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MINIMAX THEOREM FOR FRACTIONAL EXPECTATION ON A TWO-PERSON ZERO-SUM DYNAMIC GAME

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Abstract. Let X and Y be the stochastic strategy spaces of players I and II, respectively, in a two-person zero-sum dynamic game. In this paper, we prove that the fractional function of the total conditional expectations of players I and II which satisfies the Ky Fan type minimax theorem under some reasonable conditions.

1. Introduction

This paper is a continuous work of our previous paper Lai and Yu [14] for non-fractional total expectation function, satisfying the minimax theorem could extend to fractional total expectation function of players I and II on a two-person zero-sum dynamic game.

To this aim, we recall the two-person zero-sum dynamic game with a parameter θ by the same assertion as the seven elements:

$$(DGP_\theta) \qquad (S_n, A_n, B_n, t_{n+1}, u_n, v_n, \theta).$$

For convenience, we recall some results with some suitable explanations in Lai and Yu [14] as follows. In (DGP_θ), S_n, A_n, and B_n are state space, and action spaces of players I and II, at time $n \in \mathbb{N}$, respectively, they are assumed metrizable separable complete Borel spaces. $\{t_{n+1}\}$ is the sequence of transition probability from state S_n moving to S_{n+1} for $n \in \mathbb{N}$. The histories of the game system from $n = 1, 2, \ldots$ are denoted by

$$H_1 = S_1, \quad H_2 = H_1 \times A_1 \times B_1 \times S_2, \ldots$$
$$H_n = S_1A_1B_1S_2A_2B_2 \ldots S_{n-1}A_{n-1}B_{n-1}S_n = H_{n-1}A_{n-1}B_{n-1}S_n, \ n = 2, 3, \ldots.$$

We assume that the reward functions u_n and v_n are both bounded Borel measurable functions defined on the strategy spaces at time $n \in \mathbb{N}$ which are represented by

$$u_n : \ H_nA_nB_n \to \mathbb{R} \text{ and } v_n : \ H_nA_nB_n \to \mathbb{R}_+.$$

As the time n goes to infinity, they have the limit functions:

$$\lim_{n\to\infty} u_n = u : \ H_\infty \to \mathbb{R} \text{ and } \lim_{n\to\infty} v_n = v : \ H_\infty \to \mathbb{R}_+.$$

Let $h \in H_\infty$ be a stochastic variable in H_n with time n going to infinity.

If the strategy spaces X and Y of players I and II, respectively, then for any pair of strategies $(x, y) \in X \times Y$, there is a unique universal measurable transition

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probability $P_{xy}(\cdot\,|\,\cdot)$ from S_1 to the game process: $A_1B_1S_2A_2B_2S_3\ldots = H_\infty$, such that for the two bounded Borel measurable (reward) functions $\{u_n, v_n\}$ defined on $H_nA_nB_n$ $(n \in \mathbb{N}) \subset H_\infty$ are the density functions for the expectations of players I and II, respectively.

As the previous paper [14], for each $s_1 \in S_1$ and the history (stochastic) variable $h \in H_\infty$, we obtain the total conditional expectations at time $n \in \mathbb{N}$ as the following form:

$$\begin{aligned} E(u_n, x, y)(s_1) &= \int_{H_\infty} u_n(h)P_{xy}(dh\,|\,s_1) \\ &= E_{x_1}E_{y_1}E_{t_2}\ldots E_{x_{n-1}}E_{y_{n-1}}E_{t_n}E_{x_n}E_{y_n}u_n(s_1) \\ &\equiv E_{xy}u_n(s_1), \end{aligned}$$

and

$$\begin{aligned} E(v_n, x, y)(s_1) &= \int_{H_\infty} v_n(h)P_{xy}(dh\,|\,s_1) \\ &= E_{x_1}E_{y_1}E_{t_2}\ldots E_{x_{n-1}}E_{y_{n-1}}E_{t_n}E_{x_n}E_{y_n}v_n(s_1) \\ &\equiv E_{xy}v_n(s_1), \end{aligned}$$

where $E_{x_n},\ E_{y_n}$, and $E_{t_{n+1}},\ n = 1, 2, \ldots$ are the conditional expectation operators and transition probability with respect to $x_n \in X_n,\ y_n \in Y_n$, and $\{x_n\} \subset X,\ \{y_n\} \subset Y$.

Consequently, the total stochastic reward conditional expectational functions of players I and II, respectively, are given by

$$U(x,y)(s_1) = \lim_{n\to\infty} E_{xy}\ u_n(s_1)\ \in \mathbb{R}, \tag{1.1}$$

and

$$V(x,y)(s_1) = \lim_{n\to\infty} E_{xy}\ v_n(s_1)\ \in \mathbb{R}_+. \tag{1.2}$$

In the game system (DGP_θ), the loss (gain) function of player I at $n \in \mathbb{N}$ obeys the law of function

$$F_\theta^n = u_n - \theta v_n, \tag{1.3}$$

and the player II at time $n \in \mathbb{N}$ has his gain (loss) function

$$-F_\theta^n. \tag{1.4}$$

Thus, at any time $n \in \mathbb{N}$, the two players are playing zero sum rewards. Since the reward functions $\{u_n, v_n\}$ are bounded, by dominated convergence theorem employing in (1.1) and (1.2), as well as Fubini theorem in (1.1) and (1.2).

Consequently, one can get the total conditional expectation of the loss function of player I is given by

$$\begin{aligned} F_\theta(x,y)(s_1) &\equiv \lim_{n\to\infty} E_{xy}\ F_\theta^n(x,y)(s_1) \\ &= \lim_{n\to\infty} E_{xy}[u_n(x,y) - \theta v_n(x,y)](s_1) \\ &= U(x,y)(s_1) - \theta(s_1)V(x,y)(s_1) \qquad (1.5) \end{aligned}$$

and shows that

$$\overline{F}_\theta(s_1) = \inf_{x\in X}\sup_{y\in Y} F_\theta(x,y)(s_1) = \sup_{y\in Y}\inf_{x\in X} F_\theta(x,y)(s_1) = \underline{F}_\theta(s_1) \tag{1.6}$$

holds in the game system (DGP_θ).

Since $V(x,y)(s_1) \in \mathbb{R}_+$, the quotient of functions:

$$W(x,y)(s_1) = \frac{U(x,y)(s_1)}{V(x,y)(s_1)},\ (x,y) \in X \times Y, \tag{1.7}$$

is well defined and makes a plausible problem, that is, how to derive the minimax theorem as

$$\inf_{x\in X}\sup_{y\in Y} W(x,y)(s_1) = \sup_{y\in Y}\inf_{x\in X} W(x,y)(s_1) \tag{1.8}$$

holds.

Actually, the right hand side of (1.8) is similar to a duality problem with respect to the minimax programming problem of the left hand side of (1.8) with some constraints. There are many research articles in minimax programming problems (cf. Lai et al. [4–9]).

Usually a minimax programming with an objective function $f:\ X \times Y \to \mathbb{R}$ is as the form:

$$(P_0) \qquad \inf_{x\in X}\sup_{y\in Y} f(x,y)$$

satisfying some constraints,

and most of the variables $(x,y) \in X \times Y$ in (P_0) are reals. There are also complex variables in engineering (cf. Lai et al. [4,5,8]), or set variables in measurable spaces X and Y (cf. Lai et al. [6,7]).

Minimax theorem has many applications. For instance, in variational inequality, mathematic economic for minimum cost and maximum profit, or minimum loss and maximum gain in game theory, etc..

In game theory, there are many types of games. Several types of game systems can be referred to Lai and Tanaka in [10–13,17]. See also the related works in [2,3,14]. These games involved n-person noncooperative dynamic systems in various spaces (cf. Lai [10–13]) and two-person zero-sum dynamic games (see [2,3,14–17]).

In this paper we focus on an objective function $W(x,y)$ which is a ratio of fractional conditional expectation functions of players I and II in the two-person zero-sum dynamic game and is defined in the expression (1.8), and hence we may regard our game system as a two-person zero-sum dynamic fractional game. However we are only focus on the expression (1.8).

We infer that it will be consistent with Ky Fan's [1] minimax theorem for the strategy spaces of the zero-sum dynamic fractional game, that is,

$$\inf_{x\in X}\sup_{y\in Y} W(x,y)(s_1) = \sup_{y\in Y}\inf_{x\in X} W(x,y)(s_1) \text{ holds.}$$

In other words, we will investigate an existence theorem for the saddle value function so that the two players can obtain an equilibrium point in the dynamic fractional game system.

2. Preliminary for a two-person zero-sum dynamic fractional game

In this paper we recall a two-person zero-sum dynamic fractional game (DFG) with the strategy spaces X and Y for players I and II, respectively. It is performed by the following six elements:

$$(DFG) \qquad (S_n, A_n, B_n, t_{n+1}, u_n, v_n) \text{ at time } n \in \mathbb{N}.$$

According to the same assumptions of the previous work [14] we can start to discuss the theory of the dynamic game system.

Recall that X and Y are metrizable separable spaces employed as the strategy spaces of players I and II, respectively. Hence we can denote by X_n (resp. Y_n) the set of all universal measurable transition mappings from history H_n to action A_n of player I (resp. B_n of player II).

By the separability of the metric spaces, for any $x \in X$ and $y \in Y$, we can consider the sequences $x = \{x_n\},\ y = \{y_n\}$ with $x_n \in X_n,\ y_n \in Y_n$ at each time $n \in \mathbb{N}$.

Then for each pair $(x, y) \in X \times Y$ and transition probabilities $\{t_{n+1}\}_{n=1}^{\infty}$, there exists a unique universal measurable transition probability $P_{xy}(\,\cdot\,|\,\cdot\,)$ from S_1 to the game process: $A_1B_1S_2A_2B_2S_3\cdots = H_\infty$, such that for the two bounded Borel measurable functions $\{u_n, v_n\}$ defined on $H_nA_nB_n$
$(n \in \mathbb{N})$ are the density functions for the expectations of players I and II, respectively.

Let $E_{x_n},\ E_{y_n}$, and $E_{t_{n+1}}$ denote the conditional expectation operators with respect to $x_n \in X_n,\ y_n \in Y_n$, and transition probability t_{n+1}, respectively.

Then for each $s_1 \in S_1$ and the history (stochastic) variable $h \in H_\infty$, the total conditional expectation at time $n \in \mathbb{N}$ follows from [14], we restate it as the following form S_1 to

$$\begin{aligned} E(u_n, x, y)(s_1) &= \int_{H_\infty} u_n(h) P_{xy}(dh \,|\, s_1) \\ &\equiv E_{xy} u_n(s_1), \end{aligned}$$

and

$$\begin{aligned} E(v_n, x, y)(s_1) &= \int_{H_\infty} v_n(h) P_{xy}(dh \,|\, s_1) \\ &\equiv E_{xy} v_n(s_1). \end{aligned}$$

Consequently, the total stochastic reward conditional expectational functions of players I and II, respectively, are performed as the following limits:

$$U(x, y)(s_1) = \lim_{n\to\infty} E_{xy}\ u_n(s_1)\ \in \mathbb{R},$$

and

$$V(x, y)(s_1) = \lim_{n\to\infty} E_{xy}\ v_n(s_1)\ \in \mathbb{R}_+.$$

Since $V(x, y)(s_1) > 0$, the total expectation of the considered dynamic fractional game (DFG) is well defined, and is given by the form:

$$W(x, y)(s_1) = \frac{U(x, y)(s_1)}{V(x, y)(s_1)},\ (x, y) \in X \times Y.$$

The minimax problem is natural to inqire that whether the saddle value function exists on the dynamic fractional game while the players have chosen their strategies from both the numerator and denominator. The outcome response will eventually have a saddle point under some reasonable conditions.

3. Existence of saddle value function

For each $y \in Y$, there exists a real number λ depending on x $(x \in X)$ such that the reward functions $u_n \in \mathbb{R}$ and $v_n \in \mathbb{R}_+$ in (DGP_θ), having the following relations

$$\sup_{y\in Y} \frac{u_n(x,y)(s_1)}{v_n(x,y)(s_1)} = \lambda \ (x,y) \in X \times Y,$$

and so

$$\frac{u_n(x,y)(s_1)}{v_n(x,y)(s_1)} \le \lambda \text{ for all } y \in Y,$$

or equivalently

$$u_n(x,y)(s_1) - \lambda v_n(x,y)(s_1) \le 0 \text{ for all } (x,y) \in X \times Y.$$

Consequently, we can reduce the fractional problem to an equivalent nonfractional parametric problem:

$$(P_\lambda) \qquad \inf_{x\in X} \sup_{y\in Y} \, [u_n(x,y)(s_1) - \lambda v_n(x,y)(s_1)] \quad (\le 0)$$

where λ is a parameter. It is the same as minimax fractional programming deduce to nonfractional case. That is, the minimax fractional programming can be deduced to a nonfractional programming problem having equivalence solution (without additional assumptions) (cf. Lai et al. [9]).

The expression (1.5) is deduced to the minimax type theorem for the game system (DGP_θ) as the expression (1.6)

$$\inf_{x\in X} \sup_{y\in Y} F_\theta(x,y)(s_1) = \sup_{y\in Y} \inf_{x\in X} F_\theta(x,y)(s_1)$$

which is equivalent to the dynamic fractional game system (DFG) by using the total expectation ratio (1.7) of the two players as the minimax type theorem:

$$\inf_{x\in X} \sup_{y\in Y} W(x,y)(s_1) = \sup_{y\in Y} \inf_{x\in X} W(x,y)(s_1)$$

under certain conditions. We want to inquire that whether (1.6) and (1.8) are both solvable.

4. Upper and lower value functions of the game system (DFG)

To this purpose, we define the upper and lower value functions of the game system (DFG) by

$$\overline{\omega}(s_1) = \inf_{x\in X} \sup_{y\in Y} W(x,y)(s_1) \text{ and}$$

$$\underline{\omega}(s_1) = \sup_{y\in Y} \inf_{x\in X} W(x,y)(s_1), \text{ respectively.}$$

Of course, $\underline{\omega}(s_1) \leq \overline{\omega}(s_1)$ for all $s_1 \in S_1$, and call the interval $[\underline{\omega}(s_1), \overline{\omega}(s_1)]$ as the duality gap of the game system (DFG).

If there is a saddle value function $\omega^*(s_1)$ such that

$$\underline{\omega}(s_1) = \overline{\omega}(s_1) = \omega^*(s_1),$$

then the dual gap is equal zero, and call $\omega^*(s_1)$ a saddle value function of the game system (DFG).

Indeed, if there exists $y^* \in Y$ such that

$$\overline{\omega}(s_1) = \inf_{x\in X} \sup_{y\in Y} W(x,y)(s_1) = \inf_{x\in X} W(x,y^*)(s_1),$$

namely y^* as a *maximizer* of $W(x,y)(s_1)$ over $y \in Y$ for each $x \in X$ in the game system (DFG).

Similarly, if there exists $x^* \in X$ such that

$$\underline{\omega}(s_1) = \sup_{y\in Y} \inf_{x\in X} W(x,y)(s_1) = \sup_{y\in Y} W(x^*,y)(s_1),$$

we call x^* as a *minimizer* of $W(x,y)(s_1)$ over $x \in X$ for each $y \in Y$ in the game system (DFG).

It is remarkable that in minimax fractional programming, the denominator is always positive, and the numerator may assume to be nonnegative.

In the dynamic fractional game system, the objective ratio function $W(x,y)(s_1)$ could have a saddle value (point) function and needs some extra assumptions since the law of motion $\overline{F}_\theta(\cdot)$ can not guarantee the numerator's reward function to be nonnegative.

Hence, we need to analyze some relations between the parametric upper value function as well as the lower value function for (DGP_θ) and (DFG):

$\overline{F}_\theta(s_1)$, $\underline{F}_\theta(s_1)$ in (DGP$_\theta$) and $\overline{\omega}(s_1)$, $\underline{\omega}(s_1)$ in (DFG) (cf. Lai [3]).

At first, we state some properties for $\overline{F}_\theta(s_1)$ as follows:

Proposition 4.1.

(i) *For two parameter functions $\theta_1(s_1)$ and $\theta_2(s_1)$, if $\theta_1(s_1) > \theta_2(s_1) \geq 0$, then $\overline{F}_{\theta_1}(s_1) \leq \overline{F}_{\theta_2}(s_1)$,*

(ii) *If $\overline{F}_\theta(s_1) > 0$, then $\theta(s_1) \leq \overline{\omega}(s_1)$,*

(iii) *If $\overline{F}_\theta(s_1) < 0$, then $\theta(s_1) \geq \overline{\omega}(s_1)$,*

(iv) *If $\theta(s_1) < \overline{\omega}(s_1)$, then $\overline{F}_\theta(s_1) \geq 0$,*

(v) *If $\theta(s_1) > \overline{\omega}(s_1)$, then $\overline{F}_\theta(s_1) \leq 0$.*

Proof. (i) If $\theta_1(s_1) > \theta_2(s_1) \geq 0$, then for any $(x,y) \in X \times Y$, since $V(x,y)(s_1) > 0$, we have $\theta_1(s_1)V(x,y)(s_1) > \theta_2(s_1)V(x,y)(s_1)$, it yields

$$U(x,y)(s_1) - \theta_1(s_1)V(x,y)(s_1) < U(x,y)(s_1) - \theta_2(s_1)V(x,y)(s_1).$$

That is,

$$F_{\theta_1}(x,y)(s_1) < F_{\theta_2}(x,y)(s_1).$$

Hence

$$\overline{F}_{\theta_1}(s_1) = \inf_{x\in X}\sup_{y\in Y} F_{\theta_1}(x,y)(s_1) \le \inf_{x\in X}\sup_{y\in Y} F_{\theta_2}(x,y)(s_1) = \overline{F}_{\theta_2}(s_1),$$

or

$$\overline{F}_{\theta_1}(s_1) \le \overline{F}_{\theta_2}(s_1), \quad \textit{provided } \theta_1(s_1) > \theta_2(s_1) \ge 0.$$

(ii) If $\overline{F}_{\theta}(s_1) > 0$, then for any $x \in X$, by definition of infimum

$$\sup_{y\in Y} F_{\theta}(x,y)(s_1) > 0,$$

it follows that, there exists $y_x \in Y$ depending on x, such that

$$F_{\theta}(x,y_x)(s_1) = U(x,y_x)(s_1) - \theta(s_1)V(x,y_x)(s_1) > 0.$$

Since $V(x,y_x)(s_1) > 0$, we obtain

$$W(x,y_x)(s_1) = \frac{U(x,y_x)(s_1)}{V(x,y_x)(s_1)} > \theta(s_1).$$

Hence, for each $x \in X$,

$$\sup_{y\in Y} W(x,y)(s_1) \ge W(x,y_x)(s_1) = \frac{U(x,y_x)(s_1)}{V(x,y_x)(s_1)} > \theta(s_1),$$

we have

$$\overline{\omega}(s_1) = \inf_{x\in X}\sup_{y\in Y} W(x,y)(s_1) \ge \theta(s_1),$$

that is, $\theta(s_1) \le \overline{\omega}(s_1)$.

(iii) If $\overline{F}_{\theta}(s_1) < 0$, then there exists $\overline{x} \in X$ such that

$$\sup_{y\in Y} F_{\theta}(\overline{x},y)(s_1) < 0,$$

that is, for any $y \in Y$,

$$F_{\theta}(\overline{x},y)(s_1) = U(\overline{x},y)(s_1) - \theta(s_1)V(\overline{x},y)(s_1) < 0.$$

Since $V(\overline{x},y)(s_1) > 0$,
this implies that

$$W(\overline{x},y)(s_1) = \frac{U(\overline{x},y)(s_1)}{V(\overline{x},y)(s_1)} < \theta(s_1),$$

and so

$$\sup_{y\in Y} W(\overline{x},y)(s_1) \le \theta(s_1).$$

Therefore

$$\overline{\omega}(s_1) = \inf_{x\in X}\sup_{y\in Y} W(x,y)(s_1) \le \theta(s_1),$$

that is, $\theta(s_1) \ge \overline{\omega}(s_1)$.

(iv) If $\overline{\omega}(s_1) > \theta(s_1)$, then for any $x \in X$,

$$\sup_{y\in Y} W(x,y)(s_1) > \theta(s_1).$$

It follows that, there exists $y_x \in Y$ depending on x, such that

$$W(x,y_x)(s_1) = \frac{U(x,y_x)(s_1)}{V(x,y_x)(s_1)} > \theta(s_1).$$

This implies that

$$F_\theta(x,y_x)(s_1) = U(x,y_x)(s_1) - \theta(s_1)V(x,y_x)(s_1) > 0,$$

and satisfying

$$\sup_{y\in Y} F_\theta(x,y)(s_1) \geq F_\theta(x,y_x)(s_1) > 0.$$

Hence

$$\overline{F}_\theta(s_1) = \inf_{x\in X}\sup_{y\in Y} F_\theta(x,y)(s_1) \geq 0.$$

(v) If $\theta(s_1) > \overline{\omega}(s_1)$, then by definition of $\overline{\omega}(s_1)$, there exists $\overline{x} \in X$ such that

$$\theta(s_1) > \sup_{y\in Y} W(\overline{x},y)(s_1),$$

or

$$\theta(s_1) > W(\overline{x},y)(s_1), \text{ for all } y \in Y.$$

This implies that, for all $y \in Y$,

$$F_\theta(\overline{x},y)(s_1) = U(\overline{x},y)(s_1) - \theta(s_1)V(\overline{x},y)(s_1) < 0,$$

and satisfying

$$\sup_{y\in Y} F_\theta(\overline{x},y)(s_1) \leq 0.$$

Therefore

$$\overline{F}_\theta(s_1) = \inf_{x\in X}\sup_{y\in Y} F_\theta(x,y)(s_1) \leq \sup_{y\in Y} F_\theta(\overline{x},y)(s_1) \leq 0.$$

□

By the similar arguments as Proposition 4.1 for $\overline{F}_\theta(x,y)(s_1)$, we have the following proposition for $\underline{F}_\theta(x,y)(s_1)$.

Proposition 4.2.

(i) *For two parameter functions $\theta_1(s_1)$ and $\theta_2(s_1)$, if $\theta_1(s_1) > \theta_2(s_1) \geq 0$, then $\underline{F}_{\theta_1}(s_1) \leq \underline{F}_{\theta_2}(s_1)$,*
(ii) *If $\underline{F}_\theta(s_1) > 0$, then $\theta(s_1) \leq \underline{\omega}(s_1)$,*
(iii) *If $\underline{F}_\theta(s_1) < 0$, then $\theta(s_1) \geq \underline{\omega}(s_1)$,*
(iv) *If $\theta(s_1) < \underline{\omega}(s_1)$, then $\underline{F}_\theta(s_1) \geq 0$,*
(v) *If $\theta(s_1) > \underline{\omega}(s_1)$, then $\underline{F}_\theta(s_1) \leq 0$.*

Proof. Using $\underline{F}_\theta(s_1)$ and $\underline{\omega}(s_1)$ instead of $\overline{F}_\theta(s_1)$ and $\overline{\omega}(s_1)$, respectively, we can prove this proposition by the similar arguments as in the previous proof. □

5. The saddle value function of the game system (DFG)

We recall that in previous section 4, if there exists $y^* \in Y$ such that

$$\overline{\omega}(s_1) = \inf_{x\in X} \sup_{y\in Y} W(x,y)(s_1) = \inf_{x\in X} W(x,y^*)(s_1),$$

namely y^* as a *maximizer* of $W(x,y)(s_1)$ over $y \in Y$ for each $x \in X$ in the game system (DFG).

Similarly, if there exists $x^* \in X$ such that

$$\underline{\omega}(s_1) = \sup_{y\in Y} \inf_{x\in X} W(x,y)(s_1) = \sup_{y\in Y} W(x^*,y)(s_1),$$

we call x^* as a *minimizer* of $W(x,y)(s_1)$ over $x \in X$ for each $y \in Y$ in the game system (DFG).

The existence theorems for saddle value function between (DFG) and (DGP_θ) have the following theorems:

Theorem 5.1. (i) *Let $y^* \in Y$ be a maximizer of $W(x,y)(s_1)$ over $y \in Y$ for each $x \in X$ in the game system (DFG) , then we have*

$$\overline{\omega}(s_1) = \underline{\omega}(s_1) \equiv \omega^*(s_1).$$

That is,

$$\inf_{x\in X} \sup_{y\in Y} W(x,y)(s_1) = \sup_{y\in Y} \inf_{x\in X} W(x,y)(s_1) \text{ holds for } (DFG).$$

(The saddle value function of (DFG) exists.)

(ii) *Suppose that $y^* \in Y$ is a maximizer of $W(x,y)(s_1)$ in the game system (DFG). If $\overline{F}_\theta(s_1) \le 0$, then $y^* \in Y$ is also a maximizer of $F_\theta(x,y)(s_1)$ in the game system (DGP_θ).*

Proof. (i) By definitions, we see that $\overline{\omega}(s_1) \ge \underline{\omega}(s_1)$.
As $y^* \in Y$ is a maximizer of $W(x,y)(s_1)$, we have

$$\begin{aligned}\overline{\omega}(s_1) &= \inf_{x\in X} \sup_{y\in Y} W(x,y)(s_1) = \inf_{x\in X} W(x,y^*)(s_1)\\ &\le \sup_{y\in Y} \inf_{x\in X} W(x,y)(s_1) = \underline{\omega}(s_1)\end{aligned}$$

This implies that

$$\overline{\omega}(s_1) = \underline{\omega}(s_1) \equiv \omega^*(s_1).$$

Hence, the game system (DFG) has a saddle value function, that is,

$$\overline{\omega}(s_1) = \underline{\omega}(s_1).$$

(ii) If $y^* \in Y$ is a maximizer of $W(x,y)(s_1)$ in the game system (DFG), then for all $x \in X$,

$$\omega^*(s_1) = \overline{\omega}(s_1) = \inf_{x\in X}\sup_{y\in Y} W(x,y)(s_1) = \inf_{x\in X} W(x,y^*)(s_1) \le W(x,y^*)(s_1).$$

That is, for all $x \in X$,

$$\omega^*(s_1) \le W(x,y^*)(s_1) = \frac{U(x,y^*)(s_1)}{V(x,y^*)(s_1)}.$$

This implies that, for all $x \in X$,

$$0 \le U(x,y^*)(s_1) - \omega^*(s_1)V(x,y^*)(s_1) = F_\theta(x,y^*)(s_1) = \sup_{y\in Y} F_\theta(x,y)(s_1).$$

Thus

$$0 \le \inf_{x\in X} F_\theta(x,y^*)(s_1) \le \inf_{x\in X}\sup_{y\in Y} F_\theta(x,y)(s_1) = \overline{F}_\theta(s_1) \le 0 \quad \text{(by assumption)}.$$

This shows that,

$$\inf_{x\in X} F_\theta(x,y^*)(s_1) = \inf_{x\in X}\sup_{y\in Y} F_\theta(x,y)(s_1).$$

Therefore, $y^* \in Y$ is also a maximizer of $F_\theta(x,y)(s_1)$ in the game system (DGP_θ).

□

Definition 5.2. We say that (x^*,y^*) is a saddle point of $W(x,y)(s_1)$, if

$$W(x^*,y)(s_1) \le W(x^*,y^*)(s_1) \le W(x,y^*)(s_1) \quad \text{for all } x \in X \text{ and } y \in Y.$$

That is,

$$\sup_{y\in Y} W(x^*,y)(s_1) \le W(x^*,y^*)(s_1) \le \inf_{x\in X} W(x,y^*)(s_1) \quad \text{for all } x \in X \text{ and } y \in Y.$$

Corollary 5.3. *If $(x^*,y^*) \in X\times Y$ is a saddle point of the game system (DFG), then*

(i) $F_\theta(x^*,y^*)(s_1) = 0$, and

(ii) (x^*,y^*) is also a saddle point of the game system (DGP_θ).

Theorem 5.4.

(i) *Let $x^* \in X$ be a minimizer of $W(x,y)(s_1)$ over $x \in X$ for each $y \in Y$ in the game system (DFG). Then we have*

$$\overline{\omega}(s_1) \equiv \omega^*(s_1) = \underline{\omega}(s_1).$$

That is,

$$\inf_{x\in X}\sup_{y\in Y} W(x,y)(s_1) = \sup_{y\in Y}\inf_{x\in X} W(x,y)(s_1) \textit{ holds for } (DFG).$$

(The saddle value function of (DFG) exists.)

(ii) *Suppose that $x^* \in X$ is a minimizer of $W(x,y)(s_1)$ in the game system (DFG). If $\underline{F}_\theta(s_1) \ge 0$, then $x^* \in X$ is also a minimizer of $F_\theta(x,y)(s_1)$ in the game system (DGP_θ).*

Proof. (i) By definitions, we see that $\underline{\omega}(s_1) \leq \overline{\omega}(s_1)$.
On the other hand, since $x^* \in X$ is a minimizer of $W(x,y)(s_1)$, we have

$$\underline{\omega}(s_1) = \sup_{y\in Y} \inf_{x\in X} W(x,y)(s_1) = \sup_{y\in Y} W(x^*,y)(s_1) \geq \inf_{x\in X} \sup_{y\in Y} W(x,y)(s_1) = \overline{\omega}(s_1).$$

This implies that

$$\overline{\omega}(s_1) \ \equiv \omega^*(s_1) = \underline{\omega}(s_1).$$

Thus, the game (DFG) has a saddle value function, that is,

$$\overline{\omega}(s_1) = \underline{\omega}(s_1).$$

(ii) If $x^* \in X$ is a minimizer of $W(x,y)(s_1)$ in the game system (DFG), it follows that, for all $y \in Y$,

$$\omega^*(s_1) = \underline{\omega}(s_1) = \sup_{y\in Y} \inf_{x\in X} W(x,y)(s_1) = \sup_{y\in Y} W(x^*,y)(s_1) \geq W(x^*,y)(s_1).$$

That is, for all $y \in Y$,

$$\omega^*(s_1) \geq W(x^*,y)(s_1) = \frac{U(x^*,y)(s_1)}{V(x^*,y)(s_1)}.$$

This implies that, for all $x \in X$,

$$U(x^*,y)(s_1) - \omega^*(s_1)V(x^*,y)(s_1) = F_\theta(x^*,y)(s_1) = \inf_{x\in X} F_\theta(x,y)(s_1) \leq 0.$$

Thus

$$0 \geq \sup_{y\in Y} F_\theta(x^*,y)(s_1) \geq \inf_{x\in X} \sup_{y\in Y} F_\theta(x,y)(s_1) \geq \underline{F}_\theta(s_1) \geq 0 \quad \text{(by assumption)}.$$

This shows that,

$$\sup_{y\in Y} F_\theta(x^*,y)(s_1) = \sup_{y\in Y} \inf_{x\in X} F_\theta(x,y)(s_1) = \underline{F}_\theta(s_1).$$

Therefore, $x^* \in X$ is also a minimizer of $F_\theta(x,y)(s_1)$ in the game system (DGP_θ).

□

Theorem 5.5. (i) *Suppose that* $\overline{\omega}(s_1) = \underline{\omega}(s_1) \ \equiv \omega^*(s_1)$.
If $y^* \in Y$ *is a maximizer of* $F_\theta(x,y)(s_1)$ *in the game system* (DGP_θ) *with*

$$\inf_{x\in X} F_\theta(x,y^*)(s_1) = \overline{F}_\theta(s_1) = F^*_\theta(s_1) \geq 0,$$

then $y^* \in Y$ *is also a maximizer of* $W(x,y)(s_1)$ *in the game system* (DFG).

(ii) *Suppose that* $\overline{\omega}(s_1) = \underline{\omega}(s_1) \ \equiv \omega^*(s_1)$.
If $x^* \in X$ *is a minimizer of* $F_\theta(x,y)(s_1)$ *in the game system* (DGP_θ) *with*

$$\sup_{y\in Y} F_\theta(x^*,y)(s_1) = \underline{F}_\theta(s_1) = F^*_\theta(s_1) \leq 0,$$

then $x^* \in X$ *is also a minimizer of* $W(x,y)(s_1)$ *in the game system* (DFG).

Proof. (i) By the assumptions, $\overline{F}_\theta(s_1) = F^*_\theta(s_1) \geq 0$ and $y^* \in Y$ is a maximizer of $F_\theta(x,y)(s_1)$, it follows that

$$0 \leq \inf_{x\in X} \sup_{y\in Y} F_\theta(x,y)(s_1) = \inf_{x\in X} F_\theta(x,y^*)(s_1) \leq F_\theta(x,y^*)(s_1), \text{ for all } x \in X.$$

This implies that,

$$\overline{\omega}(s_1) \leq \frac{U(x,y^*)(s_1)}{V(x,y^*)(s_1)} = W(x,y^*)(s_1) \leq \sup_{y\in Y} W(x,y)(s_1), \text{ for all } x \in X.$$

Hence

$$\overline{\omega}(s_1) \leq \inf_{x\in X} W(x,y^*)(s_1) \leq \inf_{x\in X} \sup_{y\in Y} W(x,y)(s_1) = \overline{\omega}(s_1),$$

it follows that

$$\inf_{x\in X} W(x,y^*)(s_1) = \inf_{x\in X} \sup_{y\in Y} W(x,y)(s_1) = \overline{\omega}(s_1),$$

that is, $y^* \in Y$ is also a maximizer of $W(x,y)(s_1)$ in the game system (DFG).

(ii) The proof follows the same lines as the proof given for (i).

□

From the above results, we conclude that the following result holds.

Theorem 5.6. *Suppose that* $\overline{\omega}(s_1) = \underline{\omega}(s_1) \equiv \omega^*(s_1)$*, and that* $(x^*,y^*) \in X \times Y$ *is a saddle point of the game system* (DGP_θ) *with* $F_\theta(x^*,y^*)(s_1) = 0$*. Then* (x^*,y^*) *is also a saddle point of the game system* (DFG)*.*

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