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ASYMPTOTIC BEHAVIOR OF RESOLVENTS ON COMPLETE GEODESIC SPACES WITH NEGATIVE CURVATURE

YASUNORI KIMURA AND KEISUKE SHINDO

ABSTRACT. The resolvent for a proper lower semicontinuous convex function is one of the most important notions in the theory of convex analysis. Asymptotic behavior of resolvents is an important subject and there are a lot of results in various settings of underlying spaces. In this paper, we consider asymptotic behavior of resolvents on complete geodesic spaces with negative curvature.

1. Introduction

The resolvents for a proper lower semicontinuous convex function is one of the most important notions in the theory of convex analysis. The asymptotic behavior of resolvents is an important subject and there are a lot of results in various settings of underlying spaces. In a Hilbert space H, if $f\colon H \to]-\infty, \infty]$ is a proper lower semicontinuous convex function, then a resolvent J_λ is defined by

$$J_\lambda x = \operatorname*{argmin}_{y \in H} \left\{ f(y) + \frac{1}{\lambda} \|y - x\|^2 \right\}$$

for all $x \in H$. We consider the limit $J_\lambda x$ as λ tends to ∞ and 0. The following are well-known facts. See [4], for instance.

Theorem 1.1. *Let H be a Hilbert space and $f\colon H \to]-\infty, \infty]$ a proper lower semicontinuous convex function. If $\{J_{\mu_n} x\}_{n\in\mathbb{N}}$ is bounded for some sequence $\{\mu_n\} \subset \mathbb{R}$ such that $\mu_n \to \infty$, then* $\operatorname{argmin} f \neq \emptyset$ *and*

$$\lim_{\lambda\to\infty} J_\lambda x = P_{\operatorname{argmin} f}\, x.$$

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Theorem 1.2. *Let H be a Hilbert space and $f\colon H \to]-\infty, \infty]$ a proper lower semicontinuous convex function. Then*

$$\lim_{\lambda \to +0} J_\lambda x = P_{\overline{\mathrm{dom}\, f}} x.$$

Furthermore, a complete CAT(0) space is a generalization of Hilbert spaces and under the same condition as a function f on a Hilbert space, a resolvent on a complete CAT(0) space is defined by

$$J_\lambda x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{\lambda} d(y, x)^2 \right\}$$

for all $x \in X$, where the function $d(\cdot, x)^2$ is called a perturbation function. Asymptotic behavior of this resolvent is considered by [1] and the same results as the case of Hilbert spaces are obtained. On the other hand, a complete CAT(-1) space is an example of CAT(0) spaces. On a complete CAT(-1) space, a resolvent with another perturbation is defined by [3]. Its perturbation function is $\tanh d(x, \cdot) \sinh d(x, \cdot)$. In this paper, we consider the asymptotic behavior of this resolvent defined on complete CAT(-1) spaces.

2. Preliminaries

Let X be a metric space. For $x, y \in X$, a mapping $c\colon [0, l] \to X$ is called a geodesic with endpoints x, y if c satisfies $c(0) = x$, $c(l) = y$ and $d(c(u), c(v)) = |u - v|$ for $u, v \in [0, l]$. X is called a geodesic space, if there exists a geodesic for any $x, y \in X$. Moreover, if a geodesic segment exists uniquely for each $x, y \in X$, then X is called a uniquely geodesic space. In what follows, we always assume that X is a uniquely geodesic space. We call the image of c a geodesic segment joining x and y, and denote it by $[x, y]$. Then, for $t \in [0, 1]$ and $x, y \in X$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = (1-t)d(x, y)$ and $d(y, z) = td(x, y)$. We denote it by $tx \oplus (1 - t)y$. A geodesic triangle $\triangle(x_1, x_2, x_3)$ with vertices $x_1, x_2, x_3 \in X$ is the union of geodesic segments joining each pair of vertices. Let $\mathbb{H}^2$ be a two dimensional unit sphere. A comparison triangle $\overline{\triangle}(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{H}^2$ for $\triangle(x_1, x_2, x_3)$ is a triangle such that $d(x_i, x_j) = d_{\mathbb{H}^2}(\overline{x_i}, \overline{x_j})$ $(i, j = 1, 2, 3)$. A point $\overline{p} \in [\overline{x}_1, \overline{x}_2]$ is comparison point if $d(\overline{x}_1, p) = d_{\mathbb{H}^2}(\overline{x_1}, p)$. X is called a CAT(-1) space if for any $p, q \in \triangle(x_1, x_2, x_3)$, and their comparison points $\overline{p}, \overline{q} \in \overline{\triangle}(\overline{x_1}, \overline{x_2}, \overline{x_3})$, the inequality

$$d(p, q) \leq d_{\mathbb{H}^2}(\overline{p}, \overline{q})$$

is satisfied for all triangles in X.

The following inequality is a direct result from the hyperbolical law of cosines and a characterization of a CAT(-1) space; see [2].

Theorem 2.1. *Let X be a complete* CAT(-1) *space, $x, y, z \in X$, and t with $0 < t < 1$. Then*

$$\begin{aligned}&\cosh d(tx \oplus (1-t)y, z) \sinh d(x,y)\\ &\leq \cosh d(x,z) \sinh t d(x,y) + \cosh d(y,z) \sinh(1-t)d(x,y).\end{aligned}$$

In particular,

$$\cosh d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \cosh \frac{d(x,y)}{2} \leq \frac{1}{2}\cosh d(x,z) + \frac{1}{2}\cosh d(y,z).$$

This theorem is called the parallelogram law and from this result, we get the following lemma.

Lemma 2.2. *Let X be a complete* CAT(-1) *space. Then for all $x, y, z \in X$,*

$$\cosh d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \leq \frac{1}{2}\cosh d(x,z) + \frac{1}{2}\cosh d(y,z) - \left(1 - \frac{1}{\cosh \frac{d(x,y)}{2}}\right).$$

Proof. From Theorem 2.1, we have

$$\begin{aligned}&\cosh d\left(\frac{1}{2}x \oplus \frac{1}{2}y\right)\\ &\leq \frac{\frac{1}{2}\cosh d(x,z) + \frac{1}{2}\cosh d(y,z)}{\cosh \frac{d(x,y)}{2}}\\ &= \frac{1}{2}\cosh d(x,z) + \frac{1}{2}\cosh d(y,z)\\ &\quad - \left\{\frac{1}{2}\cosh d(x,z) + \frac{1}{2}\cosh d(y,z)\right\}\left(1 - \frac{1}{\cosh \frac{d(x,y)}{2}}\right)\\ &\leq \frac{1}{2}\cosh d(x,z) + \frac{1}{2}\cosh d(y,z) - \left(1 - \frac{1}{\cosh \frac{d(x,y)}{2}}\right).\end{aligned}$$

This is the desired result. □

Let X be a complete CAT(-1) space and C a nonempty closed convex subset of X. Then for each point $x \in X$, there exists a unique point $x_0 \in C$ such that $d(x, x_0) = d(x, C)$. A mapping P_C from X onto C such that $P_C x = x_0$ is called a metric projection. We know the following fact about metric projections.

Theorem 2.3. *Let X be a complete* CAT(-1) *space, C a nonempty closed convex subset of X, $x \in X$ and $y \in C$. Then,*

$$\cosh d(x, P_C x) \cosh d(P_C x, y) \leq \cosh d(x,y),$$

where P_C is the metric projection from X onto C.

Proof. Let $t \in [0,1]$. Since C is convex, we have $d(x, P_C x) \le d(x, tP_C x \oplus (1-t)y)$ and hence

$$\begin{aligned}
&\cosh d(x, P_C x) \sinh d(P_C x, y) \\
&\le \cosh d(x, tP_C x \oplus (1-t)y) \sinh d(P_C x, y) \\
&\le \cosh d(x, P_C x) \sinh t d(P_C x, y) + \cosh d(x, y) \sinh(1-t) d(P_C x, y).
\end{aligned}$$

Thus,

$$\begin{aligned}
\cosh d(x, P_C x)\{\sinh d(P_C x, y) - \sinh t d(P_C x, y)\}& \\
\le \cosh d(x, y) \sinh(1-t) d(P_C x, y)&.
\end{aligned}$$

Since

$$\sinh d(P_C x, y) - \sinh t d(P_C x, y) = 2 \cosh \frac{(1+t) d(P_C x, y)}{2} \sinh \frac{(1-t) d(P_C x, y)}{2},$$

we have

$$\begin{aligned}
&\cosh d(x, P_C x) \cosh \frac{(1+t) d(P_C x, y)}{2} \sinh(1-t) d(P_C x, y) \\
&\le 2 \cosh d(x, P_C x) \cosh \frac{(1+t) d(P_C x, y)}{2} \sinh \frac{(1-t) d(P_C x, y)}{2} \\
&= \cosh d(x, P_C x)\{\sinh d(P_C x, y) - \sinh t d(P_C x, y)\} \\
&\le \cosh d(x, y) \sinh(1-t) d(P_C x, y).
\end{aligned}$$

Dividing by $\sinh(1-t) d(P_C x, y)$ and $t \to 1$, we have

$$\cosh d(x, P_C x) \cosh d(P_C x, y) \le \cosh d(x, y).$$

This is the desired result. □

Lemma 2.4. *Let X be a complete* CAT(-1) *space, C a closed subset of X, and $x \in X$. Then*

$$\begin{aligned}
&\cosh d(x, P_C x) \cosh d(P_C x, y) - \frac{1}{\cosh d(x, P_C x) \cosh d(P_C x, y)} \\
&\le \cosh d(x, y) - \frac{1}{\cosh d(x, y)}.
\end{aligned}$$

Proof. By Theorem 2.3,

$$\cosh d(x, P_C x) \cosh d(P_C x, y) \le \cosh d(x, y).$$

Thus

$$-\frac{1}{\cosh d(x, P_C x) \cosh d(P_C x, y)} \le -\frac{1}{\cosh d(x, y)}.$$

Then we have

$$\begin{aligned}&\cosh d(x, P_C x) \cosh d(P_C x, y) - \frac{1}{\cosh d(x, P_C x) \cosh d(P_C x, y)}\\&\le \cosh d(x, y) - \frac{1}{\cosh d(x, y)}.\end{aligned}$$

This is the desired result. □

We consider a resolvent on a complete $\mathrm{CAT}(-1)$ space. Let $f\colon X \to]-\infty, \infty]$ be a proper lower semicontinuous convex function. Then for a positive real number λ, the resolvent for f of a perturbation $\tanh(\cdot)\sinh(\cdot)$ with parameter λ is defined by

$$R_\lambda x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{\lambda} \tanh d(x, y) \sinh d(x, y) \right\}.$$

It is proved that R_λ are well defined as a single-valued mapping; see [2,3]. Since f is proper, there exists a point y such that $f(y) < \infty$. That is, $R_\lambda x \in \operatorname{dom} f = \{x \in X : f(x) < \infty\}$. The following are properties of a resolvent R_λ.

Lemma 2.5. *Let X be a complete $\mathrm{CAT}(-1)$ space and $f\colon X \to]-\infty, \infty]$ a proper lower semicontinuous convex function. For $\lambda > 0$, define $R_\lambda\colon X \to X$ by*

$$R_\lambda x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{\lambda} \tanh d(y, x) \sinh d(y, x) \right\}$$

for $x \in X$. Then for $\lambda, \mu > 0$ with $\lambda \le \mu$,

$$\tanh d(R_\lambda x, x) \sinh d(R_\lambda x, x) \le \tanh d(R_\mu x, x) \sinh d(R_\mu x, x),$$

which is equivalent to

$$d(R_\lambda x, x) \le d(R_\mu x, x).$$

Proof. By the definition of the resolvent, we have

$$\begin{aligned}f(R_\lambda x) + \frac{1}{\lambda} \tanh d(R_\lambda x, x) \sinh d(R_\lambda x, x) &\le f(R_\mu x)\\&\quad + \frac{1}{\lambda} \tanh d(R_\mu x, x) \sinh d(R_\mu x, x)\end{aligned}$$

and

$$\begin{aligned}f(R_\mu x) + \frac{1}{\mu} \tanh d(R_\mu x, x) \sinh d(R_\mu x, x) &\le f(R_\lambda x)\\&\quad + \frac{1}{\mu} \tanh d(R_\lambda x, x) \sinh d(R_\lambda x, x).\end{aligned}$$

From these inequalities, we have

$$\begin{aligned}&\frac{1}{\lambda}\tanh d(R_\lambda x,x)\sinh d(R_\lambda x,x)+\frac{1}{\mu}\tanh d(R_\mu x,x)\sinh d(R_\mu x,x)\\&\le\frac{1}{\lambda}\tanh d(R_\mu x,x)\sinh d(R_\mu x,x)+\frac{1}{\mu}\tanh d(R_\lambda x,x)\sinh d(R_\lambda x,x).\end{aligned}$$

and hence

$$\begin{aligned}&\left(\frac{1}{\lambda}-\frac{1}{\mu}\right)\tanh d(R_\lambda x,x)\sinh d(R_\lambda x,x)\\&\qquad\qquad\le\left(\frac{1}{\lambda}-\frac{1}{\mu}\right)\tanh d(R_\mu x,x)\sinh d(R_\mu x,x).\end{aligned}$$

Since $\lambda\le\mu$, we obtain

$$\tanh d(R_\lambda x,x)\sinh d(R_\lambda x,x)\le\tanh d(R_\mu x,x)\sinh d(R_\mu x,x),$$

which is the desired result. □

Lemma 2.6. *Let X be a complete* CAT(−1) *space and $f\colon X\to\,]-\infty,\infty]$ a proper lower semicontinuous convex function. For $\lambda>0$, define $R_\lambda\colon X\to X$ by*

$$R_\lambda x=\operatorname*{argmin}_{y\in X}\left\{f(y)+\frac{1}{\lambda}\tanh d(y,x)\sinh d(y,x)\right\}$$

for $x\in X$. Then for $\lambda,\mu>0$ with $\lambda\le\mu$,

$$f(R_\lambda x)\ge f(R_\mu x).$$

Proof. By the definition of the resolvent, we have

$$\begin{aligned}f(R_\mu x)+\frac{1}{\mu}\tanh d(R_\mu x,x)\sinh d(R_\mu x,x)&\le f(R_\lambda x)\\&\quad+\frac{1}{\mu}\tanh d(R_\lambda x,x)\sinh d(R_\lambda x,x),\end{aligned}$$

and therefore we have

$$\begin{aligned}&f(R_\mu x)-f(R_\lambda x)\\&\le\frac{1}{\mu}\left\{\tanh d(R_\lambda x,x)\sinh d(R_\lambda x,x)-\tanh d(R_\mu x,x)\sinh d(R_\mu x,x)\right\}\le 0.\end{aligned}$$

This is the desired result. □

Lemma 2.7. *Let X be a complete* CAT(-1) *space and $f\colon X \to]-\infty, \infty]$ a proper lower semicontinuous convex function. For $\lambda > 0$, define $R_\lambda\colon X \to X$ by*

$$R_\lambda x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{\lambda} \tanh d(y,x) \sinh d(y,x) \right\}$$

for $x \in X$. If $\{R_{\mu_n} x\}_{n\in\mathbb{N}}$ is bounded for some sequence $\{\mu_n\} \subset \mathbb{R}$ such that $\mu_n \to \infty$, Then for all increasing sequences $\{\lambda_n\}$ such that $\lambda_n \to \infty$, $\{R_{\lambda_n} x\}_{n\in\mathbb{N}}$ is bounded.

Proof. Since $\{R_{\mu_n} x\}_{n\in\mathbb{N}}$ is bounded, There exists $M > 0$ such that for all $n \in \mathbb{N}$

$$d(R_{\mu_n} x, x) \le M.$$

Let $\{\lambda_n\}$ be an increasing sequence such that $\lambda_n \to \infty$. Suppose $\{R_{\lambda_n} x\}_{n\in\mathbb{N}}$ is not bounded. Then there exists a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ such that

$$d(R_{\lambda_{n_i}} x, x) > M.$$

for all $i \in \mathbb{N}$. Then there exists $l \in \mathbb{N}$ such that $\lambda_{n_1} \le \mu_l$. From Lemma 2.5, we have

$$M < d(R_{\lambda_{n_1}} x, x) \le d(R_{\mu_l} x, x) \le M.$$

This is a contradiction and we have the desired result. ∎

3. Main Theorems

First we consider the asymptotic behavior at infinity of a resolvent R_λ on CAT(-1) spaces.

Theorem 3.1. *Let X be a complete* CAT(-1) *space and $f\colon X \to]-\infty, \infty]$ a proper lower semicontinuous convex function. For $\lambda > 0$, define $R_\lambda\colon X \to X$ by*

$$R_\lambda x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{\lambda} \tanh d(y,x) \sinh d(y,x) \right\}$$

for $x \in X$. If $\{R_{\mu_n} x\}_{n\in\mathbb{N}}$ is bounded for some sequence $\{\mu_n\} \subset \mathbb{R}$ such that $\mu_n \to \infty$, then $\operatorname{argmin} f \ne \emptyset$ *and*

$$\lim_{\lambda\to\infty} R_\lambda x = P_{\operatorname{argmin} f} x.$$

Proof. Let $\{\lambda_n\} \subset \mathbb{R}$ be an increasing sequence such that $\lambda_n \to \infty$ and denote $R_{\lambda_n} x$ by x_n. First we show that $\operatorname{argmin} f$ is nonempty. Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded by Lemma 2.7, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}_{n\in\mathbb{N}}$ and $x_0 \in X$ such that $\{x_{n_i}\}$ is Δ-convergent to x_0. For all $y \in X$,

$$f(x_{n_i}) + \frac{1}{\lambda_{n_i}} \tanh d(x_{n_i}, x) \sinh d(x_{n_i}, x) \le f(y) + \frac{1}{\lambda_{n_i}} \tanh d(y,x) \sinh d(y,x).$$

Tending $i \to \infty$, we have

$$\liminf_{i\to\infty} f(x_{n_i}) \le f(y).$$

Since f is Δ-lower semicontinuous,

$$f(x_0) \le \liminf_{i\to\infty} f(x_{n_i}).$$

Therefore $f(x_0) \le f(y)$ for all $y \in X$, and hence we obtain $\operatorname{argmin} f \neq \emptyset$.

Next we show $x_n \to P_{\operatorname{argmin} f}x$. Suppose $n \le m$. From Lemma 2.2, we have

$$\begin{aligned}
&f(x_n) + \frac{1}{\lambda_n}\left\{\cosh d\,(x_n, x) - \frac{1}{\cosh d\,(x_n, x)}\right\}\\
&\le f\left(\frac{1}{2}x_n \oplus \frac{1}{2}x_m\right) + \frac{1}{\lambda_n}\left\{\cosh d\left(\frac{1}{2}x_n \oplus \frac{1}{2}x_m, x\right) - \frac{1}{\cosh d\left(\frac{1}{2}x_n \oplus \frac{1}{2}x_m, x\right)}\right\}\\
&\le \frac{1}{2}f(x_n) + \frac{1}{2}f(x_m)\\
&\quad + \frac{1}{\lambda_n}\left\{\frac{1}{2}\cosh d(x_n, x) + \frac{1}{2}\cosh d(x_m, x) - \left(1 - \frac{1}{\cosh \frac{d(x_n, x_m)}{2}}\right)\right\}\\
&\quad - \frac{1}{\lambda_n}\frac{1}{\frac{1}{2}\cosh d(x_n, x) + \frac{1}{2}\cosh d(x_m, x)}\\
&\le f(x_n) + \frac{1}{\lambda_n}\left\{\frac{1}{2}\cosh d(x_n, x) + \frac{1}{2}\cosh d(x_m, x) - \left(1 - \frac{1}{\cosh \frac{d(x_n, x_m)}{2}}\right)\right\}\\
&\quad - \frac{1}{\lambda_n}\frac{1}{\cosh d(x_m, x)},
\end{aligned}$$

and therefore

$$\begin{aligned}
&1 - \frac{1}{\cosh \frac{d(x_n, x_m)}{2}}\\
&\le \left\{\frac{1}{2}\cosh d(x_m, x) - \frac{1}{2}\cosh d(x_n, x)\right\} + \left\{\frac{1}{\cosh d\,(x_n, x)} - \frac{1}{\cosh d(x_m, x)}\right\}\\
&\le \{\cosh d(x_m, x) - \cosh d(x_n, x)\} + \left\{\frac{1}{\cosh d\,(x_n, x)} - \frac{1}{\cosh d(x_m, x)}\right\}\\
&= \left\{\cosh d(x_m, x) - \frac{1}{\cosh d\,(x_m, x)}\right\} - \left\{\cosh d(x_n, x) - \frac{1}{\cosh d\,(x_n, x)}\right\}\\
&= \tanh d(x_m, x)\sinh d(x_m, x) - \tanh d(x_n, x)\sinh d(x_n, x).
\end{aligned}$$

Since $\{\tanh d(x_n, x)\sinh d(x_n, x)\}_{n\in\mathbb{N}}$ is increasing and bounded, it is convergent. Moreover since it is a Cauchy sequence, there exists a sequence $\{\alpha_n\}$

such that $\alpha_n \to 0$ and

$$\tanh d(x_m, x) \sinh d(x_m, x) - \tanh d(x_n, x) \sinh d(x_n, x) \le \alpha_n \to 0.$$

Then we have

$$1 - \frac{1}{\cosh \frac{d(x_n, x_m)}{2}} \le \alpha_n.$$

This implies that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of X, $\{x_n\}_{n\in\mathbb{N}}$ is convergent. Suppose $x_n \to p$. Then from the former part of this proof, $p \in \operatorname{argmin} f$ and we show $p = P_{\operatorname{argmin} f} x$. For all $y \in \operatorname{argmin} f$,

$$\begin{aligned}
&f(x_n) + \frac{1}{\lambda_n}\left\{\cosh d\left(x_n, x\right) - \frac{1}{\cosh d\left(x_n, x\right)}\right\} \\
&\le f\left(\frac{1}{2}x_n \oplus \frac{1}{2}y\right) + \frac{1}{\lambda_n}\left\{\cosh d\left(\frac{1}{2}x_n \oplus \frac{1}{2}y, x\right) - \frac{1}{\cosh d\left(\frac{1}{2}x_n \oplus \frac{1}{2}y, x\right)}\right\} \\
&\le f(x_n) \\
&\quad + \frac{1}{\lambda_n}\left\{\frac{1}{2}\cosh d(x_n, x) + \frac{1}{2}\cosh d(y, x) - \left(1 - \frac{1}{\cosh \frac{d(x_n, y)}{2}}\right) - \frac{1}{\cosh d(y, x)}\right\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
0 &\le 1 - \frac{1}{\cosh \frac{d(x_n, y)}{2}} \\
&\le \left\{\frac{1}{2}\cosh d\left(y, x\right) - \frac{1}{\cosh d\left(y, x\right)}\right\} - \left\{\frac{1}{2}\cosh d\left(x_n, x\right) - \frac{1}{\cosh d\left(x_n, x\right)}\right\}.
\end{aligned}$$

This implies $d(x_n, x) \le d(y, x)$ and tending $n \to \infty$, we obtain that

$$d(p, x) \le d(y, x)$$

for all $y \in \operatorname{argmin} f$. This is the desired result. □

Next we consider the asymptotic behavior of R_λ at 0.

Theorem 3.2. *Let X be a complete* CAT(-1) *space and $f\colon X \to]-\infty, \infty]$ a proper lower semicontinuous convex function. For $\lambda > 0$, define $R_\lambda\colon X \to X$ by*

$$R_\lambda x = \operatorname*{argmin}_{y\in X}\left\{f(y) + \frac{1}{\lambda}\tanh d(y, x) \sinh d(y, x)\right\}$$

for $x \in X$. Then

$$\lim_{\lambda\to+0} R_\lambda x = P_{\overline{\operatorname{dom} f}} x.$$

Proof. Let $\{\lambda_n\} \subset \mathbb{R}$ be a decreasing sequence such that $\lambda_n \to 0$ and $\{z_m\} \subset X$ such that $z_m \to P_{\overline{\operatorname{dom} f}} x$. Put $x_n = R_{\lambda_n} x$ and $p = P_{\overline{\operatorname{dom} f}} x$. From the Lemma 2.4,

$$\begin{aligned}
& f(x_n) + \frac{1}{\lambda_n}\left\{\cosh d(x_n,p)\cosh d(p,x) - \frac{1}{\cosh d(x_n,p)\cosh d(p,x)}\right\} \\
& \le f(x_n) + \frac{1}{\lambda_n}\left\{\cosh d(x_n,x) - \frac{1}{\cosh d(x_n,x)}\right\} \\
& \le f(z_m) + \frac{1}{\lambda_n}\left\{\cosh d(z_m,x) - \frac{1}{\cosh d(z_m,x)}\right\}.
\end{aligned}$$

Then

$$\begin{aligned}
\cosh d(x_n,p)\cosh d(p,x) \le{}& \lambda_n(f(z_m) - f(x_n)) + \cosh d(z_m,x) \\
& - \left\{\frac{1}{\cosh d(z_m,x)} - \frac{1}{\cosh d(x_n,p)\cosh d(p,x)}\right\}.
\end{aligned}$$

By the monotonicity of $\{f(x_n)\}_{n\in\mathbb{N}}$, there exists $\alpha \in \mathbb{R}$ such that

$$\begin{aligned}
\cosh d(x_n,p)\cosh d(p,x) &\le \lambda_n(f(z_m) - \alpha) + \cosh d(z_m,x) \\
&\quad - \left\{\frac{1}{\cosh d(z_m,x)} - \frac{1}{\cosh d(x_n,p)\cosh d(p,x)}\right\} \\
&\le \lambda_n(f(z_m) - \alpha) + \cosh d(z_m,x) \\
&\quad - \left\{\frac{1}{\cosh d(z_m,x)} - \frac{1}{\cosh d(p,x)}\right\}.
\end{aligned}$$

Dividing by $\cosh d(p,x)$ and tending $n \to \infty$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \cosh d(x_n,p) \le{}& \frac{\cosh d(z_m,x)}{\cosh d(p,x)} \\
& - \left\{\frac{1}{\cosh d(z_m,x)} - \frac{1}{\cosh d(p,x)}\right\}\frac{1}{\cosh d(p,x)}.
\end{aligned}$$

Tending $m \to \infty$, we have

$$\limsup_{n\to\infty} \cosh d(x_n,p) \le 1.$$

and therefore $d(x_n,p) \to 0$. Then we obtain

$$\lim_{n\to\infty} R_{\lambda_n} x = p.$$

Since $\{R_{\lambda_n} x\}_{n\in\mathbb{N}}$ converges to p for any decreasing sequences converging to 0, we conclude $\lim_{\lambda\to 0} R_\lambda x = p$, which is the desired result. □

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Y. Kimura
Department of Information Science, Toho University, Miyama, Funabashi, Chiba, 274-8510, Japan
E-mail address: yasunori@is.sci.toho-u.ac.jp

K. Shindo
Department of Information Science, Toho University, Miyama, Funabashi, Chiba, 274-8510, Japan
E-mail address: 7520001s@st.toho-u.jp