): $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii): $\alpha_n + \beta_n \leq \gamma_n$;

(iii): $\liminf_{n\to\infty} \beta_n > 0$ *and* $\limsup_{n\to\infty} \gamma_n < \frac{1}{\sqrt{1+L^2}+1}$.

Then the sequence $\{x_n\}$ defined by (3.10) converges strongly to $P_{Fix(T)}(0)$, the minimum-norm fixed point of T.

We observe that related results have been established recently in [25]- [28].

Remark 3.5. *We can find the minimum norm fixed point of the pseudocontractive mapping T by using the algorithm (3.10). In contrast to Theorem 1.1, Theorem 3.4 may be effective due to the weaker parameters restrictions.*

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