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A HALPERN'S ITERATIVE SCHEME WITH MULTIPLE ANCHOR POINTS IN COMPLETE GEODESIC SPACES WITH CURVATURE BOUNDED ABOVE

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Abstract. In this paper, we define a new convex combination specific to geodesic spaces with curvature bounded above by one, and we show an approximation theorem for finding a common fixed point of mappings using a Halpern's iterative scheme with multiple anchor points on complete CAT(1) spaces.

1. Introduction

Approximating a common fixed point of mappings is one of the important topics in convex analysis and it has been studied in various spaces by many mathematicians. Halpern's iterative scheme is one of the popular methods to find a common fixed point of mappings; see [2, 10, 9]. In 2010, Saejung [7] proved a Halpern type approximation theorem with a nonexpansive mapping in a complete CAT(0) space. In 2013, Kimura and Satô [5] obtained a convergence theorem using a Halpern type iteration with a strongly quasinonexpansive and Δ-demiclosed mapping in a complete CAT(1) space. In 2015, Kimura and Wada [6] showed that a Halpern type iteration with nonexpansive mappings and multiple anchor points converges to a common fixed points of their mappings in a complete CAT(0) space.

Theorem 1.1 (Kimura and Wada [6])**.** *Let X be a complete* CAT(0) *space and R, S, T nonexpansive mappings from X into itself with $F = F(R) \cap F(S) \cap F(T) \neq \varnothing$. Let $\{\alpha_n\} \subset]0,1[$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\{\beta_n\}, \{\gamma_n\} \subset]a,b[\subset]0,1[$ such that $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\lim_{n\to\infty} \beta_n = \beta \in]0,1[$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ and $\lim_{n\to\infty} \gamma_n =$*

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$\gamma \in]0,1[$. *Let* $u, v, w, x_1 \in X$ *and define an iterative sequence* $\{x_n\} \subset X$ *by*

$$\begin{cases} r_n = \alpha_n u \oplus (1-\alpha_n) R x_n, \\ s_n = \alpha_n v \oplus (1-\alpha_n) S x_n, \\ t_n = \alpha_n w \oplus (1-\alpha_n) T x_n, \\ x_{n+1} = \beta_n r_n \oplus (1-\beta_n)(\gamma_n s_n \oplus (1-\gamma_n) t_n) \end{cases}$$

for all $n \in \mathbb{N}$. *Then the sequence* $\{x_n\}$ *converges to a point* $p \in F$, *which is a minimizer of the function* $g(x) = \beta d(u,x)^2 + (1-\beta)(\gamma d(v,x)^2 + (1-\gamma) d(w,x)^2)$ *on* F.

In this theorem, the function g can be defined as

$$g(x) = \lambda d(u,x)^2 + \mu d(v,x)^2 + \nu d(w,x)^2,$$

where $\lambda, \mu, \nu > 0$ satisfy $\lambda + \mu + \nu = 1$. This function is inspired by the following inequality, which is obtained a CAT(0) space X: for any $x, y, z \in X$ and $\alpha \in [0,1]$,

$$d(\alpha x \oplus (1-\alpha) y, z)^2 \leq \alpha d(x,z)^2 + (1-\alpha) d(y,z)^2.$$

On the other hand, if X is a CAT(-1) space, the following inequality holds for any $x, y, z \in X$ and $\alpha \in [0,1]$:

$$\cosh d(\alpha x \oplus (1-\alpha) y, z) \leq \alpha \cosh d(x,z) + (1-\alpha) \cosh d(y,z).$$

From this observation, it seems to be appropriate to use the following function in CAT(-1) spaces:

$$h(x) = \lambda \cosh d(u,x) + \mu \cosh d(v,x) + \nu \cosh d(w,x).$$

It is known that all CAT(-1) spaces are also CAT(0) space. Therefore, the sequence generated by the same method as in Theorem 1.1 converges to the same point p also in a CAT(-1) space. That point is a minimizer of the function g, however, it is not a minimizer of the following function:

$$h(x) = \beta \cosh d(u,x) + (1-\beta)(\gamma \cosh d(v,x) + (1-\gamma) \cosh d(w,x)).$$

Similarly, the limit p of the iteration in Theorem 1.1 does not maximize the following function:

$$i(x) = \beta \cos d(u,x) + (1-\beta)(\gamma \cos d(v,x) + (1-\gamma) \cos d(w,x)).$$

To solve this problem, we need to redefine the notion of convex combination in a different way.

In this paper, we define a new convex combination on a CAT(1) space in order to resolve that problem, and show that a sequence generated by using that convex combination converges to a maximizer of i in CAT(1) spaces.

2. Preliminaries

Let (X,d) be a metric space. For $x,y \in X$, a mapping $\gamma\colon [0,l] \to X$ is called a geodesic joining x and y if γ satisfies $\gamma(0)=x$, $\gamma(l)=y$ and $d(\gamma(s),\gamma(t)) = |s-t|$ for $s,t \in [0,l]$, where $l = d(x,y)$. For $D \in \,]0,\infty]$, X is said to be a D-geodesic space if for any two points $x,y \in X$ satisfying $d(x,y) < D$, there exists a geodesic joining x and y. Furthermore, if a geodesic exists uniquely for any two points $x,y \in X$ such that $d(x,y) < D$, then X is called a uniquely D-geodesic space. In a uniquely D-geodesic space, the image of a geodesic joining x and y is said to be a geodesic segment and is denoted by $[x,y]$.

Let X be a uniquely D-geodesic space. For $x,y \in X$ with $d(x,y) < D$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that $d(x,z) = (1-t)d(x,y)$ and $d(y,z) = td(x,y)$. The point z is called a convex combination of x and y, and is denoted by $tx \oplus (1-t)y$. For three points $x,y,z \in X$, a geodesic triangle $\triangle(x,y,z) \subset X$ is defined by the union of geodesic segments joining each two points. A subset $C \subset X$ is said to be convex if $[x,y] \subset C$ for all $x,y \in C$.

For $\kappa \in \mathbb{R}$, let M_κ be a two-dimensional model space with curvature κ. In particular, M_0 is the two-dimensional Euclidean space $\mathbb{R}^2$, M_1 is the two-dimensional unit sphere $\mathbb{S}^2$, and M_{-1} is the two-dimensional hyperbolic space $\mathbb{H}^2$. The diameter of M_κ is denoted by D_κ, that is, $D_\kappa = \infty$ for $\kappa \le 0$ and $D_\kappa = \pi/\sqrt{\kappa}$ otherwise.

Let $\kappa \in \mathbb{R}$ and let X be a uniquely D_κ-geodesic space. For a geodesic triangle $\triangle(x,y,z) \subset X$ with $d(x,y)+d(y,z)+d(z,x) < 2D_\kappa$, a comparison triangle $\overline{\triangle}(\bar{x},\bar{y},\bar{z}) \subset M_\kappa$ is defined by $[\bar{x},\bar{y}] \cup [\bar{y},\bar{z}] \cup [\bar{z},\bar{x}]$, where $\bar{x},\bar{y},\bar{z}$ are comparison points on M_κ which satisfies $d(x,y) = d(\bar{x},\bar{y})$, $d(y,z) = d(\bar{y},\bar{z})$, and $d(z,x) = d(\bar{z},\bar{x})$. X is called a CAT(κ) space if for any two points $p,q \in \triangle(x,y,z)$ and their comparison points $\bar{p},\bar{q} \in \overline{\triangle}(\bar{x},\bar{y},\bar{z})$, the inequality $d(p,q) \le d(\bar{p},\bar{q})$, which is called a CAT($\kappa$) inequality, is satisfied for any $\triangle(x,y,z) \subset X$ and its comparison triangle $\overline{\triangle}(\bar{x},\bar{y},\bar{z}) \subset M_\kappa$. It is well known that any CAT(κ) space is also a CAT(κ') space whenever $\kappa < \kappa'$. A CAT(κ) space X is said to be admissible if $d(x,y) < D_\kappa/2$ for any $x,y \in X$.

Let X be an admissible CAT(1) space. Then the following inequality always holds for any $x,y,z \in X$ and $\alpha \in [0,1]$:

$$\begin{aligned}\cos d(\alpha x \oplus (1-\alpha)y, z) \sin d(x,y) \\ \ge \cos d(x,z)\sin(\alpha d(x,y)) + \cos d(y,z)\sin((1-\alpha)d(x,y)).\end{aligned}$$

This inequality is often called the parallelogram law on CAT(1) spaces. The following inequality is easily obtained by this inequality:

$$\cos d(\alpha x \oplus (1-\alpha)y, z) \ge \alpha \cos d(x,z) + (1-\alpha)\cos d(y,z).$$

Let C be a nonempty set. For $f\colon C \to \mathbb{R}$, the set of all maximizers and all minimizers of f is denoted by $\operatorname{argmax}_{x\in C} f(x)$ and $\operatorname{argmin}_{x\in C} f(x)$, respectively. In this paper, if $\operatorname{argmax}_{x\in C} f(x)$ consists of exactly one point p, it is denoted by $p = \operatorname{argmax}_{x\in C} f(x)$.

Let X be a set and C a nonempty subset of X. For $T\colon C \to X$, the set of all fixed points of T is denoted by $F(T)$.

Let X be a metric space. An asymptotic center of a sequence $\{x_n\} \subset X$ is defined by $\operatorname{argmin}_{x\in X}(\limsup_{n\to\infty} d(x, x_n))$. If the asymptotic center of any subsequences of $\{x_n\}$ is just one point $x \in X$, then $\{x_n\}$ is said to Δ-converge to x, and we denote it by $x_n \overset{\Delta}{\rightharpoonup} x$. A mapping T from X into itself is said to be Δ-demiclosed if for any sequences $\{x_n\} \subset X$ with $x_n \overset{\Delta}{\rightharpoonup} x$, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ implies $x \in F(T)$.

Theorem 2.1 (Espínola and Fernández-León [3])**.** *Let X be a complete* CAT(1) *space. If a sequence $\{x_n\} \subset X$ satisfies $\inf_{y\in X} \limsup_{n\to\infty} d(x_n, y) < \pi/2$, then there exists a Δ-convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$.*

Theorem 2.2 (He, Fang, Lopez and Li [4])**.** *Let X be a complete* CAT(1) *space and $\{x_n\}$ a sequence on X such that $x_n \overset{\Delta}{\rightharpoonup} x \in X$. Then for any $u \in X$ with $\limsup_{n\to\infty} d(u, x_n) < \pi/2$, the following inequality holds:*

$$d(u, x) \le \liminf_{n\to\infty} d(u, x_n).$$

Let X be a CAT(1) space and T a mapping from X into itself with $F(T) \neq \varnothing$. T is said to be quasinonexpansive if it satisfies $d(Tx, z) \le d(x, z)$ for all $x \in X$ and $z \in F(T)$. We know that the set of all fixed points of quasinonexpansive mapping is closed and convex. Further, T is said to be strongly quasinonexpansive if it is quasinonexpansive and, for any sequence $\{x_n\} \subset X$, $\lim_{n\to\infty}(d(x_n, z) - d(Tx_n, z)) = 0$ for some $z \in F(T)$ implies that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Let X be a complete CAT(1) space and C a nonempty closed convex subset of X. Then there exists a unique point $p_x \in C$ such that $d(x, p_x) = \inf_{y\in C} d(x, y)$ for each $x \in X$. We define a metric projection P_C from X onto C by $P_C x = p_x$ for any $x \in X$.

Next, we introduce some properties of trigonometric functions and their inverses. In this paper, we assume that $\sin^{-1}\colon [-1, 1] \to [-\pi/2, \pi/2]$, $\cos^{-1}\colon [-1, 1] \to [0, \pi]$, $\tan^{-1}\colon \mathbb{R} \to]-\pi/2, \pi/2[$ are inverses of the trigonometric functions.

Lemma 2.3. *For any $a \in \mathbb{R}$,*

$$\sin(\tan^{-1} a) = \frac{a}{\sqrt{1 + a^2}}, \quad \cos(\tan^{-1} a) = \frac{1}{\sqrt{1 + a^2}}.$$

Proof. For any $A \in \,]-\pi/2, \pi/2[$, we have $\sin A = (\tan A)/\sqrt{1+\tan^2 A}$, $\cos A = 1/\sqrt{1+\tan^2 A}$. So putting $A = \tan^{-1} a$, we get the desired result. □

Lemma 2.4. *For any $a, b \in \mathbb{R}$ with $ab > -1$,*

$$\tan^{-1} a - \tan^{-1} b = \tan^{-1} \frac{a-b}{1+ab}.$$

Proof. Let $a, b \in \mathbb{R}$ and suppose that $ab > -1$. From the definition of $\tan^{-1}$, we have $\tan^{-1} a - \tan^{-1} b \in \,]-\pi, \pi[$. Moreover, from Lemma 2.3, we obtain $\cos(\tan^{-1} a - \tan^{-1} b) = (1+ab)/(\sqrt{1+a^2}\sqrt{1+b^2}) > 0$ and hence $\tan^{-1} a - \tan^{-1} b \in \,]-\pi/2, \pi/2[$. Therefore, since $\tan(\tan^{-1} a - \tan^{-1} b) = (a-b)/(1+ab)$, we get the conclusion. □

The following is an important lemma that forms the basis of the proof of the main result.

Lemma 2.5 (Aoyama, Kimura and Kohsaka [1]; Saejung and Yotkaew [8])**.** *Let $\{a_n\}$ be a sequence of non-negative real numbers and $\{t_n\}$ a sequence of real numbers. Let $\{\beta_n\}$ be a sequence in $]0,1[$ such that $\sum_{n=1}^{\infty} \beta_n = \infty$. Suppose that $a_{n+1} \leq (1-\beta_n)a_n + \beta_n t_n$ for all $n \in \mathbb{N}$. If $\liminf_{i\to\infty} \left(a_{\varphi(i)+1} - a_{\varphi(i)}\right) \geq 0$ implies $\limsup_{i\to\infty} t_{\varphi(i)} \leq 0$ for any nondecreasing $\varphi\colon \mathbb{N} \to \mathbb{N}$ with $\lim_{i\to\infty} \varphi(i) = \infty$, then $a_n \to 0$.*

3. Main result

In this section, we prove a Halpern type approximation theorem with multiple anchor points for strongly quasinonexpansive and Δ-demiclosed mappings on admissible complete CAT(1) spaces. To prove the main result, we define a new convex combination and show that its properties.

Lemma 3.1. *Let $\{s_n\}, \{t_n\}, \{u_n\}$ be sequences of non-positive real numbers. Then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = \lim_{n\to\infty} u_n = 0$ whenever $\lim_{n\to\infty}(s_n + t_n + u_n) = 0$.*

Lemma 3.2. *Let X be an admissible* CAT(1) *space. For $u_1, u_2, u_3 \in X$ and $\beta_1, \beta_2, \beta_3 \in [0,1]$ with $\beta_1 + \beta_2 + \beta_3 = 1$, define a function $g\colon X \to \,]0,1]$ by*

$$g(x) = \beta_1 \cos d(u_1, x) + \beta_2 \cos d(u_2, x) + \beta_3 \cos d(u_3, x)$$

for all $x \in X$. Then for any $x, y \in X$,

$$g\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) \cos \frac{d(x,y)}{2} \geq \frac{g(x) + g(y)}{2}.$$

Proof. Let x, y be elements of X. It is obvious if $x = y$. Otherwise, we have

$$\begin{aligned} g\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) \sin d(x,y) &= \sum_{i=1}^{3} \beta_i \cos d\left(u_i, \frac{1}{2}x \oplus \frac{1}{2}y\right) \sin d(x,y) \\ &\geq \sum_{i=1}^{3} \beta_i (\cos d(u_i, x) + \cos d(u_i, y)) \sin \frac{d(x,y)}{2} \\ &= (g(x) + g(y)) \sin \frac{d(x,y)}{2}. \end{aligned}$$

Since $\sin d(x,y) = 2 \sin \frac{d(x,y)}{2} \cos \frac{d(x,y)}{2}$, we obtain the desired result. □

Lemma 3.3. *Let X be a complete admissible* CAT(1) *space and C a nonempty closed convex subset of X. For $u_1, u_2, u_3 \in X$ and $\beta_1, \beta_2, \beta_3 \in [0,1]$ with $\beta_1 + \beta_2 + \beta_3 = 1$, define a function $g \colon X \to]0,1]$ by*

$$g(x) = \beta_1 \cos d(u_1, x) + \beta_2 \cos d(u_2, x) + \beta_3 \cos d(u_3, x)$$

for all $x \in X$. Then g has a unique maximizer on C.

Proof. Put $L = \sup_{x \in C} g(x)$ and take a sequence $\{z_n\} \subset C$ with $L - 1/n \leq g(z_n) \leq L$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} g(z_n) = L$.

We show that $\{z_n\}$ is a Cauchy sequence on C. Let $m, n \in \mathbb{N}$ with $m \geq n$. From Lemma 3.2, we get $g(\frac{1}{2}z_m \oplus \frac{1}{2}z_n) \cos \frac{d(z_m, z_n)}{2} \geq (g(z_m) + g(z_n))/2$. Since $g\big(\frac{1}{2}z_m \oplus \frac{1}{2}z_n\big) \leq L \leq 1$,

$$\cos \frac{d(z_m, z_n)}{2} \geq \frac{g(z_m) + g(z_n)}{2g\big(\frac{1}{2}z_m \oplus \frac{1}{2}z_n\big)} \geq \frac{g(z_m) + g(z_n)}{2L} \geq \frac{L - 1/n}{L} \to 1 \quad (n \to \infty).$$

Therefore $\{z_n\}$ is a Cauchy sequence on C. From completeness of X and closedness of C, there exists $z \in C$ such that $z_n \to z$ and hence $g(z) = L = \sup_{x \in C} g(x)$. So z is a maximizer of g on C.

Next, we prove its uniqueness. Let $z, z' \in C$ satisfying $g(z) = g(z') = L$. From Lemma 3.2, we have

$$L \cos \frac{d(z,z')}{2} \geq g\left(\frac{1}{2}z \oplus \frac{1}{2}z'\right) \cos \frac{d(z,z')}{2} \geq \frac{g(z) + g(z')}{2} = L.$$

Since $L > 0$, we get $\cos \frac{d(z,z')}{2} \geq 1$ and hence $z = z'$. Therefore we get the conclusion. □

Lemma 3.4. *Let X be a uniquely geodesic space. Then for $u, v \in X$ with $0 < d(u,v) < \pi/2$ and $\beta \in [0,1]$,*

$$\sigma u \oplus (1 - \sigma) v = \operatorname*{argmax}_{x \in [u,v]} (\beta \cos d(u,x) + (1 - \beta) \cos d(v,x))$$

if and only if

$$\sigma = \frac{1}{d(u,v)} \tan^{-1} \frac{\beta \sin d(u,v)}{1-\beta+\beta\cos d(u,v)}.$$

Proof. It is obvious if $\beta = 0$ or $\beta = 1$. For $u, v \in X$ with $u \neq v$ and $\beta \in]0,1[$, put $d = d(u,v)$, $A = \operatorname{argmax}_{x\in[u,v]}(\beta\cos d(u,x) + (1-\beta)\cos d(v,x))$, $B = \operatorname{argmax}_{0\le t\le 1}(\beta\cos((1-t)d) + (1-\beta)\cos td)$, and $C = \operatorname{argmax}_{0\le t\le d}(\beta\cos(d-t) + (1-\beta)\cos t)$. Then the sets A, B and C consist of one point, respectively. We also have $A = \{tu \oplus (1-t)v \mid t \in B\}$ and $B = \operatorname{argmax}_{0\le t\le 1}(\beta\cos(d-td) + (1-\beta)\cos td) = \{\frac{1}{d}t \mid t \in C\}$.

Define a function $f\colon]-\pi/2,\pi/2[\to]0,1[$ by $f(t) = \beta\cos(d-t) + (1-\beta)\cos t$ for all $t \in]-\pi/2,\pi/2[$, then f is infinitely differentiable and $f'(0) > 0$, $f'(d) < 0$ and $f''(t) < 0$ for all $t \in [0,d]$. So there exists a unique real number t such that $f'(t) = 0$ and $t \in]0,d[$, that is, there exists a unique maximizer t of f on $]0,d[$ and it satisfies $f'(t) = 0$. Then we have $f'(t) = 0$ if and only if $t = \tan^{-1}\frac{\beta\sin d}{1-\beta+\beta\cos d}$.

Thus we get $C = \{\tan^{-1}\frac{\beta\sin d}{1-\beta+\beta\cos d}\}$ and $B = \{\frac{1}{d}\tan^{-1}\frac{\beta\sin d}{1-\beta+\beta\cos d}\}$. So putting $\sigma = \frac{1}{d}\tan^{-1}\frac{\beta\sin d}{1-\beta+\beta\cos d}$, we get $A = \{\sigma u \oplus (1-\sigma)v\}$, that is, $\sigma u \oplus (1-\sigma)v = \operatorname{argmax}_{x\in[u,v]}(\beta\cos d(u,x) + (1-\beta)\cos d(v,x))$. □

Lemma 3.5. *Let X be a uniquely geodesic space. Then for $u, v \in X$ with $d(u,v) < \pi/2$ and $\beta \in [0,1]$,*

$$\operatorname*{argmax}_{x\in[u,v]}(\beta\cos d(u,x) + (1-\beta)\cos d(v,x)) = \operatorname*{argmax}_{x\in X}(\beta\cos d(u,x) + (1-\beta)\cos d(v,x)).$$

Proof. If $u = v$, it is obvious. Let $u, v \in X$ with $u \neq v$ and $\beta \in [0,1]$, and define a function $f\colon X \to \mathbb{R}$ by $f(x) = \beta\cos d(u,x) + (1-\beta)\cos d(v,x)$ for all $x \in X$. Put $z = \operatorname{argmax}_{x\in[u,v]} f(x)$ and let $w \in X$. Further, put $t = d(v,w)/(d(u,w) + d(v,w))$ and $z' = tu \oplus (1-t)v \in [u,v]$. Then we get $f(z) \ge f(z')$ and $d(u,w) = (1-t)(d(u,w)+d(v,w)) \ge (1-t)d(u,v) = d(u,z')$. Similarly, we also have $d(v,w) \ge d(v,z')$. Therefore we get $f(z') \ge f(w)$ and hence $f(z) = \max_{w\in X} f(w)$. □

Using Lemma 3.4 and Lemma 3.5, we define a new convex combination.

Definition 3.6. Let X be a uniquely geodesic space. For $u, v \in X$ with $d(u,v) < \pi/2$ and $\alpha \in [0,1]$, we define a 1-convex combination of u and v by

$$\alpha u \overset{1}{\oplus} (1-\alpha)v \overset{\text{def}}{=} \operatorname*{argmax}_{x\in X}(\alpha\cos d(u,x) + (1-\alpha)\cos d(v,x)).$$

From Lemma 3.4 and Lemma 3.5, it can be expressed by $\alpha u \overset{1}{\oplus} (1-\alpha)v = \sigma u \oplus (1-\sigma)v$, where $\sigma = \frac{1}{d(u,v)} \tan^{-1} \frac{\alpha \sin d(u,v)}{1-\alpha+\alpha\cos d(u,v)}$ whenever $u \neq v$.

Lemma 3.7. *For any* $\alpha \in [0,1]$ *and* $d \in \,]0, \pi/2[$,

$$\frac{1}{d} \tan^{-1} \frac{\alpha \sin d}{1-\alpha+\alpha\cos d} + \frac{1}{d} \tan^{-1} \frac{(1-\alpha)\sin d}{\alpha+(1-\alpha)\cos d} = 1.$$

Proof. It is obvious if $\alpha = 0$ or $\alpha = 1$.

We consider the case where $\alpha \in \,]0,1[$. Let $\alpha \in \,]0,1[$ and $d \in \,]0,\pi/2[$. Using Lemma 2.4, we have

$$\begin{aligned} 1 - \frac{1}{d} \tan^{-1} \frac{\alpha \sin d}{1-\alpha+\alpha\cos d} &= \frac{1}{d} \left(\tan^{-1} \frac{\sin d}{\cos d} - \tan^{-1} \frac{\alpha \sin d}{1-\alpha+\alpha\cos d} \right) \\ &= \frac{1}{d} \tan^{-1} \frac{\frac{\sin d}{\cos d} - \frac{\alpha \sin d}{1-\alpha+\alpha\cos d}}{1 + \frac{\sin d}{\cos d} \cdot \frac{\alpha \sin d}{1-\alpha+\alpha\cos d}} \\ &= \frac{1}{d} \tan^{-1} \frac{(1-\alpha)\sin d}{\alpha+(1-\alpha)\cos d}. \end{aligned}$$

So we get the desired result. □

Lemma 3.8. *Let* X *be an admissible* CAT(1) *space and* $x, y, z \in X$, $\alpha \in [0,1]$. *Then*

$$\cos d(\alpha x \overset{1}{\oplus} (1-\alpha)y, z) \geq \frac{\alpha \cos d(x,z) + (1-\alpha)\cos d(y,z)}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cos d(x,y) + (1-\alpha)^2}}.$$

Proof. Let $x, y, z \in X$ and $\alpha \in [0,1]$. It is obvious if $x = y$. Suppose that $x \neq y$ and put

$$\sigma = \frac{1}{d(x,y)} \tan^{-1} \frac{\alpha \sin d(x,y)}{1-\alpha+\alpha\cos d(x,y)}.$$

From Lemma 3.4, Lemma 3.7 and Lemma 2.3, we have

$$\begin{aligned}
&\cos d(\alpha x \overset{1}{\oplus} (1-\alpha)y, z)\sin d(x,y)\\
&= \cos d(\sigma x \oplus (1-\sigma)y, z)\sin d(x,y)\\
&\geq \cos d(x,z)\sin\left(\tan^{-1}\frac{\alpha\sin d(x,y)}{1-\alpha+\alpha\cos d(x,y)}\right)\\
&\quad + \cos d(y,z)\sin\left(\tan^{-1}\frac{(1-\alpha)\sin d(x,y)}{\alpha+(1-\alpha)\cos d(x,y)}\right)\\
&= \cos d(x,z)\cdot\frac{\alpha\sin d(x,y)}{\sqrt{\alpha^2+2\alpha(1-\alpha)\cos d(x,y)+(1-\alpha)^2}}\\
&\quad + \cos d(y,z)\cdot\frac{(1-\alpha)\sin d(x,y)}{\sqrt{\alpha^2+2\alpha(1-\alpha)\cos d(x,y)+(1-\alpha)^2}}\\
&= \frac{\alpha\cos d(x,z)+(1-\alpha)\cos d(y,z)}{\sqrt{\alpha^2+2\alpha(1-\alpha)\cos d(x,y)+(1-\alpha)^2}}\cdot\sin d(x,y)
\end{aligned}$$

and hence

$$\cos d(\alpha x \overset{1}{\oplus} (1-\alpha)y, z) \geq \frac{\alpha\cos d(x,z)+(1-\alpha)\cos d(y,z)}{\sqrt{\alpha^2+2\alpha(1-\alpha)\cos d(x,y)+(1-\alpha)^2}}.$$

Thus we get the desired result. □

Corollary 3.9. *Let X be an admissible* CAT(1) *space and $x, y, z \in X$, $\alpha \in [0,1]$. Then*

$$\cos d(\alpha x \overset{1}{\oplus} (1-\alpha)y, z) \geq \alpha\cos d(x,z) + (1-\alpha)\cos d(y,z).$$

Proof. Since $\sqrt{\alpha^2+2\alpha(1-\alpha)\cos d(x,y)+(1-\alpha)^2} \leq 1$, Lemma 3.8 implies the conclusion. □

Lemma 3.10. *Let $\alpha \in]0,1[$ and $d \in [0,\pi/2[$. Define $\beta \in \mathbb{R}$ by*

$$\beta = 1 - \frac{1-\alpha}{\sqrt{\alpha^2+2\alpha(1-\alpha)\cos d+(1-\alpha)^2}}.$$

Then $\frac{\alpha^2}{2} < \beta < 1$.

Proof. Since $\alpha^2 + (1-\alpha)^2 < 1$, we get

$$1 - \frac{1-\alpha}{\sqrt{\alpha^2+2\alpha(1-\alpha)\cos d+(1-\alpha)^2}} \geq 1 - \frac{1-\alpha}{\sqrt{\alpha^2+(1-\alpha)^2}} > \frac{\alpha^2}{2}.$$

It is obvious that $\beta < 1$. □

Lemma 3.11. *Let X be an admissible* CAT(1) *space. Then for $u, y, z \in X$ and $\alpha \in]0,1[$,*

$$\begin{aligned}
&1 - \cos d(\alpha u \overset{1}{\oplus} (1-\alpha)y, z) \\
&\le (1-\beta)(1-\cos d(y,z)) \\
&\quad + \beta \left(1 - \frac{\left(1-\alpha+\sqrt{\alpha^2+2\alpha(1-\alpha)\cos d(u,y)+(1-\alpha)^2}\right)\cos d(u,z)}{\alpha+2(1-\alpha)\cos d(u,y)} \right),
\end{aligned}$$

where

$$\beta = 1 - \frac{1-\alpha}{\sqrt{\alpha^2+2\alpha(1-\alpha)\cos d(u,y)+(1-\alpha)^2}}.$$

Proof. It is obvious if $u = y$. Otherwise, from Lemma 3.8, we have

$$\begin{aligned}
&1 - \cos d(\alpha u \overset{1}{\oplus} (1-\alpha)y, z) \\
&\le 1 - (1-\beta)\cos d(y,z) - \frac{\alpha \cos d(u,z)}{\sqrt{\alpha^2+2\alpha(1-\alpha)\cos d(u,y)+(1-\alpha)^2}} \\
&= (1-\beta)(1-\cos d(y,z)) + \beta \left(1 - \frac{\alpha \cos d(u,z)}{\beta\sqrt{\alpha^2+2\alpha(1-\alpha)\cos d(u,y)+(1-\alpha)^2}} \right).
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{\alpha \cos d(u,z)}{\beta\sqrt{\alpha^2+2\alpha(1-\alpha)\cos d(u,y)+(1-\alpha)^2}} \\
&= \frac{\alpha \cos d(u,z)}{\sqrt{\alpha^2+2\alpha(1-\alpha)\cos d(u,y)+(1-\alpha)^2}-(1-\alpha)} \\
&= \frac{\left(1-\alpha+\sqrt{\alpha^2+2\alpha(1-\alpha)\cos d(u,y)+(1-\alpha)^2}\right)\cos d(u,z)}{\alpha+2(1-\alpha)\cos d(u,y)},
\end{aligned}$$

we get the conclusion. □

Lemma 3.12. *Let $\{\alpha_n\}$ be a sequence on $]0,1[$ such that $\lim_{n\to\infty} \alpha_n = 0$, and $\{s_n\}, \{t_n\}$ sequences on $[0,\pi/2[$ such that $\lim_{n\to\infty} s_n = d_1 \in [0,\pi/2[$, $\lim_{n\to\infty} t_n = d_2 \in [0,\pi/2[$. Define sequences $\{\sigma_n\}, \{\tau_n\} \subset]0,1[$ by*

$$\begin{aligned}
\sigma_n &= 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)\cos s_n+(1-\alpha_n)^2}}, \\
\tau_n &= 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)\cos t_n+(1-\alpha_n)^2}}
\end{aligned}$$

for all $n \in \mathbb{N}$, respectively. Then

$$\lim_{n\to\infty} \frac{\tau_n}{\sigma_n} = \frac{\cos d_2}{\cos d_1}.$$

Proof. Put

$$p_n = \sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cos s_n + (1-\alpha_n)^2},$$
$$q_n = \sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cos t_n + (1-\alpha_n)^2}$$

for all $n \in \mathbb{N}$. Then we have

$$\frac{\tau_n}{\sigma_n} = \frac{1 - \dfrac{1-\alpha_n}{\sqrt{{\alpha_n}^2 + 2\alpha_n(1-\alpha_n)\cos t_n + (1-\alpha_n)^2}}}{1 - \dfrac{1-\alpha_n}{\sqrt{{\alpha_n}^2 + 2\alpha_n(1-\alpha_n)\cos s_n + (1-\alpha_n)^2}}} = \frac{\dfrac{\alpha_n + 2(1-\alpha_n)\cos t_n}{q_n(q_n + 1 - \alpha_n)}}{\dfrac{\alpha_n + 2(1-\alpha_n)\cos s_n}{p_n(p_n + 1 - \alpha_n)}}.$$

Since $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = 1$, we get the conclusion. □

Now, we show the main result.

Theorem 3.13. *Let X be an admissible complete* CAT(1) *space and suppose that $\sup_{u,v\in X} d(u,v) < \pi/2$. Let R, S, T be strongly quasinonexpansive and Δ-demiclosed mappings from X into itself with $F = F(R) \cap F(S) \cap F(T) \neq \varnothing$. Let $\{\alpha_n\} \subset]0,1[$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$, and $\{\beta_n\}, \{\gamma_n\} \subset]0,1[$ such that $\lim_{n\to\infty} \beta_n = \beta \in]0,1[$, $\lim_{n\to\infty} \gamma_n = \gamma \in]0,1[$.*

Let $u, v, w, x_1 \in X$ and define a iterative sequence $\{x_n\} \subset X$ by

$$\begin{cases} r_n = \alpha_n u \overset{1}{\oplus} (1-\alpha_n)Rx_n, \\ s_n = \alpha_n v \overset{1}{\oplus} (1-\alpha_n)Sx_n, \\ t_n = \alpha_n w \overset{1}{\oplus} (1-\alpha_n)Tx_n, \\ x_{n+1} = \beta_n r_n \overset{1}{\oplus} (1-\beta_n)(\gamma_n s_n \overset{1}{\oplus} (1-\gamma_n)t_n) \end{cases}$$

for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges to a point $p \in F$, which is a maximizer of the function $g(x) = \beta \cos d(u,x) + (1-\beta)(\gamma \cos d(v,x) + (1-\gamma)\cos d(w,x))$ on F.

Proof. Since F is a closed convex subset of X and from Lemma 3.3, the existence and uniqueness of the elements of the set $\operatorname{argmax}_{x\in F} g(x)$ are guaranteed.

Let $p = \operatorname{argmax}_{x\in F} g(x)$ and put

$$
\begin{aligned}
a_n &= 1 - \cos d(x_n, p),\\
b_n^R &= 1 - \frac{\left(1-\alpha_n + \sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cos d(u, Rx_n) + (1-\alpha_n)^2}\right)\cos d(u,p)}{\alpha_n + 2(1-\alpha_n)\cos d(u, Rx_n)},\\
b_n^S &= 1 - \frac{\left(1-\alpha_n + \sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cos d(v, Sx_n) + (1-\alpha_n)^2}\right)\cos d(v,p)}{\alpha_n + 2(1-\alpha_n)\cos d(v, Sx_n)},\\
b_n^T &= 1 - \frac{\left(1-\alpha_n + \sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cos d(w, Tx_n) + (1-\alpha_n)^2}\right)\cos d(w,p)}{\alpha_n + 2(1-\alpha_n)\cos d(w, Tx_n)},\\
\gamma_n^R &= 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cos d(u, Rx_n) + (1-\alpha_n)^2}},\\
\gamma_n^S &= 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cos d(v, Sx_n) + (1-\alpha_n)^2}},\\
\gamma_n^T &= 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cos d(w, Tx_n) + (1-\alpha_n)^2}}
\end{aligned}
$$

for all $n \in \mathbb{N}$. Moreover, put $\beta_n^R = \beta_n$, $\beta_n^S = (1-\beta_n)\gamma_n$, $\beta_n^T = (1-\beta_n)(1-\gamma_n)$ for all $n \in \mathbb{N}$ and put $\beta^R = \beta$, $\beta^S = (1-\beta)\gamma$, $\beta^T = (1-\beta)(1-\gamma)$. Then $\{\gamma_n^R\}$, $\{\gamma_n^S\}$ and $\{\gamma_n^T\}$ are sequences on $]0,1[$. From Lemmas 3.8 and 3.11, we have

$$
\begin{aligned}
a_{n+1} &\le 1 - \beta_n \cos d(r_n, p) - (1-\beta_n)\cos d(\gamma_n s_n \overset{1}{\oplus} (1-\gamma_n)t_n, p)\\
&\le 1 - \beta_n^R \cos d(r_n, p) - \beta_n^S \cos d(s_n, p) - \beta_n^T \cos d(t_n, p)\\
&= \beta_n^R(1-\cos d(r_n,p)) + \beta_n^S(1-\cos d(s_n,p)) + \beta_n^T(1-\cos d(t_n,p))\\
&\le \beta_n^R\left((1-\gamma_n^R)(1-\cos d(Rx_n,p)) + \gamma_n^R b_n^R\right)\\
&\quad + \beta_n^S\left((1-\gamma_n^S)(1-\cos d(Sx_n,p)) + \gamma_n^S b_n^S\right)\\
&\quad + \beta_n^T\left((1-\gamma_n^T)(1-\cos d(Tx_n,p)) + \gamma_n^T b_n^T\right)\\
&\le \left(\beta_n^R(1-\gamma_n^R) + \beta_n^S(1-\gamma_n^S) + \beta_n^T(1-\gamma_n^T)\right) a_n\\
&\quad + \beta_n^R\gamma_n^R b_n^R + \beta_n^S\gamma_n^S b_n^S + \beta_n^T\gamma_n^T b_n^T\\
&= \left(1 - \left(\beta_n^R\gamma_n^R + \beta_n^S\gamma_n^S + \beta_n^T\gamma_n^T\right)\right) a_n\\
&\quad + \left(\beta_n^R\gamma_n^R + \beta_n^S\gamma_n^S + \beta_n^T\gamma_n^T\right)\cdot \frac{\beta_n^R\gamma_n^R b_n^R + \beta_n^S\gamma_n^S b_n^S + \beta_n^T\gamma_n^T b_n^T}{\beta_n^R\gamma_n^R + \beta_n^S\gamma_n^S + \beta_n^T\gamma_n^T}
\end{aligned}
$$

for all $n \in \mathbb{N}$.

Now we show that the following conditions hold:

(i) $\sum_{n=1}^{\infty}(\beta_n^R\gamma_n^R + \beta_n^S\gamma_n^S + \beta_n^T\gamma_n^T) = \infty$,

(ii) for any $\varphi\colon \mathbb{N} \to \mathbb{N}$ satisfying that φ is nondecreasing and $\lim_{i\to\infty}\varphi(i) = \infty$, $\liminf_{i\to\infty}\bigl(a_{\varphi(i)+1} - a_{\varphi(i)}\bigr) \geq 0$ implies

$$\limsup_{i\to\infty} \frac{\beta_{\varphi(i)}^R\gamma_{\varphi(i)}^R b_{\varphi(i)}^R + \beta_{\varphi(i)}^S\gamma_{\varphi(i)}^S b_{\varphi(i)}^S + \beta_{\varphi(i)}^T\gamma_{\varphi(i)}^T b_{\varphi(i)}^T}{\beta_{\varphi(i)}^R\gamma_{\varphi(i)}^R + \beta_{\varphi(i)}^S\gamma_{\varphi(i)}^S + \beta_{\varphi(i)}^T\gamma_{\varphi(i)}^T} \leq 0.$$

First, we show (i). From Lemma 3.10, we have $\gamma_n^R \geq \alpha_n^2/2$, $\gamma_n^S \geq \alpha_n^2/2$ and $\gamma_n^T \geq \alpha_n^2/2$. Thus we get

$$\sum_{n=1}^{\infty}(\beta_n^R\gamma_n^R + \beta_n^S\gamma_n^S + \beta_n^T\gamma_n^T) \geq \sum_{n=1}^{\infty}\frac{\beta_n^R\alpha_n^2 + \beta_n^S\alpha_n^2 + \beta_n^T\alpha_n^2}{2} = \sum_{n=1}^{\infty}\frac{\alpha_n^2}{2} = \infty.$$

Next, we consider (ii). Let φ be a nondecreasing function from $\mathbb{N}$ into itself with $\lim_{i\to\infty}\varphi(i) = \infty$ and put $n_i = \varphi(i)$ for all $i \in \mathbb{N}$. Assume $\liminf_{i\to\infty}(a_{n_i+1} - a_{n_i}) \geq 0$. Then we get

$$\begin{aligned}
0 &\leq \liminf_{i\to\infty}(a_{n_i+1} - a_{n_i}) \\
&= \liminf_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(x_{n_i+1}, p)) \\
&\leq \liminf_{i\to\infty}\bigl(\cos d(x_{n_i}, p) - \beta_{n_i}^R \cos d(r_{n_i}, p) - \beta_{n_i}^S \cos d(s_{n_i}, p) \\
&\quad - \beta_{n_i}^T \cos d(t_{n_i}, p)\bigr) \\
&\leq \liminf_{i\to\infty}\bigl(\cos d(x_{n_i}, p) - \beta_{n_i}^R(\alpha_{n_i}\cos d(u, p) + (1-\alpha_{n_i})\cos d(Rx_{n_i}, p)) \\
&\quad - \beta_{n_i}^S(\alpha_{n_i}\cos d(v, p) + (1-\alpha_{n_i})\cos d(Sx_{n_i}, p)) \\
&\quad - \beta_{n_i}^T(\alpha_{n_i}\cos d(w, p) + (1-\alpha_{n_i})\cos d(Tx_{n_i}, p))\bigr) \\
&= \liminf_{i\to\infty}\bigl(\beta^R(\cos d(x_{n_i}, p) - \cos d(Rx_{n_i}, p)) \\
&\quad + \beta^S(\cos d(x_{n_i}, p) - \cos d(Sx_{n_i}, p)) \\
&\quad + \beta^T(\cos d(x_{n_i}, p) - \cos d(Tx_{n_i}, p))\bigr) \\
&\leq \limsup_{i\to\infty}\bigl(\beta^R(\cos d(x_{n_i}, p) - \cos d(Rx_{n_i}, p)) \\
&\quad + \beta^S(\cos d(x_{n_i}, p) - \cos d(Sx_{n_i}, p)) \\
&\quad + \beta^T(\cos d(x_{n_i}, p) - \cos d(Tx_{n_i}, p))\bigr) \\
&\leq 0.
\end{aligned}$$

Thus we obtain

$$\lim_{i\to\infty}\big(\beta^R(\cos d(x_{n_i},p)-\cos d(Rx_{n_i},p))+\beta^S(\cos d(x_{n_i},p)-\cos d(Sx_{n_i},p))\\ +\beta^T(\cos d(x_{n_i},p)-\cos d(Tx_{n_i},p))\big)=0.$$

From Lemma 3.1, we have

$$\lim_{i\to\infty}(\cos d(x_{n_i},p)-\cos d(Rx_{n_i},p))=0,$$
$$\lim_{i\to\infty}(\cos d(x_{n_i},p)-\cos d(Sx_{n_i},p))=0,$$
$$\lim_{i\to\infty}(\cos d(x_{n_i},p)-\cos d(Tx_{n_i},p))=0.$$

Since R,S,T are strongly quasinonexpansive, we obtain

$$\lim_{i\to\infty}d(x_{n_i},Rx_{n_i})=\lim_{i\to\infty}d(x_{n_i},Sx_{n_i})=\lim_{i\to\infty}d(x_{n_i},Tx_{n_i})=0. \tag{1}$$

Take a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ satisfying

$$\limsup_{i\to\infty}\frac{\beta^R_{n_i}\gamma^R_{n_i}b^R_{n_i}+\beta^S_{n_i}\gamma^S_{n_i}b^S_{n_i}+\beta^T_{n_i}\gamma^T_{n_i}b^T_{n_i}}{\beta^R_{n_i}\gamma^R_{n_i}+\beta^S_{n_i}\gamma^S_{n_i}+\beta^T_{n_i}\gamma^T_{n_i}}\\ =\lim_{j\to\infty}\frac{\beta^R_{n_{i_j}}\gamma^R_{n_{i_j}}b^R_{n_{i_j}}+\beta^S_{n_{i_j}}\gamma^S_{n_{i_j}}b^S_{n_{i_j}}+\beta^T_{n_{i_j}}\gamma^T_{n_{i_j}}b^T_{n_{i_j}}}{\beta^R_{n_{i_j}}\gamma^R_{n_{i_j}}+\beta^S_{n_{i_j}}\gamma^S_{n_{i_j}}+\beta^T_{n_{i_j}}\gamma^T_{n_{i_j}}}.$$

Moreover, take a subsequence $\{z_r\}$ of $\{x_{n_{i_j}}\}$ satisfying

$$\liminf_{j\to\infty}d(u,x_{n_{i_j}})=\lim_{r\to\infty}d(u,z_r)$$

and a subsequence $\{z_{r_s}\}$ of $\{z_r\}$ with

$$\liminf_{r\to\infty}d(v,z_r)=\lim_{s\to\infty}d(v,z_{r_s}).$$

Furthermore, take a subsequence $\{z_{r_{s_t}}\}$ of $\{z_{r_s}\}$ satisfying

$$\liminf_{s\to\infty}d(w,z_{r_s})=\lim_{t\to\infty}d(w,z_{r_{s_t}})$$

and a subsequence $\{v_k\}$ of $\{z_{r_{s_t}}\}$ which satisfies $v_k \overset{\Delta}{\rightharpoonup} z\in X$. Then from the formula (1), we have $\lim_{k\to\infty}d(v_k,Rv_k)=\lim_{k\to\infty}d(v_k,Sv_k)=$

$\lim_{k\to\infty} d(v_k, Tv_k) = 0$ and hence $z \in F$. Further, since

$$\begin{aligned}\lim_{k\to\infty} d(u, v_k) = \liminf_{j\to\infty} d(u, x_{n_{i_j}}) &\le \liminf_{j\to\infty}(d(u, Rx_{n_{i_j}}) + d(Rx_{n_{i_j}}, x_{n_{i_j}}))\\ &= \liminf_{j\to\infty} d(u, Rx_{n_{i_j}})\\ &\le \liminf_{k\to\infty} d(u, Rv_k)\\ &\le \limsup_{k\to\infty} d(u, Rv_k)\\ &\le \limsup_{k\to\infty}(d(u, v_k) + d(v_k, Rv_k))\\ &= \lim_{k\to\infty} d(u, v_k),\end{aligned}$$

we get $\lim_{k\to\infty} d(u, v_k) = \lim_{k\to\infty} d(u, Rv_k)$. In the same way, we also obtain $\lim_{k\to\infty} d(v, v_k) = \lim_{k\to\infty} d(v, Sv_k)$ and $\lim_{k\to\infty} d(w, v_k) = \lim_{k\to\infty} d(w, Tv_k)$. By Theorem 2.2, we have

$$\begin{aligned}\lim_{k\to\infty} d(u, Rv_k) &= \lim_{k\to\infty} d(u, v_k) \ge d(u, z),\\ \lim_{k\to\infty} d(v, Sv_k) &= \lim_{k\to\infty} d(v, v_k) \ge d(v, z),\\ \lim_{k\to\infty} d(w, Tv_k) &= \lim_{k\to\infty} d(w, v_k) \ge d(w, z)\end{aligned}$$

and hence

$$\begin{aligned}&\lim_{k\to\infty} \left(\beta^R \cos d(u, Rv_k) + \beta^S \cos d(v, Sv_k) + \beta^T \cos d(w, Tv_k)\right)\\ &\le \beta^R \cos d(u, z) + \beta^S \cos d(v, z) + \beta^T \cos d(w, z)\\ &\le \beta^R \cos d(u, p) + \beta^S \cos d(v, p) + \beta^T \cos d(w, p).\end{aligned}$$

Put $d_1 = \lim_{k\to\infty} d(u, Rv_k)$, $d_2 = \lim_{k\to\infty} d(v, Sv_k)$, $d_3 = \lim_{k\to\infty} d(w, Tv_k)$ and put $m_k = n_{i_{j_{r_{s_{t_k}}}}}$ for all $k \in \mathbb{N}$. Then from Lemma 3.12, we obtain

$$\lim_{k\to\infty} \frac{\gamma^S_{m_k}}{\gamma^R_{m_k}} = \frac{\cos d_2}{\cos d_1}, \quad \lim_{k\to\infty} \frac{\gamma^T_{m_k}}{\gamma^R_{m_k}} = \frac{\cos d_3}{\cos d_1}.$$

Put

$$\begin{aligned}\mu^R &= \frac{\beta^R \cos d_1}{\beta^R \cos d_1 + \beta^S \cos d_2 + \beta^T \cos d_3},\\ \mu^S &= \frac{\beta^S \cos d_2}{\beta^R \cos d_1 + \beta^S \cos d_2 + \beta^T \cos d_3},\\ \mu^T &= \frac{\beta^T \cos d_3}{\beta^R \cos d_1 + \beta^S \cos d_2 + \beta^T \cos d_3}.\end{aligned}$$

Then we get

$$\begin{aligned}
&\limsup_{i\to\infty} \frac{\beta_{n_i}^R \gamma_{n_i}^R b_{n_i}^R + \beta_{n_i}^S \gamma_{n_i}^S b_{n_i}^S + \beta_{n_i}^T \gamma_{n_i}^T b_{n_i}^T}{\beta_{n_i}^R \gamma_{n_i}^R + \beta_{n_i}^S \gamma_{n_i}^S + \beta_{n_i}^T \gamma_{n_i}^T} \\
&= \lim_{k\to\infty} \frac{\beta_{m_k}^R \gamma_{m_k}^R b_{m_k}^R + \beta_{m_k}^S \gamma_{m_k}^S b_{m_k}^S + \beta_{m_k}^T \gamma_{m_k}^T b_{m_k}^T}{\beta_{m_k}^R \gamma_{m_k}^R + \beta_{m_k}^S \gamma_{m_k}^S + \beta_{m_k}^T \gamma_{m_k}^T} \\
&= \lim_{k\to\infty} \frac{\beta^R b_{m_k}^R + \beta^S \cdot \dfrac{\cos d_2}{\cos d_1} \cdot b_{m_k}^S + \beta^T \cdot \dfrac{\cos d_3}{\cos d_1} \cdot b_{m_k}^T}{\beta^R + \beta^S \cdot \dfrac{\cos d_2}{\cos d_1} + \beta^T \cdot \dfrac{\cos d_3}{\cos d_1}} \\
&= \lim_{k\to\infty} \left(\mu^R b_{m_k}^R + \mu^S b_{m_k}^S + \mu^T b_{m_k}^T\right) \\
&= \lim_{k\to\infty} \left(\mu^R \left(1 - \frac{\cos d(u,p)}{\cos d(u, Rv_k)}\right) + \mu^S \left(1 - \frac{\cos d(v,p)}{\cos d(v, Sv_k)}\right)\right. \\
&\quad \left. + \mu^T \left(1 - \frac{\cos d(w,p)}{\cos d(w, Tv_k)}\right)\right) \\
&= \lim_{k\to\infty} \left(\mu^R \cdot \frac{\cos d(u, Rv_k) - \cos d(u,p)}{\cos d_1} + \mu^S \cdot \frac{\cos d(v, Sv_k) - \cos d(v,p)}{\cos d_2}\right. \\
&\quad \left. + \mu^T \cdot \frac{\cos d(w, Tv_k) - \cos d(w,p)}{\cos d_3}\right) \\
&= \lim_{k\to\infty} \left(\frac{\beta^R \cos d(u, Rv_k) + \beta^S \cos d(v, Sv_k) + \beta^T \cos d(w, Tv_k)}{\beta^R \cos d_1 + \beta^S \cos d_2 + \beta^T \cos d_3}\right. \\
&\quad \left. - \frac{\beta^R \cos d(u,p) + \beta^S \cos d(v,p) + \beta^T \cos d(w,p)}{\beta^R \cos d_1 + \beta^S \cos d_2 + \beta^T \cos d_3}\right) \\
&\le 0.
\end{aligned}$$

Thus we have (ii). Hence, using Lemma 2.5, we obtain the desired result. □

In the formula

$$x_{n+1} = \beta_n s_n \overset{1}{\oplus} (1-\beta_n)(\gamma_n r_n \overset{1}{\oplus} (1-\gamma_n) t_n)$$

of Theorem 3.13, the limit of the sequence $\{x_n\}$ does not depend on the order of the convex combination of r_n, s_n and t_n unless the weight of the coefficient of the convex combination of r_n, s_n and t_n is changed. For example, the sequence

$\{x_n\}$ defined by

$$\begin{cases} r_n = \alpha_n u \overset{1}{\oplus} (1-\alpha_n) R x_n, \\ s_n = \alpha_n v \overset{1}{\oplus} (1-\alpha_n) S x_n, \\ t_n = \alpha_n w \overset{1}{\oplus} (1-\alpha_n) T x_n, \\ x_{n+1} = \beta'_n s_n \overset{1}{\oplus} (1-\beta'_n)(\gamma'_n r_n \overset{1}{\oplus} (1-\gamma'_n) t_n) \end{cases}$$

for all $n \in \mathbb{N}$ converges to

$$p = \operatorname*{argmax}_{x\in F} \left(\beta \cos d(u,x) + (1-\beta)(\gamma \cos d(v,x) + (1-\gamma)\cos d(w,x))\right),$$

where $\beta'_n = (1-\beta_n)\gamma_n$, $(1-\beta'_n)\gamma'_n = \beta_n$.

References

[1] K. Aoyama, Y. Kimura and F. Kohsaka, *Strong convergence theorems for strongly relatively nonexpansive sequences and applications*, J. Nonlinear Anal. Optim. **3** (2012), 67–77.

[2] B. Halpern, *Fixed points of nonexpansive maps*, Bull. Am. Math. Soc. **73** (1967), 957–961.

[3] R. Espínola and A. Fernández-León, *CAT(k)-spaces, weak convergence and fixed points*, J. Math. Anal. Appl. **353** (2009), 410–427.

[4] J. S. He, D. H. Fang, G. Lopez, and C. Li, *Mann's algorithm for nonexpansive mappings in CAT(κ) spaces*, Nonlinear Anal. **75** (2012), 445–452.

[5] Y. Kimura and K. Satô, *Halpern iteration for strongly quasinonexpansive mappings on a geodesic space with curvature bounded above by one*, Fixed Point Theory Appl. **2013** (2013), 14pages.

[6] Y. Kimura and H. Wada, *Halpern type iteration with multiple anchor points in a Hadamard space*, J. Inequal. Appl. **2015** (2015), 11 pages.

[7] S. Saejung, *Halpern's iteration in CAT(0) spaces*, Fixed Point Theory Appl. **2010** (2010), 13 pages.

[8] S. Saejung and P. Yotkaew, *Approximation of zeros of inverse strongly monotone operators in Banach spaces*, Nonlinear Anal. **75** (2012), 742–750.

[9] N. Shioji and W. Takahashi, *Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces*, Proc. Amer. Math. Soc. **125** (1997), 3641–3645.

[10] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. (Basel) **58** (1992), 486–491.

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