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PROXIMAL QUASI-NORMAL STRUCTURE AND A BEST PROXIMITY POINT THEOREM

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Abstract. In this paper we prove the existence of best proximity points for weak cyclic Kannan contraction mappings in Banach spaces. Moreover, we introduce a notion of quasi-proximal normal structure and study the existence of best proximity points for relatively Kannan nonexpansive mappings which are relatively nonexpansive.

1. Introduction and Preliminaries

Let X be a Banach space, and let A, B be nonempty subsets of X. A mapping $T : A \cup B \to A \cup B$ is said to be *cyclic* provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. For this class of mappings the fixed point equation $x = Tx$ may not have solution. Thus we can consider the minimization problem

$$\min_{x \in A\cup B} d(x, Tx). \tag{1.1}$$

Each solution of (1.1) is called a *best proximity point* of the cyclic mapping T.

Definition 1.1. Let T be a cyclic mapping. A point $x \in A \cup B$ is said to be a best proximity point for T provided that $d(x, Tx) = dist(A, B)$, where $dist(A, B) := inf\{d(x, y) : x \in A, y \in B\}$.

Existence and convergence of best proximity points for various classes of mappings is an interesting subject which was studied by several authors; see for example [1, 2, 3, 4, 13, 14, 15].

Let (A, B) be a nonempty pair in a Banach space X. We say that the pair (A, B) of subsets in a Banach space X satisfies a property if both A and B satisfy that property. For example, (A, B) is convex if and only if both A and B are convex. Moreover, throughout this paper we shall use the following notations and definitions:

$$(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C, \quad \text{and} \quad B \subseteq D,$$
$$\delta_x(A) = \sup\{d(x, y) : y \in A\} \text{ for all } x \in X,$$
$$\delta(A, B) = \sup\{d(x, y) : x \in A,\ y \in B\}.$$

2. Weak cyclic Kannan contraction mappings

Definition 2.1. ([10]) Let (A, B) be a nonempty pair in a Banach space X and $T : A \cup B \to A \cup B$ be a cyclic mapping. We say that T is a weak cyclic Kannan

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contraction mapping if

$$\|Tx - Ty\| \leq \alpha\{\|x - Tx\| + \|y - Ty\|\} + (1 - 2\alpha)dist(A, B), \tag{2.1}$$

for some $\alpha \in (0, \frac{1}{2})$ and for all $(x, y) \in A \times B$.

The following theorem was established in [10].

Theorem 2.2. *Let (A, B) be a nonempty closed convex pair in a uniformly convex Banach space X. Suppose that $T : A \cup B \to A \cup B$ is a weak cyclic Kannan contraction mapping. Then T has a unique best proximity point $z \in A$. Moreover, the sequence $\{T^{2n}x\}$ converges to z for any $x \in A$.*

We emphasize that the geometric property of X, that is, being a uniformly convex Banach space, plays an important role in the proof of Theorem 2.2.
In this section we prove the existence of best proximity points for weak cyclic Kannan contractions in Banach spaces.

Theorem 2.3. *Let (A, B) be a nonempty weakly compact convex pair in a Banach space X. Assume that $T : A \cup B \to A \cup B$ is a weak cyclic Kannan contraction mapping. If $B.P.P(T)$ denotes the set of all best proximity points of T, then both $B.P.P(T) \cap A$ and $B.P.P(T) \cap B$ are nonempty subsets of X.*

Proof. Let Σ denote the collection of all nonempty weakly compact convex pairs (E, F) which are subsets of (A, B) and such that T is cyclic on $E \cup F$. Then Σ is nonempty, since $(A, B) \in \Sigma$. Note that Σ is partially ordered by the revers inclusion, that is $(A, B) \leq (C, D) \Leftrightarrow (C, D) \subseteq (A, B)$. It is easy to check that every increasing chain in Σ is bounded above. Hence by Zorn's lemma we can get a minimal element, say $(K_1, K_2) \in \Sigma$. We have

$$(\overline{co}(T(K_2)), \overline{co}(T(K_1))) \subseteq (K_1, K_2).$$

Moreover

$$T(\overline{co}(T(K_2))) \subseteq T(K_1) \subseteq \overline{co}(T(K_1)),$$

and also

$$T(\overline{co}(T(K_1))) \subseteq \overline{co}(T(K_2)).$$

Now by the minimality of (K_1, K_2), we have $\overline{co}(T(K_2)) = K_1$, $\overline{co}(T(K_1)) = K_2$. Let $a \in K_1$, then $K_2 \subseteq B(a; \delta_a(K_2))$. Now if $y \in K_2$, then

$$\|Ta - Ty\| \leq \alpha\{\|a - Ta\| + \|Ty - y\|\} + (1 - 2\alpha)dist(A, B)$$

$$\leq 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B).$$

Therefore for all $y \in K_2$, we have

$$T(K_2) \subseteq B(Ta; 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B)).$$

Hence,

$$K_1 = \overline{co}(T(K_2)) \subseteq B(Ta; 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B)).$$

This implies that

$$\|x - Ta\| \leq 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B) \quad \text{for all} \quad x \in K_1,$$

so that

$$\delta_{Ta}(K_1) \leq 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B). \tag{2.2}$$

Similarly, if $b \in K_2$, we see that

$$\delta_{Tb}(K_2) \leq 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B). \tag{2.3}$$

Now, we put

$$E_1 := \{x \in K_1 : \delta_x(K_2) \leq 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B)\},$$
$$E_2 := \{y \in K_2 : \delta_y(K_1) \leq 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B)\}.$$

Then (E_1, E_2) is nonempty, and

$$E_1 = \bigcap_{y \in K_2} B(y; 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B)) \cap K_1,$$

$$E_2 = \bigcap_{x \in K_1} B(x; 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B)) \cap K_2.$$

Moreover, by (2.2) and (2.3) it is easy to check that T is cyclic on $E_1 \cup E_2$. Now by the minimality of (K_1, K_2) we must have $E_1 = K_1$ and $E_2 = K_2$. Thus we obtain

$$\delta_x(K_2) \leq 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B), \quad \text{for all} \quad x \in K_1,$$

which implies that

$$\delta(K_1, K_2) = \sup_{x \in K_1} \delta_x(K_2) \leq 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B).$$

Therefore

$$\delta(K_1, K_2) = dist(A, B).$$

Now we have

$$\|x - Tx\| \leq \delta_x(K_2) = dist(A, B) \quad \text{for all} \quad x \in K_1.$$

This shows that each point in K_1 is a best proximity point of T, so that $K_1 \subseteq B.P.P(T) \cap A$. Similarly, we see that $K_2 \subseteq B.P.P(T) \cap B$. ☐

Corollary 2.4. *Let (A, B) be a nonempty bounded closed convex pair in a reflexive Banach space X. Assume that $T : A \cup B \to A \cup B$ is a weak cyclic Kannan contraction mapping. Then both $B.P.P(T) \cap A$ and $B.P.P(T) \cap B$ are nonempty subsets of X.*

3. Proximal quasi-normal structure

The notion of normal structure for Banach spaces was introduced by Brodskii and Milman in [5], where it was shown that every weakly compact convex set which has this property contains a point which is fixed under surjective isometry. More information on normal structure, can be found in [7, 8, 9, 11]. In [16] Wong proved that X has the weak fixed point property for Kannan maps if and only if it has weak normal structure. Also it is shown in [17] that spaces which are separable or strictly convex have normal structure. After that a concept weaker than that of normal structure was introduced by Soardi in [12]. It was announced in [17] that if X is a Banach space with quasi-weak normal structure if and only if every Kannan mapping T of a non-empty weakly compact convex subset K of X into itself has a fixed point.

The notion of proximal normal structure was introduced in [6] as follows:

Definition 3.1. A pair (A, B) of subsets of a linear space X is said to be a proximal pair if for each $(x, y) \in A \times B$ there exists $(\acute{x}, \acute{y}) \in A \times B$ such that

$$\|x - \acute{y}\| = \|\acute{x} - y\| = dist(A, B).$$

Definition 3.2. A convex pair (K_1, K_2) in a Banach space X is said to have proximal normal structure if for any bounded, closed and convex proximal pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which $dist(H_1, H_2) = dist(K_1, K_2)$ and $\delta(H_1, H_2) > dist(H_1, H_2)$, there exits $(x_1, x_2) \in H_1 \times H_2$ such that

$$\delta_{x_1}(H_2) < \delta(H_1, H_2), \quad \delta_{x_2}(H_1) < \delta(H_1, H_2).$$

By using this geometric property Eldred et. al established the following theorem. Before we mention the main theorem of [6], we recall the following definition.

Definition 3.3. Let (A, B) be a nonempty pair of a normed linear space X and $T : A \cup B \to A \cup B$ be a mapping. We say that T is relatively nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $(x, y) \in A \times B$.

Theorem 3.4. *([6]) Let (A, B) be a nonempty, weakly compact convex pair in a Banach space X, and suppose (A, B) has proximal normal structure. Assume that $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping. Then T has a best proximity point in both A and B, that is, there exists $(x^*, y^*) \in A \times B$ such that $\|x^* - Tx^*\| = \|Ty^* - y^*\| = dist(A, B)$.*

We now introduce a notion of proximal quasi-normal structure.

Definition 3.5. A convex pair (K_1, K_2) in a Banach space X is said to have proximal quasi-normal structure if for any bounded, closed and convex proximal pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which $dist(H_1, H_2) = dist(K_1, K_2)$ and $\delta(H_1, H_2) > dist(H_1, H_2)$, there exits $(p_1, p_2) \in H_1 \times H_2$ such that

$$d(p_1, y) < \delta(H_1, H_2), \quad d(x, p_2) < \delta(H_1, H_2),$$

for all $(x, y) \in H_1 \times H_2$.

It follows from Definition 3.6 that for a convex subset K of a Banach space X, the pair (K, K) has proximal quasi-normal structure if and only if K has quasi-normal structure. Moreover,

$$\text{proximal normal structure} \Rightarrow \text{proximal qausi-normal structure.}$$

Example 3.6. Let (A, B) be a bounded closed convex pair in a uniformly convex Banach space X. Then (A, B) has proximal quasi-normal structure.

Proof. By Proposition 2.1 of [6], (A, B) has proximal normal structure and thus has proximal quasi-normal structure. □

Before we shift to the main result of this section, we prove the following lemma.

Lemma 3.7. *Let (K_1, K_2) be a nonempty pair of a normed linear space X. Then $\delta(K_1, K_2) = \delta(\overline{co}(K_1), \overline{co}(K_2))$.*

Proof. It is sufficient to show that $\delta(\overline{co}(K_1), \overline{co}(K_2)) \leq \delta(K_1, K_2)$. Let $x \in K_2$. For all $y \in K_1$ we have $y \in B(x; \delta_x(K_1))$. Then $K_1 \subseteq \bigcap_{x \in K_2} B(x; \delta_x(K_1))$ and hence $\overline{co}(K_1) \subseteq \bigcap_{x \in K_2} B(x; \delta_x(K_1))$. Now if $z \in \overline{co}(K_1)$, it is easy to see that $\overline{co}(K_2) \subseteq B(z; \delta(K_1, K_2))$. Thus $\overline{co}(K_2) \subseteq \bigcap_{z \in \overline{co}(K_1)} B(z; \delta(K_1, K_2))$, and the result follows. □

Theorem 3.8. *Let (A, B) be a nonempty, weakly compact convex pair in a Banach space X and suppose (A, B) has proximal quasi-normal structure. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that*

$$\|Tx - Ty\| \leq \min\left\{\frac{1}{2}(\|x - Tx\| + \|y - Ty\|), \|x - y\|\right\}, \tag{3.1}$$

for all $(x, y) \in A \times B$. Then there exists $(x^*, y^*) \in A \times B$ such that

$$\|x^* - Tx^*\| = \|Ty^* - y^*\| = dist(A, B).$$

Proof. It is not difficult to see that (A_0, B_0) is a nonempty, weakly compact convex pair and $dist(A, B) = dist(A_0, B_0)$. Moreover, T is cyclic on $A_0 \cup B_0$ (for more details see [6]). Let Σ denote the collection of all nonempty, weakly compact convex pairs (E, F) which are subsets of (A, B) and $dist(E, F) = dist(A, B)$ and T is cyclic on $E \cup F$. Then Σ is nonempty, since $(A_0, B_0) \in \Sigma$. By using Zorn's lemma we can see that Σ has a minimal element, say (K_1, K_2) with respect to revers inclusion relation and $dist(K_1, K_2) = dist(A, B)$ (see Theorem 2.1 of [6]). Now let r be a real positive number such that $r \geq dist(A, B)$ and let $(p, q) \in K_1 \times K_2$ is such that

$$\|p - q\| = dist(A, B), \|p - Tp\| \leq r$$

and $\|Tq - q\| \leq r$. Put

$$K_1^r = \{x \in K_1 : \|x - Tx\| \leq r\},\ K_2^r = \{x \in K_2 : \|Tx - x\| \leq r\},$$

and set $C_1^r := \overline{co}(T(K_1^r))$, $C_2^r := \overline{co}(T(K_2^r))$. We claim that T is cyclic on $C_1^r \cup C_2^r$. Firstly, we prove $C_1^r \subseteq K_2^r$. Let $x \in C_1^r$. If $\|Tx - x\| = dist(A, B)$, then $x \in K_2^r$. Suppose $\|Tx - x\| > dist(A, B)$. Set $s := \sup\{\|Tw - Tx\| : w \in K_1^r\}$. Then $B(Tx; s) \supseteq T(K_1^r)$. This implies that $C_1^r \subseteq B(Tx; s)$. Since $x \in C_1^r$, we have $\|Tx - x\| \leq s$. By the definition of s, for each $\varepsilon > 0$ there exists $w \in K_1^r$ such that $s - \varepsilon \leq \|Tw - Tx\|$. Hence

$$\|Tx - x\| - \varepsilon \leq s - \varepsilon \leq \|Tw - Tx\| \leq \frac{1}{2}[\|w - Tw\| + \|Tx - x\|] \leq \frac{1}{2}\|Tx - x\| + \frac{1}{2}r.$$

This implies that $\|Tx - x\| \leq r + 2\varepsilon$. Thus $x \in K_2^r$ and hence $C_1^r \subseteq K_2^r$. Therefore

$$T(C_1^r) \subseteq T(K_2^r) \subseteq \overline{co}(T(K_2^r)) = C_2^r.$$

Similar argument shows that $T(C_2^r) \subseteq C_1^r$. Thus T is cyclic on $C_1^r \cup C_2^r$. We now prove that $\delta(C_1^r, C_2^r) \leq r$. By Lemma 3.7 we have

$$\begin{aligned}
\delta(C_1^r, C_2^r) &= \delta(\overline{co}(T(K_1^r), T(K_2^r))) \\
&= \delta(T(K_1^r), T(K_2^r)) = \sup\{\|Tx - Ty\| : x \in K_1^r, y \in K_2^r\} \\
&\leq \sup\{\frac{1}{2}[\|x - Tx\| + \|Ty - y\|] : x \in K_1^r, y \in K_2^r\} \\
&\leq \frac{1}{2}[r + r] = r.
\end{aligned}$$

On the other hand since $p \in K_1^r, q \in K_2^r$ and $\|p-q\| = dist(A,B)$, we conclude that

$$dist(A,B) \leq dist(C_2^r, C_1^r) \leq \|Tq - Tp\| \leq \|p-q\| = dist(A,B),$$

that is, $dist(C_2^r, C_1^r) = dist(A,B)$.
Put

$$r_0 = \inf\{\|x - Tx\| : x \in K_1 \cup K_2\}.$$

Then $r_0 \geq dist(A,B)$. Let $\{r_n\}$ be a nonnegative sequence such that $r_n \downarrow r_0$. Thus $\{C_1^{r_n}\}, \{C_2^{r_n}\}$ are descending sequences of nonempty, weakly compact convex subsets of K_1, K_2 respectively. By the weakly compactness of K_1, K_2 we must have

$$C_1^{r_0} = \bigcap_{n=1}^{\infty} C_1^{r_n} \neq \emptyset,\ C_2^{r_0} = \bigcap_{n=1}^{\infty} C_2^{r_n} \neq \emptyset.$$

Also by the preceding argument $T : C_1^{r_0} \cup C_2^{r_0} \to C_1^{r_0} \cup C_2^{r_0}$ is a cyclic mapping. Moreover, since $dist(C_2^{r_n}, C_1^{r_n}) = dist(A,B)$ for all $n \in \mathbb{N}$, we conclude that $dist(C_2^{r_0}, C_1^{r_0}) = dist(A,B)$. The minimality of (K_1, K_2) implies that $C_2^{r_0} = K_1$ and $C_1^{r_0} = K_2$. Hence $\|x - Tx\| \leq r_0$ for all $x \in K_1 \cup K_2$. Now let $r_0 > dist(A,B)$. Since (A,B) has proximal quasi-normal structure, there exists $(p_1, q_1) \in K_1 \times K_2$ such that

$$\|p_1 - y\| < \delta(K_1, K_2) \leq r_0,\ \|x - q_1\| < \delta(K_1, K_2) \leq r_0,$$

for all $(x,y) \in K_1 \times K_2$. Therefore

$$\|p_1 - Tp_1\| < \delta(K_1, K_2) \leq r_0 \ \&\ \|Tq_1 - q_1\| < \delta(K_1, K_2) \leq r_0,$$

which is a contradiction. This implies that $r_0 = dist(A,B)$ and hence

$$\|x - Tx\| = \|Ty - y\| = dist(A,B)$$

for all $(x,y) \in K_1 \times K_2$. □

Remark 3.9. Let (A,B) be a nonempty pair of a normed linear space X and $T : A \cup B \to A \cup B$ be a mapping. We say that T is relatively Kannan nonexpansive if

$$\|Tx - Ty\| \leq \frac{1}{2}[\|x - Tx\| + \|y - Ty\|]$$

for all $(x,y) \in A \times B$. We note that the class of mappings introduced in (3.1) consists of relatively Kannan nonexpansive mappings which are relatively nonexpansive as well.

Example 3.10. Let $X = \mathbb{R}$ with the usual metric and $A = [0,1], B = [\frac{3}{2}, 2]$. Define $T : A \cup B \to A \cup B$ with

$$T(x) = \begin{cases} 2 & \text{if } \ x = 0, \\ \frac{3}{2} & \text{if } \ x \neq 0, x \in A, \\ 1 & \text{if } \ x \in B. \end{cases}$$

It is easy to check that T is cyclic on $A \cup B$ and satisfies the condition (3.1). Thus by Theorem 3.8, T has a best proximity point in $A \cup B$.

References

[1] A. Abkar and M. Gabeleh, *Results on the existence and convergence of best proximity points*, Fixed Point Theory and Appl. 2010, Art ID 386037, 10 pages.

[2] A. Abkar and M. Gabeleh, *Generalized cyclic contractions in partially ordered metric spaces*, Optim. Lett., In press, DOI 10.1007/s11590-011-0379-y.

[3] A. Abkar and M. Gabeleh, *Best proximity points for asymptotic cyclic contraction mappings*, Nonlinear Anal. **74** (2011), 7261–7268.

[4] M. A. Al-Thagafi and N. Shahzad, *Convergence and existence results for best proximity points*, Nonlinear Anal. **70** (2009), 3665–3671.

[5] M. S. Brodskii and D. P. Milman, *On the center of a convex set*, Dokl. Akad. Nauk. **USSR** **59** (1948), 837–840.

[6] A. A. Eldred, W. A. Kirk and P. Veeramani, *Proximal normal structure and relatively nonexpansive mappings*, Studia Math. **171** (2005), 283–293.

[7] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004–1006.

[8] W. A. Kirk, *Nonexpansive mappings and normal structure in Banach spaces*, Proceedings of the research workshop on Banach space theory, B. L. Lin (ed.), University of Iowa, 1981.

[9] M. A. Khamsi, *On metric spaces with uniform normal structure*, Proc. Amer. Math. Soc. **106** (1989), 723–726.

[10] M. A. Petric, *Best proximity point theorems for weak cyclic kannan contractions*, Filomat, 25:1 DOI 10.2298/FIL1101145P (2011).

[11] S. Swaminathan, *Normal structures in Banach spaces and its generalizations, fixed points and nonexpansive mappings*, Contemp. Math. Amer. Math. Soc. **18** (1983), 201–215.

[12] P. M. Soardi, *Struttura quasi normale e teoremi di punto unito*, Rend. 1st Mat. Univ. Trieste, **4** (1972), 105–114.

[13] S. S. Basha,*Best proximity points: global optimal approximate solutions*, J. Glob. Optim. DOI 10.1007/s10898-009-9521-0 (2011)

[14] V. S. Raj, *A best proximity point theorem for weakly contractive non-self-mappings*, Nonlinear Anal. **74** (2011), 4804–4808.

[15] T. Suzuki, M. Kikkawa and C. Vetro, *The existence of best proximity points in metric spaces with the property UC*, Nonlinear Anal. **71** (2009), 2918–2926.

[16] C. Wong, *On Kannan maps*, Proc. Amer. Math. Soc., **47** (1975), 105–111.

[17] C. Wong, *Close-to-normal structure and its applications*, J. Funct. Anal. **16** (1974), 353–358.

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