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INFINITE-HORIZON PONTRYAGIN PRINCIPLES WITHOUT INVERTIBILITY

JOËL BLOT

Abstract. We give new statements of Pontryagin principles in the setting of infinite-horizon discrete-time optimal control problems with various criteria, when the system is governed by a difference equation or by a difference inequation. In a previous work [3] an invertibility condition of the vector field with respect to the state variable was used. In the present work, after a section devoted to the comparison between the problems governed by difference equations and problems governed by difference inequations, we give new conditions to obtain Pontryagin principles without the using of the above-mentionned invertibility condition: a positivity condition and a partial surjectivity condition.

1. Introduction

Our aim is to establish Pontryagin principles for discrete-time infinite-horizon optimal control problems with various criteria, as to maximize $\sum_{t=0}^{\infty} f_t^0(x_t, u_t)$, or as to find $((\hat{x}_t)_t, (\hat{u}_t)_t)$ such that $\liminf_{h\to\infty} \sum_{t=0}^{h}(f_t^0(\hat{x}_t, \hat{u}_t) - f_t^0(x_t, u_t)) \geq 0$ for all admissible processes $((x_t)_t, (u_t)_t)$, or as to find $((\hat{x}_t)_t, (\hat{u}_t)_t)$ such that $\limsup_{h\to\infty} \sum_{t=0}^{h}(f_t^0(\hat{x}_t, \hat{u}_t) - f_t^0(x_t, u_t)) \geq 0$ for all admissible processes $((x_t)_t, (u_t)_t)$, when the system is governed by a difference equation $x_{t+1} = f_t(x_t, u_t)$ or by a difference inequation $x_{t+1} \leq f_t(x_t, u_t)$.

Such problems are fundamental in the macroeconomic optimal growth theory, see for instance [17], [14] and the references in [4]. In [18] we can find motivations provided by other scientific fields. Note that, following [17], Chapter I, Section 2, the difference inequations are more suitable than difference equations in the macroeconomic models.

Recall that the pre-Hamiltonian associated to these problems is

$$H_t(x_t, u_t, \lambda^0, p_{t+1}) := \lambda^0 f_t^0(x_t, u_t) + \langle p_{t+1}, f_t(x_t, u_t)\rangle$$

where x_t is the state variable, u_t is the control variable, λ^0 is a multiplier, p_{t+1} is the adjoint variable, and $\langle ., . \rangle$ is the duality bracket on $R^{n*} \times R^n$.

An optimal process $((\hat{x}_t)_t, (\hat{u}_t)_t)$ being given, we say that a Pontryagin principle is *strong* when, beside other conditions notably the adjoint equation, for all $t \in N$, we have

$$H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1}) \geq H_t(\hat{x}_t, u_t, \lambda^0, p_{t+1})$$

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for all the possible values u_t of the control variable. We say that a Pontryagin principle is *weak* when instead of this maximality condition we have only a tangency condition (like a first-order necessary condition of optimality in Mathematical Programming) as, for instance, a generalized differential with respect to u_t, $\partial_{u_t} H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1})$, contains normal vectors to the set of the possible values of u_t.

For continuous-time problems, we have not need special conditions to obtain a strong Pontryagin principle; see for instance [1] in the finite-horizon setting, and [6] in the infinite-horizon setting. But as Boltyanski emphazises it in [5] by providing counter-examples, for the (finite-horizon) discrete-time problems, strong Pontryagin principle cannot hold without additional assumptions which are, in a more or less sophisticated form, convexity conditions. This is a great difference between the continuous-time optimal control problems and the discrete-time optimal control problems. In this paper, for the strong Pontryagin principles, we use a condition due to Michel [12], so-called the condition of *mixed* problem.

Here we describe the sketch of the methods used to prove our theorems. An optimal process being given, we consider a family of finite-horizon problems for which the restrictions of the optimal process are optimal solutions. Then by using known results on the finite-horizon problems, for each finite horizon $h \in N$, we obtain a multiplier $\lambda^{0,h} \in R^1_+$ and an adjoint variable $(p^h_1, ..., p^h_h) \in (R^{n*})^h$ or $(R^{n*}_+)^h$ which satisfy the adjoint equation and the maximum principle in a weak form or in a strong form. After, for the infinite-horizon problem, we obtain a multiplier λ^0 and an adjoint variable p_t, when $t \geq 1$, as limit points of the sequences $(\lambda^{0,h})_h$ and $(p^h_t)_{h>t}$ respectively (when $h \to \infty$). The main question is to prove that λ^0 and the p_t are not simultaneously equal to zero. A technical difficulty related to this question is the character backward of the adjoint equation; this is also a difference with the continuous-time setting. In [3] an invertibility condition permits to transform the adjoint equation in a forward equation. In this paper we give new conditions to avoid to use the invertibility condition of [3].

Now we briefly describe the contents of the paper. In Section 2 we present our settings and the problems considered. In Section 3 we establish results on the comparison between problems governed by difference equations and problems governed by difference inequations. In Section 4 we establish new weak Pontryagin principles. In Section 5 we establish new strong Pontryagin principles under a positivity condition. In Section 6 we establish new strong Pontryagin principles under a partial surjectivity condition.

2. The settings

R^n and R^m are endowed with their natural order: when $d = n$ or m, $x = (x^1, ..., x^d) \leq y = (y^1, ..., y^d)$ means that $x^i \leq y^i$ for all $i = 1, ...d$. The writing $x < y$ means that $x \leq y$ and $x \neq y$ that is equivalent to say that $x^i \leq y^i$ for all $i = 1, ..., d$ and there exists $j \in \{1, ..., d\}$ such that $x^j < y^j$. N denotes the set of the nonnegative integer numbers, and we set $N_* := N \setminus \{0\}$. Following [8] p. 224, a function $\phi : R^d \to R^e$, where $e \in N_*$, is called *increasing* when $x \leq y$ implies $\phi(x) \leq \phi(y)$.

To abridge the writing inside the proofs, we will denote a sequence by an underlined letter, for instance $\underline{x} := (x_t)_t = (x_t)_{t\in N}$ and $\underline{u} := (u_t)_t = (u_t)_{t\in N}$.

For each integer $t \in N$, X_t denotes a nonempty open subset of R^n, U_t denotes a nonempty subset of R^m, $f_t^0 : X_t \times U_t \to R^1$ and $f_t : X_t \times U_t \to X_{t+1}$ denote functions, and η is an element of X_0. Let $(x_t)_t \in \prod_{t\in N} X_t$ and $(u_t)_t \in \prod_{t\in N} U_t$; we say that $((x_t)_t, (u_t)_t)$ belongs to dom$J$ when the following series sum $J((x_t)_t, (u_t)_t) := \sum_{t\geq 0} f_t^0(x_t, u_t)$ exists in R^1 (i.e. this series is convergent toward a finite real number).

We denote by Adm(η) the set of the processes $((x_t)_t, (u_t)_t) \in \prod_{t\in N} X_t \times \prod_{t\in N} U_t$ such that $x_{t+1} = f_t(x_t, u_t)$ for all $t \in N$ and such that $x_0 = \eta$. We denote by Iad(η) the set of the processes $((x_t)_t, (u_t)_t) \in \prod_{t\in N} X_t \times \prod_{t\in N} U_t$ such that $x_{t+1} \leq f_t(x_t, u_t)$ for all $t \in N$ and such that $x_0 = \eta$.

With these notations we can define the following problems.

- **($\mathcal{P}_{I,\eta}$):** Maximize $\sum_{t=0}^{\infty} f_t^0(x_t, u_t) =: J((x_t)_t, (u_t)_t)$ when $((x_t)_t, (u_t)_t) \in \text{dom}J \cap \text{Adm}(\eta)$.
- **($\mathcal{P}_{II,\eta}$):** Find $((\overline{x}_t)_t, (\overline{u}_t)_t) \in \text{Adm}(\eta)$ such that $\liminf_{h\to\infty} \sum_{t=0}^{h} (f_t^0(\overline{x}_t, \overline{u}_t) - f_t^0(x_t, u_t)) \geq 0$ for all $((x_t)_t, (u_t)_t) \in \text{Adm}(\eta)$.
- **($\mathcal{P}_{III,\eta}$):** Find $((\overline{x}_t)_t, (\overline{u}_t)_t) \in \text{Adm}(\eta)$ such that $\limsup_{h\to\infty} \sum_{t=0}^{h} (f_t^0(\overline{x}_t, \overline{u}_t) - f_t^0(x_t, u_t)) \geq 0$ for all $((x_t)_t, (u_t)_t) \in \text{Adm}(\eta)$.
- **($\mathcal{Q}_{I,\eta}$):** Maximize $\sum_{t=0}^{\infty} f_t^0(x_t, u_t) =: J((x_t)_t, (u_t)_t)$ when $((x_t)_t, (u_t)_t) \in \text{dom}J \cap \text{Iad}(\eta)$.
- **($\mathcal{Q}_{II,\eta}$):** Find $((\overline{x}_t)_t, (\overline{u}_t)_t) \in \text{Iad}(\eta)$ such that $\liminf_{h\to\infty} \sum_{t=0}^{h} (f_t^0(\overline{x}_t, \overline{u}_t) - f_t^0(x_t, u_t)) \geq 0$ for all $((x_t)_t, (u_t)_t) \in \text{Iad}(\eta)$.
- **($\mathcal{Q}_{III,\eta}$):** Find $((\overline{x}_t)_t, (\overline{u}_t)_t) \in \text{Iad}(\eta)$ such that $\limsup_{h\to\infty} \sum_{t=0}^{h} (f_t^0(\overline{x}_t, \overline{u}_t) - f_t^0(x_t, u_t)) \geq 0$ for all $((x_t)_t, (u_t)_t) \in \text{Iad}(\eta)$.

Remark 2.1. We define the following binary relation on Adm(η) or on Iad(η) : $((x_t)_t, (u_t)_t) \ \mathcal{S}\ ((y_t)_t, (v_t)_t)$ which means $\liminf_{h\to\infty} \sum_{t=0}^{h} (f_t^0(x_t, u_t) - f_t^0(y_t, v_t)) \geq 0$. We easily verify that $\mathcal{S}$ is reflexive and transitive, that is called a *pre-ordering* in [2] p. 28. Note that these properties do not hold when we replace liminf by limsup.

3. Some relations between problems governed by difference equations and problems governed by difference inequations

We begin by giving a list of conditions which will be used into the results of this section.

- **(1):** For all $t \in N$ and for all $u_t \in U_t$, the partial function $f_t^0(., u_t)$ is increasing.
- **(2):** For all $t \in N$ and for all $u_t \in U_t$, the partial function $f_t(., u_t)$ is increasing.
- **(3):** For all $t \in N$, $f_t^0 \geq 0$.
- **(4):** For all $t \in N$, for all $(y_{t+1}, y_t, u_t) \in X_{t+1} \times X_t \times U_t$ such that $y_{t+1} \leq f_t(y_t, u_t)$, there exists $v_t \in U_t$ such that $v_t \geq u_t$ and $y_{t+1} = f_t(y_t, v_t)$.
- **(5):** For all $t \in N$ and for all $x_t \in X_t$, the partial function $f_t^0(x_t, .)$ is increasing.

When $(\underline{\hat{x}}, \underline{\hat{u}}) \in \text{Adm}(\eta)$, we consider the following condition.

(6): For all $t \in N$ and for all $z_t \in X_t$, there exist $s \in N_*$ and $(v_t, v_{t+1}, ..., v_{t+s-1}) \in \prod_{0 \leq i \leq s-1} U_{t+i}$ such that, by setting $z_{t+i+1} := f_{t+i}(z_{t+i}, v_{t+i})$ for $i = 0, ..., s-1$, we have $z_{t+s} = \hat{x}_{t+s}$.

Remark 3.1. This condition (6) is a condition of *reachability in finite time.* For instance, when f_t, X_t and U_t are independent of t, the usual condition of *controllability in finite time*, as defined in [16] p. 81 or in [10] p. 98, implies the condition (6).

Lemma 3.2. *We assume (1) and (2) fulfilled. Let* $((y_t)_t, (u_t)_t) \in \mathrm{Iad}(\eta)$. *Then there exists* $(x_t)_t \in \prod_{t \in N} X_t$ *such that* $((x_t)_t, (u_t)_t) \in \mathrm{Adm}(\eta)$ *and such that* $\liminf_{h \to \infty} \sum_{t=0}^{h} (f_t^0(x_t, u_t) - f_t^0(y_t, u_t)) \geq 0$.

Proof. If $((y_t)_t, (u_t)_t) \in \mathrm{Adm}(\eta)$ it suffices to take $\underline{x} = \underline{y}$. Now we assume that $((y_t)_t, (u_t)_t) \notin \mathrm{Adm}(\eta)$ and we define $T \in N_*$ as the minimum of the set of the $t \in N_*$ such that $y_{t+1} < f_t(y_t, u_t)$. We define $x_t := y_t$ when $t \leq T$, $x_{T+1} := f_T(y_T, u_T)$, and by induction $x_{t+1} := f_t(x_t, u_t)$ when $t > T + 1$. And so we have $(\underline{x}, \underline{u}) \in \mathrm{Adm}(\eta)$. Moreover we have $x_{T+1} = f_T(y_T, u_T) > y_{T+1}$, and by using (2) we obtain $x_{T+2} = f_{T+1}(x_{T+1}, u_{T+1}) \geq f_{T+1}(y_{T+1}, u_{T+1}) \geq y_{T+2}$, and proceeding by induction we obtain $x_t \geq y_t$ for all $t \in \mathbf{N}$. Then by using (1) we obtain $f_t^0(x_t, u_t) \geq f_t^0(y_t, u_t)$ for all $t \in N$, and consequently we have $\liminf_{h \to \infty} \sum_{t=0}^{h} (f_t^0(x_t, u_t) - f_t^0(y_t, u_t)) \geq 0$. □

Theorem 3.3. *We assume (1) and (2) fulfilled. If* $((\hat{x}_t)_t, (\hat{u}_t)_t)$ *is an optimal solution of* $(\mathcal{P}_{II,\eta})$ *then it is also an optimal solution of* $(\mathcal{Q}_{II,\eta})$.

Proof. Let $(\underline{y}, \underline{u}) \in \mathrm{Iad}(\eta)$. By using Lemma 3.2 we know that there exists $\underline{x} \in \prod_{t \in N} X_t$ such that $(\underline{x}, \underline{u}) \in \mathrm{Adm}(\eta)$ and such that $\liminf_{h \to \infty} \sum_{t=0}^{h} (f_t^0(x_t, u_t) - f_t^0(y_t, u_t)) \geq 0$. By using Remark 2.1, the transitivity ensures $\liminf_{h \to \infty} \sum_{t=0}^{h} (f_t^0(\hat{x}_t, \hat{u}_t) - f_t^0(y_t, u_t)) \geq 0$. □

Lemma 3.4. *Let* $((\hat{x}_t)_t, (\hat{u}_t)_t)$ *be an optimal solution of* $(\mathcal{P}_{I,\eta})$. *We assume (3) and (6) fulfilled. Then we have* $((x_t)_t, (u_t)_t) \in \mathrm{dom}(J)$ *for all* $((x_t)_t, (u_t)_t) \in \mathrm{Adm}(\eta)$.

Proof. We proceed by contradiction, we assume that there exists $((x_t)_t, (u_t)_t) \in \mathrm{Adm}(\eta)$ such that $((x_t)_t, (u_t)_t) \notin \mathrm{dom}(J)$. Because of (3), we necessarily have $\sum_{t=0}^{\infty} f_t^0(x_t, u_t) = \infty$, and consequently there exists $h > 0$ such that $\sum_{t=0}^{h} f_t^0(x_t, u_t) > J(\underline{\hat{x}}, \underline{\hat{u}})$. Now we use (6) with $t = h$, $z_t = x_h$ and we define

$$y_t := \begin{cases} x_t & \text{if} \quad t \leq h \\ z_t & \text{if} \quad h+1 \leq t \leq h+1+s \\ \hat{x}_t & \text{if} \quad t \geq h+1+s \end{cases}$$

and

$$w_t := \begin{cases} u_t & \text{if} \quad t \leq h \\ v_t & \text{if} \quad h+1 \leq t \leq h+1+s \\ \hat{u}_t & \text{if} \quad t \geq h+1+s \end{cases}$$

And so we have $(\underline{y}, \underline{w}) \in \mathrm{Adm}(\eta)$, and since we have $\sum_{t \geq h+1+s} f_t^0(y_t, w_t) = \sum_{t \geq h+1+s} f_t^0(\hat{x}_t, \hat{u}_t) < \infty$ we can assert that $(\underline{y}, \underline{w}) \in \mathrm{dom}(J)$. By using (3) we have $J(\underline{y}, \underline{w}) \geq \sum_{t=0}^{h} f_t^0(x_t, u_t) > J(\underline{\hat{x}}, \underline{\hat{u}})$ that is a contradiction with the optimality of $(\underline{\hat{x}}, \underline{\hat{u}})$. □

Theorem 3.5. *Let $((\hat{x}_t)_t, (\hat{u}_t)_t)$ be an optimal solution of $(\mathcal{P}_{I,\eta})$. We assume (1-3) and (6) fulfilled. Then $((\hat{x}_t)_t, (\hat{u}_t)_t)$ is also an optimal solution of $(\mathcal{Q}_{I,\eta})$.*

Proof. Let $(\underline{y}, \underline{v}) \in \mathrm{Iad}(\eta) \cap \mathrm{dom}(J)$. We define $\underline{x}$ by induction by setting $x_0 := \eta$ and $x_{t+1} := f_t(x_t, v_t)$ for all $t \in N_*$. And so we have $(\underline{x}, \underline{v}) \in \mathrm{Adm}(\eta)$. By using (2) we have $x_1 = f_0(\eta, v_0) \geq y_1$, and proceeding by induction we obtain $x_t \geq y_t$ for all $t \in N$. By using (1) we have $f_t^0(x_t, v_t) \geq f_t^0(y_t, v_t)$ for all $t \in N$ and consequently we have $\sum_{t=0}^{\infty} f_t^0(x_t, v_t) \geq \sum_{t=0}^{\infty} f_t^0(y_t, v_t)$. Now by using Lemma 3.4, since $(\underline{x}, \underline{v}) \in \mathrm{Adm}(\eta)$, we have $(\underline{x}, \underline{v}) \in \mathrm{dom}(J)$, and the following inequalities hold : $J(\underline{\hat{x}}, \underline{\hat{u}}) \geq J(\underline{x}, \underline{v}) \geq J(\underline{y}, \underline{v})$. □

Theorem 3.6. *We assume (1) and (4-5) fulfilled. If $((\hat{x}_t)_t, (\hat{u}_t)_t)$ is an optimal solution of $(\mathcal{P}_{II,\eta})$, then it is also an optimal solution of $(\mathcal{Q}_{II,\eta})$.*

Proof. Let $(\underline{y}, \underline{u}) \in \mathrm{Iad}(\eta)$. By using (4) we know that there exists $\underline{v} = (v_t)_t \in \prod_{t \in N} U_t$ such that $(\underline{y}, \underline{v}) \in \mathrm{Adm}(\eta)$ and such that $v_t \geq u_t$ for all $t \in N$. By using (5) we have $f_t^0(y_t, v_t) \geq f_t^0(y_t, u_t)$ for all $t \in N$, and consequently we obtain $\liminf_{h \to \infty} \sum_{t=0}^{h} (f_t^0(y_t, v_t) - f_t^0(y_t, u_t)) \geq 0$. Since we have $\liminf_{h \to \infty} \sum_{t=0}^{h} (f_t^0(\hat{x}_t, \hat{u}_t) - f_t^0(y_t, v_t)) \geq 0$, by using the transitivity noticed in Remark 2.1, we have $\liminf_{h \to \infty} \sum_{t=0}^{h} (f_t^0(\hat{x}_t, \hat{u}_t) - f_t^0(y_t, u_t)) \geq 0$. □

Theorem 3.7. *Let $((\hat{x}_t)_t, (\hat{u}_t)_t)$ be an optimal solution of $(\mathcal{P}_{I,\eta})$. We assume (3-6) fulfilled. Then $((\hat{x}_t)_t, (\hat{u}_t)_t)$ is also an optimal solution of $(\mathcal{Q}_{I,\eta})$.*

Proof. Let $(\underline{y}, \underline{u}) \in \mathrm{Iad}(\eta) \cap \mathrm{dom}(J)$. Then by using (4) we know that there exists $\underline{v} = (v_t)_t$ such that $(\underline{y}, \underline{v}) \in \mathrm{Adm}(\eta)$ and such that $v_t \geq u_t$ for all $t \in N$. By using Lemma 3.4 we know that $(\underline{y}, \underline{v}) \in \mathrm{dom}(J)$ and by using (5) we have $f_t^0(y_t, v_t) \geq f_t^0(y_t, u_t)$ et donc $J(\underline{\hat{x}}, \underline{\hat{u}}) \geq J(\underline{y}, \underline{v}) \geq J(\underline{y}, \underline{u})$. □

4. Weak principles

We begin by giving a list of conditions used in this section.

(7): For all $t \in N$, the function f_t^0 is Lipschitzian on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and Clarke-regular at $(\hat{x}_t, \hat{u}_t)$, and the function f_t is strictly differentiable at $(\hat{x}_t, \hat{u}_t)$.

Following [7] p. 39, recall that, Z being a normed space, a function $g : Z \to R$ is Clarke-regular at z when, for all $w \in Z$, we have $\lim_{t \to 0+} \frac{1}{t}(g(z + tw) - g(z)) = \limsup_{y \to z, t \to 0+} \frac{1}{t}(g(y + tw) - g(y))$. The strict differentiability is defined in [7] p. 30 and in [1] p. 133. A particular case of (1) is the following condition.

(8): For all $t \in N$, the functions f_t^0 and f_t are of class C^1 on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and the set U_t is convex.

We introduce the new following condition that we call a *positivity* condition; in this condition the sequences $((\hat{x}_t)_t, (\hat{u}_t)_t)$ are given.

(9): For all $t \in N_*$, $\frac{\partial f_t^j(\hat{x}_t, \hat{u}_t)}{\partial x_t^i} \geq 0$ for all $i, j = 1, ..., n$ and $\frac{\partial f_t^i(\hat{x}_t, \hat{u}_t)}{\partial x_t^i} > 0$ for all $i = 1, ..., n$.

Following Samuelson, [15] p. 86, when $f_t(., u_t)$ is a production function, condition (9) is coherent with the economic theory. Condition (9) implies that the partial differential $D_{x_t} f_t(\hat{x}_t, \hat{u}_t)$ is a non negative matrix, see Chapter 13 in [9] or Chapter 9 of [13]. Condition (9) does not imply that $D_{x_t} f_t(\hat{x}_t, \hat{u}_t)$ is invertible, but it implies that $\mathrm{Ker} D_{x_t} f_t(\hat{x}_t, \hat{u}_t) \cap R^n_+ = \{0\}$. Now we give a first new weak Pontryagin principle by using this positivity condition.

Theorem 4.1. *Let $((\hat{x}_t)_t, (\hat{u}_t)_t)$ be an optimal solution of $(\mathcal{Q}_{I,\eta})$ or of $(\mathcal{Q}_{II,\eta})$ or of $(\mathcal{Q}_{III,\eta})$. We assume (7) and (9) fulfilled. Then there exist $\lambda^0 \in R^1$ and $(p_t)_{t \geq 1}$ a sequence in R^{n*} which satisfy the following conditions.*

- **(i):** *(λ^0, p_1) is non zero.*
- **(ii):** *$\lambda^0 \geq 0$.*
- **(iii):** *For all $t \in N_*$, $p_t \geq 0$.*
- **(iv):** *For all $t \in N_*$, $p_t \in \partial_{x_t} H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1})$, where ∂_{x_t} denotes the partial Clarke-differentiation with respect to x_t.*
- **(v):** *For all $t \in N$, $\partial_{u_t} H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1}) \cap \mathcal{N}_{U_t}(\hat{u}_t) \neq \emptyset$, where ∂_{u_t} denotes the partial Clarke-differentiation with respect to u_t and $\mathcal{N}_{U_t}(\hat{u}_t)$ is the Clarke-normal cone of U_t at $\hat{u}_t$.*

The definitions of the Clarke-differentiation and of the Clarke-normal cone are given in the two first chapters of [7].

The differences between Theorem 4.5 of the present paper and Theorem 1 of [3] are the following ones: here the system is governed by a difference inequation instead of a difference equation, and the condition of invertibility of $D_{x_t} f_t(\hat{x}_t, \hat{u}_t)$ in [3] is replaced by condition (9).

Proof of Theorem 4.1. We use the same method as that of First Step of the proof of Theorem 1 in [3] and by taking into account that the equality constraints are replaced by inequality constraints (that ensures non negative multipliers). Since $((\hat{x}_t)_t, (\hat{u}_t)_t)$ is an optimal process of $(\mathcal{Q}_{E,\eta})$ with $E \in \{I, II, III\}$, then, for all $h > 1$, $((\hat{x}_t)_{1 \leq t \leq h}, (\hat{u}_t)_{1 \leq t \leq h-1})$ is an optimal solution of the following finite-horizon problem:

$$(\mathcal{F}_h) \begin{cases} \text{Maximize} & \sum_{t=0}^{h-1} f_t^0(x_t, u_t) \\ \text{when} & x_t \in X_t, t = 0, ..., h \\ & u_t \in U_t, t = 0, ..., h-1 \\ & x_{t+1} \leq f_t(x_t, u_t), t = 0, ..., h-1 \\ & x_0 = \eta, x_h = \hat{x}_h \end{cases}$$

The proof of this reduction to the finite-horizon is similar as that Lemma 1 in [3]. Setting $\mathbf{x} := (x_0, x_1, ..., x_h)$, $\mathcal{X} := \prod_{t=0}^{h} X_t$, $\mathbf{u} := (u_0, u_1, ..., u_{h-1})$, $\mathcal{U} := \prod_{t=0}^{h-1} U_t$, $\pi_t : \mathcal{X} \times \mathcal{U} \to X_t$ defined by $\pi_t(x_0, ..., x_h, u_0, ..., u_{h-1}) := x_t$, $\pi'_t : \mathcal{X} \times \mathcal{U} \to U_t$ defined by $\pi'_t(x_0, ..., x_h, u_0, ..., u_{h-1}) := u_t$, $J_h(\mathbf{x}, \mathbf{u}) := \sum_{t=0}^{h-1} f_t^0(\pi_t(\mathbf{x}, \mathbf{u}), \pi'_t(\mathbf{x}, \mathbf{u}))$, we can formulate the problem $(\mathcal{F}_h)$ as the following static optimization problem:

$$(\mathcal{S}_h) \begin{cases} \text{Maximize} & J_h(\mathbf{x}, \mathbf{u}) \\ \text{when} & (\mathbf{x}, \mathbf{u}) \in \mathcal{X} \times \mathcal{U} \\ & f_t(\pi_t(\mathbf{x}, \mathbf{u}), \pi'_t(\mathbf{x}, \mathbf{u})) - \pi_{t+1}(\mathbf{x}, \mathbf{u}) \geq 0, t = 0, ..., h-1 \\ & \pi_0(\mathbf{x}, \mathbf{u}) = \eta, \pi_h(\mathbf{x}, \mathbf{u}) = \hat{x}_h \end{cases}$$

Now, ever working as in the proof of Theorem 1 in [3], we verify that we can apply the theorem on the Lagrange multipliers rule [7] p. 228, and we obtain that, for all finite horizon $h \in N_*$, there exist $\lambda^{0,h} \in R^1$ and $(p_t^h)_{1 \le t \le h} \in (R_+^{n*})^h$ which satisfy the following conditions.

(10): $(\lambda^{0,h}, p_1^h, ..., p_h^h)$ is non zero.

(11): $\lambda^{0,h} \ge 0$.

(12): For all $t = 1, ..., h$, $p_t^h \ge 0$.

(13): For all $t = 1, ..., h$, there exists $\varphi_t^{0,h} \in \partial_{x_t} f_t^0(\hat{x}_t, \hat{u}_t)$ such that $p_t^h = \lambda^{0,h}\varphi_t^{0,h} + p_{t+1}^h \circ D_{x_t} f_t(\hat{x}_t, \hat{u}_t) = \lambda^{0,h}\varphi_t^{0,h} + D_{x_t} f_t(\hat{x}_t, \hat{u}_t)^* p_{t+1}^h$, where the star denotes the transposition.

(14): For all $t = 0, ..., h-1$, there exists $\psi_t^{0,h} \in \partial_{u_t} f_t^0(\hat{x}_t, \hat{u}_t)$ such that, for all $v_t \in \mathcal{T}_{U_t}(\hat{u}_t)$, we have $\langle \lambda^{0,h}\psi_t^{0,h} + p_t^h \circ D_{u_t} f_t(\hat{x}_t, \hat{u}_t), v_t \rangle \ge 0$, where $\mathcal{T}_{U_t}(\hat{u}_t)$ denotes the tangent cone to U_t at $\hat{u}_t$.

The assumption (9) implies that there exists $c_t \in (0, \infty)$ such that $D_{x_t} f_t(\hat{x}_t, \hat{u}_t)^* v \ge c_t v$ for all $v \in R_+^{n*}$. Since a Clarke-gradient at a point is a compact subset, there exists $b_t \in (0, \infty)$ such that $\|v\| \le b_t$ for all $v \in \partial_{x_t} f_t^0(\hat{x}_t, \hat{u}_t)$. And so the equality $p_t^h = \lambda^{0,h}\varphi_t^{0,h} + D_{x_t} f_t(\hat{x}_t, \hat{u}_t)^* p_{t+1}^h$, or $p_t^h - \lambda^{0,h}\varphi_t^{0,h} = D_{x_t} f_t(\hat{x}_t, \hat{u}_t)^* p_{t+1}^h$, implies $p_t^h - \lambda^{0,h}\varphi_t^{0,h} \ge c_t p_{t+1}^h \ge 0$, and consequently we have $\|p_t^h - \lambda^{0,h}\varphi_t^{0,h}\| \ge \|c_t p_{t+1}^h\|$ that implies $c_t\|p_{t+1}^h\| \le \|p_t^h\| + \lambda^{0,h}\|\varphi_t^{0,h}\|$. And then we have $\|p_{t+1}^h\| \le c_t^{-1}\|p_t^h\| + c_t^{-1} b_t \lambda^{0,h}$. Then we can deduce from these inequalities the following assertion.

(15): For all $t \in N_*$, there exist $d_t \in (0, \infty)$ and $e_t \in (0, \infty)$ such that, for all $h > t$, $\|p_t^h\| \le d_t\|p_1^h\| + e_t\lambda^{0,h}$

where $d_t := c_t^{-1}$ and $e_t := c_t^{-1} b_t$.

From this assertion we see that $(\lambda^{0,h}, p_1^h) = (0, 0)$ implies $(\lambda^{0,h}, p_1^h, ..., p_h^h) = (0, 0, ..., 0)$, and then by contraposition and by using (10) we obtain $(\lambda^{0,h}, p_1^h) \ne (0, 0)$. Therefore, noting that when $(\lambda^{0,h}, p_1^h, ..., p_h^h)$ satisfies (10-14) then, for all $\theta \in (0, \infty)$, $(\theta\lambda^{0,h}, \theta p_1^h, ..., \theta p_h^h)$ also satisfies (10-14), we can choose $(\lambda^{0,h}, p_1^h)$ such that $|\lambda^{0,h}| + \|p_1^h\| = 1$ for all h. By using (15) we see that, for all t, the sequence $(p_t^h)_{h>t}$ is bounded by $\max\{d_t, e_t\}$. For all t, the sequences $(\varphi_t^{0,h})_{h>t}$ and $(\psi_t^{0,h})_{h>t}$ (where t is fixed) are bounded since a Clarke-gradient is compact. Then we can use Lemma 2 in [3] to assert that there exists an increasing function $\beta : N_* \to N_*$ and there exist $\lambda^0 \in R_+^1$, $(p_t)_{t \ge 1}$ a sequence in R_+^{n*}, $\varphi_t^0 \in \partial_{x_t} f_t(\hat{x}_t, \hat{u}_t)$ for all $t \in N_*$ and $\psi_t^0 \in \partial_{u_t} f_t(\hat{x}_t, \hat{u}_t)$ for all $t \in N$ such that $\lim_{h\to\infty} \lambda^{0,\beta(h)} = \lambda^0$, $\lim_{h\to\infty} p_t^{\beta(h)} = p_t$ for all $t \in N_*$, $\lim_{h\to\infty} \varphi_t^{0,\beta(h)} = \varphi_t^0$ for all $t \in N_*$, and $\lim_{h\to\infty} \psi_t^{0,\beta(h)} = \psi_t^0$ for all $t \in N$.

And so, from $|\lambda^{0,\beta(h)}| + \|p_1^{\beta(h)}\| = 1$ for all h, by taking $h \to \infty$ and by using the continuity of the norms, we obtain $|\lambda^0| + \|p_1\| = 1$ that ensures (i). Since $\lambda^{0,\beta(h)} \ge 0$ for all h and $p_t^{\beta(h)} \ge 0$ for all $h > t$, by taking $h \to \infty$ we obtain (ii) and (iii). From (13), for all $t \ge 1$, we have $p_t^{\beta(h)} = \lambda^{0,\beta(h)}\varphi_t^{0,\beta(h)} + p_{t+1}^{\beta(h)} \circ D_{x_t} f_t(\hat{x}_t, \hat{u}_t)$ for all $h > t$, and then by taking $h \to \infty$ we obtain $p_t = \lambda^0\varphi_t^0 + p_{t+1} \circ D_{x_t} f_t(\hat{x}_t, \hat{u}_t)$, and since the Clarke-gradient is a correspondence with compact values we have $p_t \in \lambda^0 \partial_{x_t} f_t^0(\overline{x}_t, \overline{u}_t) + p_{t+1} \circ D_{x_t} f_t(\hat{x}_t, \hat{u}_t)$ which is (iv). From (12), for all $t \ge 1$

and for all $v_t \in \mathcal{T}_{U_t}(\hat{u}_t)$, we have $\langle \lambda^{0,\beta(h)}\psi_t^{0,\beta(h)} + p_t^{\beta(h)} \circ D_{u_t}f_t(\hat{x}_t, \hat{u}_t), v_t\rangle \geq 0$ for all $h > t$, and then by taking $h \to \infty$ we obtain $\langle \lambda^0\psi_t^0 + p_t \circ D_{u_t}f_t(\hat{x}_t, \hat{u}_t), v_t\rangle \geq 0$ that implies $\lambda^0\psi_t^0 + p_t \circ D_{u_t}f_t(\hat{x}_t, \hat{u}_t) \in \mathcal{N}_{U_t}(\hat{u}_t)$ that ensures (v); and so Theorem 4.1 is proven. □

Theorem 4.2. *Let $((\hat{x}_t)_t, (\hat{u}_t)_t)$ be an optimal solution of $(\mathcal{Q}_{I,\eta})$ or of $(\mathcal{Q}_{II,\eta})$ or of $(\mathcal{Q}_{III,\eta})$. We assume (8-9) fulfilled. Then there exist $\lambda^0 \in R^1$ and $(p_t)_{t\geq 1}$ a sequence in R^{n*} which satisfy the following conditions.*

- **(i):** (λ^0, p_1) *is non zero.*
- **(ii):** $\lambda^0 \geq 0$.
- **(iii):** *For all $t \in N_*$, $p_t \geq 0$.*
- **(iv):** *For all $t \in N_*$, $p_t = D_{x_t}H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1})$.*
- **(v):** *For all $t \in N$, for all $u_t \in U_t$, $\langle D_{u_t}H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1}), u_t - \hat{u}_t\rangle \leq 0$.*

Theorem 4.2 is a simplified version of Theorem 4.1 whom the statement does not contain any sophisticated mathematical tools. Note also that the condition (v) appears as a variational inequation as in Section 12 in Chapter 5 of [5].

Proof of Theorem 4.2. It is clear that (8) implies (7). And so under the assumptions of Theorem 4.2 the conclusions of Theorem 4.1 hold, that justifies the conclusions (i), (ii) and (iii). Since, under (8), we have $\partial_{x_t}H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1}) = \{D_{x_t}H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1})\}$ the conclusion (iv) of Theorem 4.2 is a consequence of the conclusion (iv) of Theorem 4.1. Since U_t is convex, its tangent cone at $\hat{u}_t$ satisfies $\mathcal{T}_{U_t}(\hat{u}_t) = \{\theta(u_t - \hat{u}_t) : \theta \in R^1_+, u_t \in U_t\}$. Since $\mathcal{N}_{U_t}(\hat{u}_t)$ is the polar cone of $\mathcal{T}_{U_t}(\hat{u}_t)$, and since under (8) we have $\partial_{u_t}H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1}) = \{D_{u_t}H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1})\}$, from the conclusion (v) of Theorem 4.1, we obtain $D_{u_t}H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1}) \in \mathcal{N}_{U_t}(\hat{u}_t)$, that implies $\langle D_{u_t}H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1}), u_t - \hat{u}_t\rangle \leq 0$ for all $u_t \in U_t$ that is (v). □

Theorem 4.3. *Let $((\hat{x}_t)_t, (\hat{u}_t)_t)$ be an optimal solution of $(\mathcal{P}_{II,\eta})$. We assume that the conditions (1-2), (7), (9) are fulfilled or that the conditions (1), (4-5), (7), (9) are fulfilled. Then the conclusions of Theorem 4.1 hold, and moreover, when (8) replaces (7), the conclusions of Theorem 4.2 hold.*

Proof. By using Theorem 3.3 or Theorem 3.6, $((\hat{x}_t)_t, (\hat{u}_t)_t)$ becomes an optimal solution of $(\mathcal{Q}_{II,\eta})$, and we prove the first assertion by using Theorem 4.1, and we prove the second assertion by using Theorem 4.2. □

Theorem 4.4. *Let $((\hat{x}_t)_t, (\hat{u}_t)_t)$ be an optimal solution of $(\mathcal{P}_{I,\eta})$. We assume that the conditions (1-3), (6-7), (9) are fulfilled or that the conditions (3-7), (9) are fulfilled. Then the conclusions of Theorem 4.1 hold, and moreover, when (8) replaces (7), the conclusions of Theorem 4.2 hold.*

Proof. By using Theorem 3.5 or Theorem 3.7, $((\hat{x}_t)_t, (\hat{u}_t)_t)$ becomes an optimal solution of $(\mathcal{Q}_{I,\eta})$, and we prove the first assertion by using Theorem 4.1, and we prove the second assertion by using Theorem 4.2. □

5. Strong principles with a positivity assumption

In this section we give a new strong Pontryagin principle by using the positivity condition and the condition of mixed problem condition of Michel.

We give a list of new conditions which will be used as assumptions in the theorems of this section. In certain conditions an optimal process $((\hat{x}_t)_t, (\hat{u}_t)_t)$ is given. To use the results of Michel we need the two following conditions.

(16): For all $t \in N$, the functions f_t^0 and f_t are partially differentiable with respect to x_t.

(17): For all $t \in N$, for all $(x_t, x_{t+1}) \in X_t \times X_{t+1}$, we have $\mathrm{co}A_t(x_t, x_{t+1}) \subset B_t(x_t, x_{t+1})$, where co denotes the convex hull, where $A_t(x_t, x_{t+1})$ is the set of the $(\lambda, y) \in R^1 \times R^n$ such that there exists $u \in U_t$ satisfying $\lambda \leq f_t^0(x_t, u)$ and $y = f_t(x_t, u) - x_{t+1}$, and where $B_t(x_t, x_{t+1})$ is the set of the $(\lambda, y) \in R^1 \times R^n$ such that there exists $(u, v) \in U_t \times R^n$ satisfying $\lambda \leq f_t^0(x_t, u)$ and $v^j y^j \leq f_t^j(x_t, u) - x_{t+1}^j$ for all $j = 1, ..., n$ (the upper index denotes the coordinate).

This last condition was created by Michel [12] to establish a strong Pontryagin principle for discrete-time finite-horizon optimal control problems. For instance this condition is fulfilled when the sets $A_t(x_t, x_{t+1})$ are convex. When the sets U_t are convex, when the partial functions $f_t^0(x_t, .)$ are concave and when the partial functions $f_t(x_t, .)$ are affine, the sets $A_t(x_t, x_{t+1})$ are convex. But we note that (17) can be fulfilled without the convexity of the sets $A_t(x_t, x_{t+1})$. And so (17) appears as a weakened convexity condition to obtain a strong principle. Michel says that the problem is *mixed* when (17) is fulfilled.

Theorem 5.1. *Let $((\hat{x}_t)_t, (\hat{u}_t)_t)$ be an optimal solution of $(\mathcal{Q}_{I,\eta})$ or of $(\mathcal{Q}_{II,\eta})$ or of $(\mathcal{Q}_{III,\eta})$. We assume (9) and (16-17) fulfilled. Then there exist $\lambda^0 \in R^1$ and $(p_t)_{t \geq 1}$ a sequence in R^{n*} which satisfy the following conditions.*

(i): *(λ^0, p_1) is non zero.*

(ii): $\lambda^0 \geq 0$.

(iii): *For all $t \in N_*$, $p_t \geq 0$.*

(iv): *For all $t \in N_*$, $p_t = D_{x_t} H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1})$.*

(v): *For all $t \in N$, for all $u_t \in U_t$, $H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1}) \geq H_t(\hat{x}_t, u_t, \lambda^0, p_{t+1})$.*

The differences between this new theorem and Theorem 3 of [3] are the following ones: here the system is governed by a difference inequation instead of a difference equation, and the condition of invertibility of $D_{x_t} f_t(\hat{x}_t, \hat{u}_t)$ in [3] is replaced by condition (9).

Proof of Theorem 5.1. By using Lemma 1 in [3] and by using [12] p. 9, we can assert that there exist $\lambda^{0,h} \in R^1$ and $(p_t^h)_{1 \leq t \leq h} \in (R_+^{n*})^h$ which satisfy the conditions (8), (9), (10) and the following conditions for all $h \in N_*$.

(18): For all $t = 1, ..., h-1$, $p_t^h = D_{x_t} H_t(\hat{x}_t, \hat{u}_t, \lambda^{0,h}, p_{t+1}^h) = \lambda^{0,h} D_{x_t} f_t^0(\hat{x}_t, \hat{u}_t) + p_{t+1}^h \circ D_{x_t} f_t(\hat{x}_t, \hat{u}_t)$.

(19): For all $t = 0, ..., h-1$, for all $u_t \in U_t$, $H_t(\hat{x}_t, \hat{u}_t, \lambda^{0,h}, p_{t+1}^h) \geq H_t(\hat{x}_t, u_t, \lambda^{0,h}, p_{t+1}^h)$.

Now we proceed as in the proof of Theorem 4.1. From the equations (18) we obtain $p_t^h - \lambda^{0,h} D_{x_t} f_t^0(\hat{x}_t, \hat{u}_t) = D_{x_t} f_t(\hat{x}_t, \hat{u}_t)^* p_{t+1}^h \geq c_t p_{t+1}^h$, and therefore we have

$$\|p_t^h\| + \lambda^{0,h} \|D_{x_t} f_t^0(\hat{x}_t, \hat{u}_t)\| \geq \|p_t^h - \lambda^{0,h} D_{x_t} f_t^0(\hat{x}_t, \hat{u}_t)\| \geq c_t \|p_{t+1}^h\|.$$

And so we obtain the inequalities (15) by taking $b_t := \|D_{x_t} f_t^0(\hat{x}_t, \hat{u}_t)\|$, and the end of the proof is similar as that of Theorem 4.1. □

Theorem 5.2. *Let $((\hat{x}_t)_t, (\hat{u}_t)_t)$ be an optimal solution of $(\mathcal{P}_{II,\eta})$. We assume that the conditions (1-2), (9), (16-17) are fulfilled or that the conditions (1), (4-5), (9), (16-17) are fulfilled. Then the conclusions of Theorem 5.1 hold.*

Proof. By using Theorem 3.3 or Theorem 3.6, $((\hat{x}_t)_t, (\hat{u}_t)_t)$ becomes an optimal solution of $(\mathcal{Q}_{II,\eta})$, and we conclude by using Theorem 5.1. □

Theorem 5.3. *Let $((\hat{x}_t)_t, (\hat{u}_t)_t)$ be an optimal solution of $(\mathcal{P}_{I,\eta})$. We assume that the conditions (1-3), (6), (9), (16-17) are fulfilled or that the conditions (3-6), (9), (16-17) are fulfilled. Then the conclusions of Theorem 5.1 hold.*

Proof. By using Theorem 3.5 or Theorem 3.7, $((\hat{x}_t)_t, (\hat{u}_t)_t)$ becomes an optimal solution of $(\mathcal{Q}_{I,\eta})$, and we conclude by using Theorem 5.1. □

6. Strong principles with a partial surjectivity assumption

Now we formulate another alternative condition to replace the invertibility condition of [3]. We consider R^n as an Euclidean space by endowing it with the usual inner product. For all $t \in N$ we set $N_t := \mathrm{Ker} D_{x_t} f_t(\hat{x}_t, \hat{u}_t)$ and M_t stands for the orthogonal complement of N_t. π_{N_t} (respectively π_{M_t}) denotes the orthogonal projector on N_t (respectively M_t). When $\rho \in (0, \infty)$ we set $S_{N_t}(0, \rho) := \{x \in N_t : \|x\| = \rho\}$ where $\|.\|$ is the Euclidean norm.

(20): For all $t \in N$ there exists $P_t \subset U_t$ which satisfies the following conditions:
- **(20-i):** There exists $\rho_t \in (0, \infty)$ such that $\pi_{N_t}(\{f_t(\hat{x}_t, u_t) : u_t \in P_t\}) \supset S_{N_t}(0, \rho_t) + \pi_{N_t}(f_t(\hat{x}_t, \hat{u}_t))$.
- **(20-ii):** $\{\pi_{M_t}(f_t(\hat{x}_t, u_t)) : u_t \in P_t\}$ is bounded in $\mathbf{R}^n$.
- **(20-iii):** $\{f_t^0(\hat{x}_t, u_t) : u_t \in P_t\}$ is bounded in $\mathbf{R}$.

We call condition (20) a *partial surjectivity* condition. This condition can seem to be difficult to verify; it is why we give the following condition, more easy to verify, which is a particular case of (20).

(21): For all $t \in N$, $f_t^0(\hat{x}_t, .)$ and $f_t(\hat{x}_t, .)$ are continuous on U_t, $D_{x_t} f_t(\hat{x}_t, \hat{u}_t)$ exists, $\hat{u}_t$ belongs to the interior of U_t, $f_t(\hat{x}_t, .)$ is of class C^1 at $\hat{u}_t$, and $\mathrm{Im}(\pi_{N_t} \circ D_{u_t} f_t(\hat{x}_t, \hat{u}_t)) = N_t$.

Remark 6.1. (21) $\Longrightarrow$ (20). Applying the Surjective Mapping Theorem of Graves, [11] p. 397, since $D_{u_t}(\pi_{N_t} \circ f_t(\hat{x}_t, .))(\hat{u}_t) = \pi_{N_t} \circ D_{u_t} f_t(\hat{x}_t, \hat{u}_t)$ is surjective from R^m on N_t, there exists P_t, a closed ball centered at $\hat{u}_t$, such that $\pi_{N_t} \circ f_t(\hat{x}_t, .)(P_t)$ contains a closed ball centered at $\pi_{N_t} \circ f_t(\hat{x}_t, \hat{u}_t)$. We denote by ρ_t the radius of this last ball, and consequently we have $\{\pi_{N_t} \circ f_t(\hat{x}_t, u_t) : u_t \in P_t\} \supset S_{N_t}(0, \rho_t) + \pi_{N_t} \circ f_t(\hat{x}_t, \hat{u}_t)$, i.e. (20-i) is satisfied. Since P_t is compact and since $\pi_{M_t} \circ f_t(\hat{x}_t, .)$ is continuous, $\{\pi_{N_t} \circ f_t(\hat{x}_t, u_t) : u_t \in P_t\}$ is compact therefore bounded, and so (20-ii) is satisfied. Since $f_t^0(\hat{x}_t, .)$ is continuous, $\{f_t^0(\hat{x}_t, u_t) : u_t \in P_t\}$ is compact therefore bounded, and so (20-iii) is satisfied.

Now we give a new Pontryagin principle by using a partial surjectivity condition and the mixed problem condition of Michel.

Theorem 6.2. *Let $((\hat{x}_t)_t, (\hat{u}_t)_t)$ be an optimal solution of $(\mathcal{Q}_{E,\eta})$ or of $(\mathcal{P}_{F,\eta})$ with $E, F \in \{I, II, III\}$. We assume (16-17) and (20) fulfilled. Then there exist $\lambda^0 \in R^1$ and $(p_t)_{t\geq 1}$ a sequence in R^{n*} which satisfy the following conditions.*

(i): *(λ^0, p_1) is non zero.*
(ii): *$\lambda^0 \geq 0$.*
(iii): *For all $t \in N_*$, $p_t \geq 0$ when $((\hat{x}_t)_t, (\hat{u}_t)_t)$ is an optimal solution of $(\mathcal{Q}_{E,\eta})$.*
(iv): *For all $t \in N_*$, $p_t = D_{x_t}H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1})$.*
(v): *For all $t \in N$, for all $u_t \in U_t$, $H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1}) \geq H_t(\hat{x}_t, u_t, \lambda^0, p_{t+1})$.*

The differences between this last theorem and Theorem 3 in [3] are the following ones: here we can treat a system governed by a difference inequation or by a difference equation when Theorem 3 in [3] treats only the case of a difference equation, and the condition of invertibility of $D_{x_t}f_t(\hat{x}_t, \hat{u}_t)$ in [3] (which is omitted in [3] by mistake) is replaced by condition (20).

Proof of Theorem 6.1. By using Lemma 1 in [3] we know that $((\hat{x}_t)_{t\leq h}, (\hat{u}_t)_{t\leq h})$ is an optimal solution of the finite-horizon problem $(\mathcal{F}_h)$. Recall that the theorem in Subsection 2.3 in [12] is a strong Pontryagin principle for discrete-time finite-horizon optimal control problems; by using it on $(\mathcal{F}_h)$ we can assert that there exist $\lambda^{0,h} \in R^1$ and $(p^h_t)_{1\leq t\leq h} \in (R^{n*}_+)^h$ which satisfy the conditions (10), (11), (18), (19) and the condition (12) only when $((\hat{x}_t)_t, (\hat{u}_t)_t)$ is an optimal solution of $(\mathcal{Q}_{E,\eta})$.

Since $D_{x_t}f_t(\hat{x}_t, \hat{u}_t)^* z \neq 0$ for all nonzero $z \in M_t$, we know that there exists $a_t \in (0, \infty)$ such that $\|D_{x_t}f_t(\hat{x}_t, \hat{u}_t)^* z\| \geq a_t\|z\|$ for all $z \in M_t$. From (18) we deduce the following equations

$$\begin{aligned} p^h_t &= D_{x_t}f_t(\hat{x}_t, \hat{u}_t)^*\pi_{N_t}(p^h_{t+1}) + D_{x_t}f_t(\hat{x}_t, \hat{u}_t)^*\pi_{M_t}(p^h_{t+1}) + \lambda^{0,h}D_{x_t}f^0_t(\hat{x}_t, \hat{u}_t) \\ &= D_{x_t}f_t(\hat{x}_t, \hat{u}_t)^*\pi_{M_t}(p^h_{t+1}) + \lambda^{0,h}D_{x_t}f^0_t(\hat{x}_t, \hat{u}_t) \end{aligned}$$

since $D_{x_t}f_t(\hat{x}_t, \hat{u}_t)^* \circ \pi_{N_t} = 0$. Consequently we have

$$p^h_t - \lambda^{0,h}D_{x_t}f^0_t(\hat{x}_t, \hat{u}_t) = D_{x_t}f_t(\hat{x}_t, \hat{u}_t)^*\pi_{M_t}(p^h_{t+1}),$$

that implies

$$\begin{aligned} \|p^h_t\| + \lambda^{0,h}\|D_{x_t}f^0_t(\hat{x}_t, \hat{u}_t)\| &\geq \|p^h_t - \lambda^{0,h}D_{x_t}f^0_t(\hat{x}_t, \hat{u}_t)\| \\ &= \|D_{x_t}f_t(\overline{x}_t, \overline{u}_t)^*\pi_{M_t}(p^h_{t+1})\| \geq a_t\|\pi_{M_t}(p^h_{t+1})\|. \end{aligned}$$

And so we obtain the following inequalities:

(22): $\forall h \in N_*, \forall t < h, \|\pi_{M_t}(p^h_{t+1})\| \leq a_t^{-1}(\|p^h_t\| + \lambda^{0,h}\|D_{x_t}f^0_t(\hat{x}_t, \hat{u}_t)\|)$.

To abridge the writing we set

(23): $\Delta f^0_t(u_t) := f^0_t(\hat{x}_t, \hat{u}_t) - f^0_t(\hat{x}_t, u_t)$, $\Delta f_t(u_t) := -f_t(\hat{x}_t, u_t) + f_t(\hat{x}_t, \hat{u}_t)$.

From (19) we obtain $\lambda^{0,h}\Delta f^0_t(u_t) - \langle p^h_{t+1}, \Delta f_t(u_t)\rangle \geq 0$ for all $u_t \in U_t$, that is equivalent to

$$\lambda^{0,h}\Delta f^0_t(u_t) - \langle \pi_{N_t}(p^h_{t+1}), \pi_{N_t}(\Delta f_t(u_t))\rangle - \langle \pi_{M_t}(p^h_{t+1}), \pi_{M_t}(\Delta f_t(u_t))\rangle \geq 0$$

that implies

$$\begin{aligned} &\lambda^{0,h}|\Delta f^0_t(u_t)| + \|\pi_{M_t}(p^h_{t+1})\|.\|\pi_{M_t}(\Delta f_t(u_t))\| \geq \\ &\lambda^{0,h}\Delta f^0_t(u_t) - \langle \pi_{M_t}(p^h_{t+1}), \pi_{M_t}(\Delta f_t(u_t))\rangle \geq \langle \pi_{N_t}(p^h_{t+1}), \pi_{N_t}(\Delta f_t(u_t))\rangle, \end{aligned}$$

that gives us the following inequalities:

(24): $\lambda^{0,h}|\Delta f_t^0(u_t)|+\|\pi_{M_t}(p_{t+1}^h)\|.\|\pi_{M_t}(\Delta f_t(u_t))\| \geq \langle \pi_{N_t}(p_{t+1}^h), \pi_{N_t}(\Delta f_t(u_t))\rangle$.

We set $\xi_t := \sup\{|\Delta f_t^0(u_t)| : u_t \in P_t\}$ and $\zeta_t := \sup\{\|\pi_{M_t}(\Delta f_t(u_t))\| : u_t \in P_t\}$ where P_t is provided by (20). These two numbers are finite after (20). Now we take the lower upper bound on P_t in the inequalities (24) that gives the following inequalities:

$$\lambda^{0,h}\xi_t + \zeta_t\|\pi_{M_t}(p_{t+1}^h)\| \geq \sup_{u_t\in P_t} \langle \pi_{N_t}(p_{t+1}^h), \pi_{N_t}(\Delta f_t(u_t))\rangle$$

$$\geq \sup_{z_t\in S_{N_t}(0,\rho_t)} \langle \pi_{N_t}(p_{t+1}^h), z_t\rangle = \rho_t \sup_{w_t\in S_{N_t}(0,1)} \langle \pi_{N_t}(p_{t+1}^h), w_t\rangle = \rho_t\|\pi_{N_t}(p_{t+1}^h)\|.$$

Therefore we have proven:

(25): $\forall t, \forall h > t, \|\pi_{N_t}(p_{t+1}^h)\| \leq (\rho_t^{-1}\xi_t)\lambda^{0,h} + (\rho_t^{-1}\zeta_t)\|\pi_{M_t}(p_{t+1}^h)\|$.

From (21) and (25) we deduce

$$\begin{aligned}\|\pi_{N_t}(p_{t+1}^h)\| &\leq (\rho_t^{-1}\xi_t)\lambda^{0,h} + (\rho_t^{-1}\zeta_t a_t^{-1})(\|p_t^h\| + \lambda^{0,h}\|D_{x_t}f_t^0(\hat{x}_t,\hat{u}_t)\|)\\ &= (\rho_t^{-1}\xi_t + \rho_t^{-1}\zeta_t a_t^{-1}\|D_{x_t}f_t^0(\hat{x}_t,\hat{u}_t)\|)\lambda^{0,h} + (\rho_t^{-1}\zeta_t a_t^{-1})\|p_t^h\|.\end{aligned}$$

Consequently by using (21) and the last inequality we obtain

$$\begin{array}{lll}\|p_{t+1}^h\| & \leq & \|\pi_{N_t}(p_{t+1}^h)\| + \|\pi_{M_t}(p_{t+1}^h)\| \\ & \leq & (\rho_t^{-1}\xi_t + \rho_t^{-1}\zeta_t a_t^{-1}\|D_{x_t}f_t^0(\hat{x}_t,\hat{u}_t)\| + a_t^{-1}\|D_{x_t}f_t^0(\hat{x}_t,\hat{u}_t)\|)\lambda^{0,h} \\ & & +(\rho_t^{-1}\zeta_t a_t^{-1} + a_t^{-1})\|p_t^h\|.\end{array}$$

By setting $c_t^0 := \rho_t^{-1}\xi_t + \rho_t^{-1}\zeta_t a_t^{-1}\|D_{x_t}f_t^0(\hat{x}_t,\hat{u}_t)\| + a_t^{-1}\|D_{x_t}f_t^0(\hat{x}_t,\hat{u}_t)\|$ and $b_t^0 := \rho_t^{-1}\zeta_t a_t^{-1} + a_t^{-1}$, we have proven the following inequalities

(26): For all $t \in N_*$, there exist $c_t^0 \in (0,\infty)$ and $b_t^0 \in (0,\infty)$ such that, for all $h > t$, we have $\|p_{t+1}^h\| \leq c_t^0\lambda^{0,h} + b_t^0\|p_t^h\|$.

Note that (26) is similar to (15) and we can conclude as in the proof of Theorem 4.1. □

Theorem 6.3. *Let $((\hat{x}_t)_t, (\hat{u}_t)_t)$ be an optimal solution of $(\mathcal{Q}_{E,\eta})$ or of $(\mathcal{P}_{F,\eta})$ with $E, F \in \{I, II, III\}$. We assume (17) and (21) fulfilled. Then there exist $\lambda^0 \in R^1$ and $(p_t)_{t\geq 1}$ a sequence in R^{n*} which satisfy the following conditions.*

(i): *(λ^0, p_1) is non zero.*

(ii): $\lambda^0 \geq 0$.

(iii): *For all $t \in N_*$, $p_t \geq 0$ when $((\hat{x}_t)_t, (\hat{u}_t)_t)$ is an optimal solution of $(\mathcal{Q}_{E,\eta})$.*

(iv): *For all $t \in N_*$, $p_t = D_{x_t}H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1})$.*

(v): *For all $t \in N$, for all $u_t \in U_t$, $H_t(\hat{x}_t, \hat{u}_t, \lambda^0, p_{t+1}) \geq H_t(\hat{x}_t, u_t, \lambda^0, p_{t+1})$.*

Theorem 6.3 is a simplified version of Theorem 6.2 where the condition (21) is used instead of the condition (20); the condition (21) would be more easy to verify on explicit examples.

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Joël Blot
Laboratoire Marin Mersenne, Université Paris 1 Panthéon-Sorbonne
Centre Pierre-Mendès-France, 90 rue de Tolbiac
75643 Paris cedex 13, France

E-mail address: blot@univ-paris1.fr