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A GENERALIZED SUZUKI'S CONDITION IN THE SENSE OF RAKOTCH

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Abstract. Suzuki has recently proved a fixed point theorem that generalizes the classical Banach-Caccioppoli fixed point theorem. A somewhat different generalization of this classical result was given by Dugundji and Granas, who extended Banach-Caccioppoli theorem to the class of maps which are contractive in a certain weak sense. It turns out that the class of weakly contractive maps in the sense of Dugundji and Granas coincides with the class of maps which are contractive in a weak sense introduced previously by Rakotch. In this paper we generalize Suzuki's fixed point theorem by extending his result to the class of maps in the weak sense of Rakotch. Additionally, we obtain a Cauchy rate as well as a rate of convergence for Picard iterates and an explicit expression of uniqueness modulus for this new class of maps.

1. Introduction

Suppose that (X, d) is a complete metric space and that $T : X \to X$ is a map. The well known Banach-Caccioppoli fixed point theorem [3, 5] states that T has a fixed point if it is contractive, that is, if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in X$.

Since then, many generalizations of this theorem have been given, and some of these extensions have been achieved by replacing the constant α by a function $\alpha = \alpha(x, y)$ with certain properties. As far as we know, the first result in this direction was given in 1962 by Rakotch [14], and reads as follows.

Theorem 1.1. Let (X, d) be a complete metric space and $T : X \to X$ a self-map of X satisfying

$$d(Tx, Ty) \leq \alpha(d(x, y))\, d(x, y) \tag{1.1}$$

for all $x, y \in X$, where $\alpha : [0, \infty) \to [0, 1]$. If α verifies the following two conditions:

(C_1) $\alpha(t) < 1$ for all $t > 0$;
(C_2) α is decreasing, that is, if $t_0 \leq t_1$ then $\alpha(t_1) \leq \alpha(t_0)$;

then T has a unique fixed point.

In the same year, Rakotch himself [15] generalized the previous result by introducing the concept of a weakly contractive map (see [7, Page 24]) as a map T for which the function α in (1.1) satisfies, instead of (C_1) and (C_2), the weaker condition

$$\sup\{\alpha(t) : a \leq t \leq b\} < 1 \tag{1.2}$$

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for all $0 < a \leq b$.

In 1969 Krasnosel'skii et al. [13] referred to a fixed point theorem, attributed to Krasnosel'skii (1964), for the class of maps $T : D \subseteq X \to X$ satisfying the condition

$$d(Tx, Ty) \leq \Theta(a, b)\, d(x, y) \qquad \text{for all } x, y \in D, \text{with } a \leq d(x, y) \leq b, \tag{1.3}$$

where $\Theta(a, b) < 1$ for all $b \geq a > 0$. Obviously, condition (1.3) is more general than the condition introduced by Rakotch.

A somewhat different condition was introduced in 1975 by Dugundji and Granas [8] who gave a fixed point theorem for the class of maps $T : D \subseteq X \to X$ satisfying the following condition:

There exists a compactly positive function $\phi : D \times D \to [0, 1]$ such that

$$d(Tx, Ty) \leq d(x, y) - \phi(x, y) \qquad \text{for all } x, y \in D. \tag{1.4}$$

Recall that a function $\phi : D \times D \to [0, 1]$ is compactly positive if

$$\inf \big\{\phi(x, y) : a \leq d(x, y) \leq b\big\} > 0 \qquad \text{for all } b \geq a > 0.$$

Although condition (1.4) looks different of condition (1.3), they are equal, since the authors themself proved in [8] that (1.4) can be rewritten as

$$d(Tx, Ty) \leq \alpha(x, y)\, d(x, y) \qquad \text{for all } x, y \in D, \tag{1.5}$$

where $\alpha : D \times D \to [0, 1]$ is a function that satisfies

$$\sup \big\{\alpha(x, y) : a \leq d(x, y) \leq b\big\} < 1 \qquad \text{for all } b \geq a > 0. \tag{1.6}$$

If the reader has not yet got lost in this forest of conditions, he may still consult some other definitions of weak contractiveness, as given, for instance, by Browder [4] in 1968 and by Alber and Guerre-Delabriere [1] in 1997.

A different kind of generalization, obtained by combining the Banach contractive condition and those introduced by Kannan [11] and Chatterjea [6], was given by Zamfirescu [22] in 1972, who proved a fixed point theorem for the class of maps $T : X \to X$ for which there exists $\zeta \in [0, 1)$ such that, for all $x, y \in X$,

$$d(Tx, Ty)) \leq \zeta \max \Big\{d(x, y),\, \frac{1}{2}\big[d(x, Tx) + d(y, Ty)\big], \frac{1}{2}\big[d(x, Ty) + d(y, Tx)\big]\Big\}. \tag{1.7}$$

In the same fashion of previous results, this theorem has recently been generalized in [2], by the authors of this paper, to the class of maps which satisfy condition (1.7) in the weak sense of Dugundji-Granas.

A new worthy contractive condition has been introduced by Suzuki [20] in 2008 in order to characterize the completeness of the space. Namely, a map T from a nonempty subset D of a metric space (X, d) into X is said to satisfy Suzuki's condition if there exists $r \in [0, 1)$ such that

$$\theta(r)\, d(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leq r\, d(x, y),$$

where $\theta : [0, 1) \to (1/2, 1]$ is defined by

$$\theta(r) := \begin{cases} 1 & \text{if } 0 \leq r \leq (\sqrt{5} - 1)/2, \\ (1 - r)r^{-2} & \text{if } (\sqrt{5} - 1)/2 \leq r \leq 1/\sqrt{2}, \\ (1 + r)^{-1} & \text{if } 1/\sqrt{2} \leq r < 1. \end{cases} \tag{1.8}$$

The aim of this paper is to generalize Suzuki's fixed point theorem by extending it to the class of maps which satisfy Suzuki's condition in a weak sense. Up to present, it seems that the more general way of weakening a strong contractive concept is that followed by Dugundji-Granas, introducing in this case the analogue condition to (1.4). On the other hand, it is very easy to prove that conditions (1.4) and (1.2) are equivalent, so that a generalization of this type should be properly named as a weak condition in the sense of Rakotch, as it is done here. As a by product of our proof, we obtain a Cauchy rate and a rate of convergence for the Picard iterates. This rate of convergence is given in terms of appropriate modulus for the class of maps considered. We point out that our results on rates of convergence are new even for Suzuki type maps.

2. A generalization of Suzuki's condition in the sense of Rakotch

We consider the family of functions $r : [0,\infty) \to [0,1]$ satisfying the following properties:

- (C_1) $r(t) < 1$ for all $t > 0$;
- (C_2) r is decreasing, that is, if $t_0 \leq t_1$ then $r(t_1) \leq r(t_0)$.
- (C_3) If $\{t_n\} \to t$ as $n \to \infty$, then $\limsup_{n\to\infty} r(t_n) \leq r(t)$;

If we check carefully the proof of Suzuki's fixed point result, we can see that the following contractive condition is essential.

(2.1)
There exists a constant $r \in [0,1)$ such that $d(Tx, T^2x) \leq r\, d(x, Tx)$ for all $x \in X$.

This condition has been studied by several authors, as for instance Hicks and Rhoades [9], Ivanov [10], Kasahara [12], Rus [17], [18], [19, Example 4.6] and Taskovic [21]. The above condition can be generalized changing the constant r with a function satisfying certain properties. To be more precise, we shall consider the self-maps T defined on a metric space (X,d) satisfying the following condition:

$$d(Tx, T^2x) \leq r(d(x,Tx))\, d(x,Tx) \tag{2.2}$$

for all $x \in X$, where $r : [0,\infty) \to [0,1]$ satisfies (C_1) and (C_2).

Remark 2.1. Suppose that $T : X \to X$ satisfies condition (2.2). If for any fixed $\sigma \in (0,1)$ we define $r_\sigma : [0,\infty) \to [0,1]$ by

$$r_\sigma(t) := \max\{r(t), \sigma\},$$

then T also satisfies the above condition with r_σ, and, moreover, we have that (C_1) and (C_2) hold for r_σ and $r_\sigma(t) > 0$ for all $t \in [0,\infty)$. Thus, from now on, when considering an r as in condition (2.2) we will assume that we have $r(t) > 0$ for all $t \geq 0$.

Definition 2.2. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a metric space (X,d) and let $h : (0,\infty) \to (0,\infty)$ be a function such that $h(t) \to 0$ as $t \to 0^+$. We say that $\gamma : (0,\infty) \to \mathbb{N}$ is an h-rate of asymptotic regularity for $\{x_n\}_{n\in\mathbb{N}}$ if for all $\varepsilon > 0$ we have that $d(x_k, x_{k+1}) \leq h(\varepsilon)$ for each $k \in \mathbb{N}$ with $k \geq \gamma(\varepsilon)$.

Lemma 2.3. Let (X,d) be a metric space and $h : (0,\infty) \to (0,\infty)$ a function with $\lim_{t\to 0^+} h(t) = 0$. Assume that $T : X \to X$ satisfies condition (2.2). Let $x_0 \in X$

be the starting point of the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by $x_{n+1} := Tx_n$. Let $b > 0$ satisfy $d(x_0, x_1) \le b$, and define $\gamma : (0,\infty) \to \mathbb{N}$ by

$$\gamma(\varepsilon) := \begin{cases} \left\lceil \frac{\log h(\varepsilon) - \log b}{\log r(h(\varepsilon))} \right\rceil + 1, & \text{if } h(\varepsilon) < b \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

where $\lceil\cdot\rceil$ denotes the ceiling function given by $\lceil x\rceil := \min\{n \in \mathbb{Z} : x \le n\}$. Then γ is a h-rate of asymptotic regularity for $\{x_n\}_{n\in\mathbb{N}}$.

Proof. Let $x_0 \in X$ be fixed such that $d(x_0, Tx_0) \le b$. First, let us remark that by (2.2) the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is decreasing. Let $\varepsilon > 0$. If $b \le h(\varepsilon)$, it follows that $d(x_n, x_{n+1}) < h(\varepsilon)$ for all $n \in \mathbb{N}$. Let us consider the case that $h(\varepsilon) < b$. We have to show that $d(x_k, x_{k+1}) < h(\varepsilon)$ for all $k \ge \gamma(\varepsilon)$. In order to do this, notice that it is sufficient to see that $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+1}) < h(\varepsilon)$ because the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is decreasing. Arguing by contradiction, we assume $h(\varepsilon) \le d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+1})$. Then, by (C_2), $r(d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+1})) \le r(h(\varepsilon))$. Hence,

$$\begin{aligned} h(\varepsilon) &\le d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+1}) \\ &\le r(d(x_{\gamma(\varepsilon)-1}, x_{\gamma(\varepsilon)}))\, d(x_{\gamma(\varepsilon)-1}, x_{\gamma(\varepsilon)}) \\ &\le \cdots \le r(h(\varepsilon))^{\gamma(\varepsilon)}\, d(x_0, x_1) \\ &\le r(h(\varepsilon))^{\gamma(\varepsilon)}\, b \\ &< h(\varepsilon), \end{aligned}$$

which is a contradiction. □

Lemma 2.4. Let D be a nonempty subset of a metric space (X, d). Assume that $T : D \to X$ satisfies the following condition:

$$d(x, Tx) \le d(x, y) \quad \text{implies} \quad d(Tx, Ty) \le r(d(x, y))\, d(x, y) \tag{2.4}$$

for all $x, y \in X$, where $r : [0,\infty) \to [0, 1]$ verifies (C_1). Then, T has at most one fixed point in D.

Proof. Suppose that p and q are fixed points of T, with $p \ne q$. Since $d(p, Tp) = 0 \le d(p, q)$, then by (2.4) and (C_1)

$$d(p, q) = d(Tp, Tq) \le r(d(p, q))\, d(p, q) < d(p, q),$$

which is a contradiction. □

We can obtain for the family of maps satisfying (2.4) an explicit expression of a modulus of uniqueness for fixed points of T. Recall that if $T : X \to X$ is a self-map on a metric space (X, d), a function $\phi : (0,\infty) \to (0,\infty)$ is called a modulus of uniqueness (for fixed points of T) if for all $\varepsilon > 0$ and $x, y \in X$ the following implication holds.

$$\left.\begin{aligned} d(x, Tx) \le \phi(\varepsilon) \\ d(y, Ty) \le \phi(\varepsilon) \end{aligned}\right\} \Rightarrow d(x, y) \le \varepsilon.$$

Proposition 2.5. Let (X, d) be a metric space. Suppose that $T : X \to X$ satisfies condition (2.4) with r verifying (C_1) and (C_2). Define $\psi_r : (0, \infty) \to (0, \infty)$ by

$$\psi_r(t) := \frac{t}{2}\left(1 - r(t)\right).$$

Then, ψ_r is a modulus of uniqueness for fixed points of T.

Proof. Notice that ψ_r is well defined because r satisfies (C_1). Let $\varepsilon > 0$ and $x, y \in X$ such that $d(x, Tx) \le \psi_r(\varepsilon)$ and $d(y, Ty) \le \psi_r(\varepsilon)$. Assume that $\varepsilon < d(x, y)$ and we shall get a contradiction as follows. Since $\psi_r(\varepsilon) \le \varepsilon$, we have $d(x, Tx) \le \psi(\varepsilon) \le \varepsilon < d(x, y)$. Then, by (2.4), $d(Tx, Ty) \le r(d(x, y))\, d(x, y)$. In this case, we obtain

$$\begin{aligned} d(x, y) &\le d(x, Tx) + d(Tx, Ty) + d(y, Ty) \\ &\le \psi_r(\varepsilon) + r(d(x, y))\, d(x, y) + \psi_r(\varepsilon) \\ &\le 2\psi_r(\varepsilon) + r(\varepsilon)\, d(x, y), \end{aligned}$$

that is,

$$d(x, y) \le \frac{2\psi_r(\varepsilon)}{1 - r(\varepsilon)} = \varepsilon,$$

which is a contradiction. □

Definition 2.6. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). We say that $\gamma : (0, \infty) \to \mathbb{N}$ is a Cauchy rate for $\{x_n\}_{n\in\mathbb{N}}$ if for all $\varepsilon > 0$ we have that $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) \le \varepsilon$ for each $k \in \mathbb{N}$.

Proposition 2.7. Let (X, d) be a metric space. Assume that $T : X \to X$ satisfies (2.4) with $r : [0, \infty) \to [0, 1]$ verifying (C_1), (C_2). Let $x_0 \in X$ be the starting point of the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by $x_{n+1} := Tx_n$. Let $b > 0$ satisfy $d(x_0, x_1) \le b$, and define $h : (0, \infty) \to (0, \infty)$ by

$$h(\varepsilon) := \min\left\{\frac{\varepsilon}{2}\left(1 - r(\tfrac{\varepsilon}{2})\right), b\right\}.$$

Then, the function $\gamma : (0, \infty) \to \mathbb{N}$ given by (2.3) is a Cauchy rate for $\{x_n\}_{n\in\mathbb{N}}$.

Proof. Let $x_0 \in X$ and $b > 0$ such that $d(x_0, x_1) \le b$. Let $\varepsilon > 0$. Note that T satisfies condition (2.2). Then, by Lemma 2.3, $d(x_k, x_{k+1}) \le h(\varepsilon)$ for each $k \in \mathbb{N}$ with $k \ge \gamma(\varepsilon)$. In particular, for $k = \gamma(\varepsilon)$, we have

$$d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+1}) < \min\left\{\frac{\varepsilon}{2}\left(1 - r(\tfrac{\varepsilon}{2})\right), b\right\}.$$

We will prove inductively that $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) < \varepsilon$ for all $k \in \mathbb{N}$. It is obvious for $k = 0$ and $k = 1$, and assuming $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) < \varepsilon$, let us see $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k+1}) < \varepsilon$.

If $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) < \frac{\varepsilon}{2}$,

$$\begin{aligned} d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+1}) &\le d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) + d(x_{\gamma(\varepsilon)+k}, x_{\gamma(\varepsilon)+k+1}) \\ &\le d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) + d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+1}) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\left(1 - r(\tfrac{\varepsilon}{2})\right) \\ &\le \varepsilon. \end{aligned}$$

And if $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) \geq \frac{\varepsilon}{2}$, applying (C_2), we have that

$$r(d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k})) \leq r(\tfrac{\varepsilon}{2}).$$

Moreover, we note that

$$d(x_{\gamma(\varepsilon)}, Tx_{\gamma(\varepsilon)}) = d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+1}) < \frac{\varepsilon}{2}\big(1 - r(\tfrac{\varepsilon}{2})\big) < \frac{\varepsilon}{2} \leq d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}).$$

Then, by (2.4),

$$\begin{aligned} d(x_{\gamma(\varepsilon)+1}, x_{\gamma(\varepsilon)+k+1}) &= d(Tx_{\gamma(\varepsilon)}, Tx_{\gamma(\varepsilon)+k}) \\ &\leq r(d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}))\, d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) \\ &\leq r(\tfrac{\varepsilon}{2})\, \varepsilon. \end{aligned}$$

Thus,

$$\begin{aligned} d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k+1}) &\leq d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+1}) + d(x_{\gamma(\varepsilon)+1}, x_{\gamma(\varepsilon)+k+1}) \\ &< \frac{\varepsilon}{2}\big(1 - r(\tfrac{\varepsilon}{2})\big) + r(\tfrac{\varepsilon}{2}) \cdot \varepsilon \\ &= \frac{\varepsilon}{2}\,\big(1 + r(\tfrac{\varepsilon}{2})\big) \\ &< \varepsilon. \end{aligned}$$

Therefore, $\{T^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence and γ is a Cauchy rate. □

In 1971, Ćirić [7] introduced the notions of orbitally continuous maps and orbitally complete spaces, which appear very often in Fixed Point Theory.

Definition 2.8. Let (X, d) be a metric space. A map $T : X \to X$ is said to be orbitally continuous if for all $x_0, z \in X$

$$\lim_{n\to\infty} T^n x_0 = z \text{ implies that } \lim_{n\to\infty} T(T^n x_0) = Tz.$$

The space X is said to be T-orbitally complete if any sequence of the form $T^n x$ that is a Cauchy sequence has a limit point in X.

Using the above definitions, we can obtain a fixed point result for maps satisfying condition (2.4).

Proposition 2.9. Let (X, d) be a metric space and $T : X \to X$ satisfying (2.4), with $r : [0, \infty) \to [0, 1]$ verifying (C_1) and (C_2). If T is orbitally continuous and X is T-orbitally complete, then T has a unique fixed point p in X and, moreover, for each $x \in X$ the Picard sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to p.

Proof. Let $x \in X$. From Proposition 2.7 we have $\{T^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is T-orbitally complete, there exists p in X such that $\{T^n x\}_{n\in\mathbb{N}}$ converges to p. By orbital continuity of T it follows

$$Tp = \lim_{n\to\infty} T(T^n x) = \lim_{n\to\infty} T^{n+1} x = p,$$

that is, p is a fixed point of T. The uniqueness of fixed point is obtained by Lemma 2.4. □

Remark 2.10. Notice that if a map $T : X \to X$ satisfies Rakotch's condition (1.1) then T verifies every hypothesis of the above result. Therefore, Rakotch's fixed point theorem is a consequence of Proposition 2.9.

Definition 2.11. Let D be a nonempty subset of a metric space (X,d). We say that a map $T : D \to X$ is a weakly Suzuki map if there exists a function $r : [0,\infty) \to [0,1]$ satisfying (C_1), (C_2) and (C_3), such that

$$(S_w^R) \qquad \theta(r(d(x,Tx)))\, d(x,Tx) \le d(x,y) \text{ implies } d(Tx,Ty) \le r(d(x,y))\, d(x,y)$$

for all $x, y \in D$, where $\theta(t)$ is defined by (1.8) for $t \in [0,1)$ and $\theta(1) = 1/2$.

Lemma 2.12. Let (X,d) be a metric space. If $T : X \to X$ is a weakly Suzuki map, then for every $x, y \in X$ with $r(d(x,Tx)) \ge \frac{1}{\sqrt{2}}$, either

$$\theta\big(r(d(x,Tx))\big)\, d(x,Tx) \le d(x,y)$$

or

$$\theta\big(r(d(x,Tx))\big)\, d(Tx,T^2x) \le d(Tx,y)$$

holds.

Proof. Assume that there exist $x, y \in X$, with $d(x,Tx) \ge \frac{1}{\sqrt{2}}$ such that

$$\theta(r(d(x,Tx)))\, d(x,Tx) > d(x,y)$$

and

$$\theta(r(d(x,Tx)))\, d(Tx,T^2x) > d(Tx,y).$$

Then, we obtain that

$$\begin{aligned} d(x,Tx) &\le d(x,y) + d(y,Tx) \\ &< \theta(r(d(x,Tx)))\, d(x,Tx) + \theta(r(d(x,Tx)))\, d(Tx,T^2x) \\ &= \theta(r(d(x,Tx)))\, \big(d(x,Tx) + d(Tx,T^2x)\big) \\ &\le \theta(r(d(x,Tx)))\, \big(d(x,Tx) + r(d(x,Tx))\, d(x,Tx)\big) \\ &= d(x,Tx), \end{aligned}$$

which is a contradiction. □

For the main result we need to recall that given a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space (X,d), the function $\gamma : (0,\infty) \to \mathbb{N}$ is a rate of convergence for $\{x_n\}_{n\in\mathbb{N}}$ to $z \in X$ if for all $\varepsilon > 0$ we have that $n \ge \gamma(\varepsilon)$ gives $d(z,x_n) \le \varepsilon$.

Theorem 2.13. Let (X,d) be a complete metric space and $T : X \to X$ a weakly Suzuki map. Then, T has a fixed point p in X and the Picard iterate $\{T^n x\}_{n\in\mathbb{N}}$ converges to p, for every $x \in X$. Moreover, the function given by (2.3) is a rate of convergence for the Picard iterates.

Proof. Our proof starts with the remark that (S_w^R) implies (2.4) and, therefore, (2.2). Then, by Proposition 2.7, we have that for each x in X the Picard iterate $\{T^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $x_0 \in X$ be fixed. Since X is complete, there exists p in X such that $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to p and moreover the Cauchy rate given by (2.3) turns into a rate of convergence.

Let us see that

$$(2.5) \qquad d(Tx,p) \le r\big(d(x,p)\big)\, d(x,p) \quad \text{for every } x \in X, x \ne p.$$

Fix $x \in X$ with $x \neq p$. Since

$$\lim_{n\to\infty} \theta\big(r(d(T^n x_0, T^{n+1}x_0))\big)\, d(T^n x_0, T^{n+1}x_0) = 0 < d(p,x) = \lim_{n\to\infty} d(T^n x_0, x),$$

there exists a natural number n_0 such that

$$\theta\big(r(d(T^n x_0, T^{n+1}x_0))\big)\, d(T^n x_0, T^{n+1}x_0) \leq d(T^n x_0, x) \quad \text{for all } n \geq n_0.$$

Using (S_w^R),

$$d(T^{n+1}x_0, Tx) \leq r\big(d(T^n x_0, x)\big)\, d(T^n x_0, x) \quad \text{for all } n \geq n_0.$$

Taking superior limit as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} d(T^{n+1}x_0, Tx) \leq \limsup_{n\to\infty} r\big(d(T^n x_0, x)\big)\, d(T^n x_0, x),$$

that is,

$$d(p, Tx) \leq r\big(d(p,x)\big)\, d(p,x),$$

because by (C_3)

$$\limsup_{n\to\infty} r\big(d(T^n x_0, x)\big) \leq r\Big(\limsup_{n\to\infty} d(T^n x_0, x)\Big) = r\big(d(p,x)\big).$$

Notice that if $T^n p = p$ for some $n \in \mathbb{N}$ then $Tp = p$, since $\{T^n p\}_{n\in\mathbb{N}}$ is a Cauchy sequence (by Proposition 2.7). Therefore, we must study the case $T^n p \neq p$ for all $n \in \mathbb{N}$. Bearing in mind (2.5), we have that

$$d(T^{n+1}p, p) \leq r\big(d(T^n p, p)\big)\, d(T^n p, p) \quad \text{for all } n \in \mathbb{N}. \tag{2.6}$$

We consider two cases.

Case 1. $r(t) < \frac{1}{\sqrt{2}}$ for all $t > 0$. Observe that

$$\theta\big(r(d(T^2p, T^3p))\big)\, d(T^2p, T^3p) \leq d(T^2p, p).$$

Indeed, if $d(T^2p, p) < \theta\big(r(d(T^2p, T^3p))\big)\, d(T^2p, T^3p)$ then using the fact that $\{d(T^n p, T^{n+1}p)\}_{n\in\mathbb{N}}$ is a decreasing sequence (see Proposition 2.7), (C_3) and $\theta(t)\, t^2 + t \leq 1$ for all $0 \leq t < \frac{1}{\sqrt{2}}$, we get

$$\begin{aligned}
d(p, Tp) &\leq d(p, T^2p) + d(T^2p, Tp)\\
&\leq d(p, T^2p) + r(d(Tp, p))\, d(Tp, p)\\
&< \theta\big(r(d(T^2p, T^3p))\big)\, d(T^2p, T^3p) + r(d(Tp,p))\, d(Tp,p)\\
&\leq \Big[\theta\big(r(d(T^2p, T^3p))\big)\, r(d(T^2p, Tp))\, r(d(Tp,p))\\
&\qquad + r(d(Tp,p))\Big]\, d(Tp,p)\\
&\leq \Big[\theta\big(r(d(T^2p, T^3p))\big)\, r(d(T^2p, T^3p))\, r(d(T^2p, T^3p))\\
&\qquad + r(d(T^2p, T^3p))\Big]\, d(Tp,p)\\
&\leq d(Tp, p),
\end{aligned}$$

which is a contradiction. Hence, using (S_w^R), we deduce that

$$d(T^3p, Tp) \leq r(d(T^2p, p))\, d(T^2p, p).$$

Thus,

$$\begin{aligned}
d(p,Tp) &\le d(p,T^3p) + d(T^3p,Tp) \\
&\le d(p,T^3p) + r(d(T^2p,p))\, d(T^2p,p) \\
&\le r(d(T^2p,p))\, d(T^2p,p) + r(d(T^2p,p))\, d(T^2p,p) \\
&= 2r(d(T^2p,p))\, d(T^2p,p) \\
&\le 2r(d(T^2p,p))\, r(d(Tp,p))\, d(Tp,p) \\
&< d(Tp,p),
\end{aligned}$$

which is a contradiction.

Case 2. There exists $t_0 > 0$ such that $\frac{1}{\sqrt{2}} \le r(t_0)$. In this case, there exists $n_0 \in \mathbb{N}$ such that $r(d(T^{2n}x_0, T^{2n+1}x_0)) \ge \frac{1}{\sqrt{2}}$ for all $n \ge n_0$. Hence, by Lemma 2.12, either

$$\theta\big(r(d(T^{2n}x_0, T^{2n+1}x_0))\big)\, d(T^{2n}x_0, T^{2n+1}x_0) \le d(T^{2n}x_0, p)$$

or

$$\theta\big(r(d(T^{2n+1}x_0, T^{2n+2}x_0))\big)\, d(T^{2n+1}x_0, T^{2n+2}x_0) \le d(T^{2n+1}x_0, p)$$

holds for every $n \ge n_0$. Then, using (S_w^R), we have that either

$$d(T^{2n+1}x_0, Tp) \le r(d(T^{2n}x_0, p))\, d(T^{2n}x_0, p)$$

or

$$d(T^{2n+2}x_0, Tp) \le r(T^{2n+1}x_0, p)\, d(T^{2n+1}x_0, p)$$

holds for every $n \ge n_0$. Since $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to p, the above inequalities imply there exists a subsequence of $\{T^n x_0\}_{n\in\mathbb{N}}$ which converges to Tp. This implies $Tp = p$.

Therefore, p is a fixed point of T. The uniqueness of fixed point is implied by Lemma 2.4. □

Remark 2.14. Suzuki fixed point result [20, Thm. 2] is a consequence of the above theorem.

The following example show that the class of weakly Suzuki maps is larger than the class of maps satisfying the hypothesis of Suzuki fixed point theorem.

Example 2.15 (See [2], Example 22)**.** Fix $\omega > 0$. Let $D = [0,\infty)$ be the subset of the metric space $X = \mathbb{R}$ with the usual metric $d(x,y) = |x-y|$. The map $T : D \to X$ defined by $Tx = \omega x/(\omega + x)$ is a weakly Suzuki map but not a Suzuki map. Indeed, if T is a Suzuki map, then there exists $r \in [0,1)$ such that for $x, y \in D$ with $\theta(r)\, d(x,Tx) \le d(x,y)$ implies $d(Tx,Ty) \le r\, d(x,y)$. Taking $x = 0$, we get $d(T0,Ty) \le r\, d(0,y)$ for all $y \in [0,\infty)$, which is impossible since

$$\lim_{y\to 0^+} \frac{d(T0,Ty)}{d(0,y)} = \lim_{y\to 0^+} \frac{\omega}{\omega + y} = 1.$$

On the other hand, T is a weakly Suzuki map because for all $x, y \in [0,\infty)$ we have

$$d(Tx,Ty) = \frac{\omega^2}{(\omega+x)(\omega+y)}\ |x-y| \le \frac{\omega}{\omega + |x-y|}\ |x-y| = r(d(x,y))\, d(x,y),$$

where the function $r(t) = \omega/(\omega + t)$ satisfies the properties (C_1), (C_2) and (C_3).

Using Example 1 of Suzuki's paper [20] together Example 21 in [2], we can show that weakly Suzuki maps and weakly Zamfirescu maps are independent. Recall that a map $T : D \subseteq X \to X$ is a weakly Zamfirescu map if there exists a functional $\alpha : D \times D \to [0, 1]$ satisfying (1.6), such that

$$(Z_w) \qquad d(Tx, Ty) \le \alpha(x, y)\, M_T(x, y)$$

for all $x, y \in D$, where

$$M_T(x, y) := \max\left\{ d(x, y), \frac{1}{2}\big[d(x, Tx) + d(y, Ty)\big], \frac{1}{2}\big[d(x, Ty) + d(y, Tx)\big] \right\}.$$

Example 2.16. Consider the metric space (X, d), where $X = [0, 1]$ and d is the usual metric. The map $T : [0, 1] \to [0, 1]$ given as

$$Tx = \begin{cases} \frac{2}{3}x & \text{if } 0 \le x < 1, \\ 0 & \text{if } x = 1, \end{cases}$$

is a weakly Zamfirescu map, (for more details, see [2, Example 21]). However, T is not a weakly Suzuki map, since T does not satisfy (2.4) for $x = 0$ and $0 < y \le \frac{2}{3}$.

Example 2.17. Define a complete metric space X by $X = \{(0, 0), (4, 0), (0, 4), (4, 5), (5, 4)\}$ and its metric d by $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$. The map T on X defined by

$$T(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if } x_1 \le x_2, \\ (0, x_2) & \text{if } x_1 > x_2. \end{cases}$$

is a weakly Suzuki map, since T satisfies Suzuki's condition with $\frac{4}{5} \le r < 1$ (see [20, Example 1]). However, T is not a weakly Zamfirescu map because

$$d\big(T(4, 5), T(5, 4)\big) = 8 > 5 = M_T\big((4, 5), (5, 4)\big).$$

Remark 2.18. The above example also shows that the class of Suzuki maps is strictly larger than that of contraction maps since every contraction map is a weakly Zamfirescu map.

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